SOME OSTROWSKI'S TYPE INEQUALITIES FOR CONVEX FUNCTIONS WITH APPLICATIONS

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Abstract. In this paper, some refitments inequalities of Ostrowski's type for the class of convex functions are introduced. Error estimates for some special means are obtained.

1. Introduction

Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I°, the interior of the interval I, such that $f' \in L[a,b]$, where $a,b \in I$ with $a < b$. If $|f'(x)| \le M$, then the following inequality,

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \le M(b-a)\left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right] \tag{1.1}$$

holds. This result is known in the literature as the *Ostrowski inequality*.

Recently, several generalisations of the Ostrowski integral inequality for mappings of bounded variation and for Lipschitzian, monotonic, absolutely continuous and n-times differentiable mappings with error estimates for some special means and for some numerical quadrature rules are considered by many authors. For recent results and generalizations concerning Ostrowski's inequality see [1]–[16] and the references therein.

The aim of this paper is to establish some Ostrowski's type inequalities for the class of convex (concave) functions.

2. Ostrowski's Type Inequalities

In order to prove our main theorems, we need the following lemma:

Lemma 1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping mapping on I° where $a,b \in I$ with $a < b$. If $f' \in L[a,b]$, then the following equality holds:*

$$f(x) - \frac{1}{b-a}\int_a^b f(u)\,du = (b-a)\int_0^1 p(t)\,f'(ta + (1-t)b)\,dt \tag{2.1}$$

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for each $t \in [0,1]$, *where*

$$p(t) = \begin{cases} t, & t \in \left[0, \frac{b-x}{b-a}\right] \\ \\ t-1, & t \in \left(\frac{b-x}{b-a}, 1\right] \end{cases},$$

for all $x \in [a,b]$.

Proof. Integrating by parts

$$\begin{aligned}
I &= \int_0^1 p(t) f'(ta+(1-t)b)\,dt \\
&= \int_0^{\frac{b-x}{b-a}} t f'(ta+(1-t)b)\,dt + \int_{\frac{b-x}{b-a}}^1 (t-1) f'(ta+(1-t)b)\,dt \\
&= t\frac{f(ta+(1-t)b)}{a-b}\bigg|_0^{\frac{b-x}{b-a}} - \int_0^{\frac{b-x}{b-a}} \frac{f(ta+(1-t)b)}{a-b}dt \\
&\qquad + (t-1)\frac{f(ta+(1-t)b)}{a-b}\bigg|_{\frac{b-x}{b-a}}^1 - \int_{\frac{b-x}{b-a}}^1 \frac{f(ta+(1-t)b)}{a-b}dt \\
&= \frac{b-x}{(b-a)^2} f(x) - \int_0^{\frac{b-x}{b-a}} \frac{f(ta+(1-t)b)}{a-b}dt + \frac{x-a}{(b-a)^2} f(x) - \int_{\frac{b-x}{b-a}}^1 \frac{f(ta+(1-t)b)}{a-b}dt \\
&= \frac{1}{b-a} f(x) - \frac{1}{(b-a)^2}\int_a^b f(u)\,du.
\end{aligned}$$

Thus, $(b-a)\cdot I$, gives the desired representation (2.1). ∎

The main result may be stated as follows:

Theorem 1. *Let* $f : I \subset [0,\infty) \to \mathbb{R}$ *be a differentiable mapping on* I° *such that* $f' \in L[a,b]$, *where* $a, b \in I$ *with* $a < b$. *If* $|f'|$ *is convex on* $[a,b]$, *then the following inequality holds:*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \le \left[\frac{1}{6} + \frac{1}{3}\left(\frac{x-a}{b-a}\right)^3\right] |f'(b)| + \left[\frac{1}{6} + \frac{1}{3}\left(\frac{b-x}{b-a}\right)^3\right] |f'(a)|, \tag{2.2}$$

for each $x \in [a,b]$.

Proof. By Lemma 1 and since $|f'|$ is convex, then we have

$$\begin{aligned}
\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right| &\leq (b-a)\int_0^{\frac{b-x}{b-a}} t\left|f'(ta+(1-t)b)\right|dt \\
&\quad + (b-a)\int_{\frac{b-x}{b-a}}^1 (1-t)\left|f'(ta+(1-t)b)\right|dt \\
&\leq (b-a)\int_0^{\frac{b-x}{b-a}} t\left[t\left|f'(a)\right|+(1-t)\left|f'(b)\right|\right]dt \\
&\quad + (b-a)\int_{\frac{b-x}{b-a}}^1 (1-t)\left[t\left|f'(a)\right|+(1-t)\left|f'(b)\right|\right]dt \\
&\leq \left[\frac{1}{6}+\frac{1}{3}\left(\frac{x-a}{b-a}\right)^3\right]\left|f'(b)\right| + \left[\frac{1}{6}+\frac{1}{3}\left(\frac{b-x}{b-a}\right)^3\right]\left|f'(a)\right|,
\end{aligned}$$

This completes the proof. ∎

One can deduce an Ostrowski like inequality for functions whose derivative absolute value is convex as follows:

Corollary 1. *In Theorem 1. Additionally, if $|f'(x)| \leq M$, $M > 0$, then inequality*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \leq M(b-a)\left[\frac{1}{3}+\frac{(b-x)^3+(x-a)^3}{3(b-a)^3}\right]. \tag{2.3}$$

holds. The constant $\frac{1}{3}$ is best possible in sense that cannot be replaced by smaller one.

The following inequality is of midpoint type.

Corollary 2. *In Theorem 1, choose $x = \frac{a+b}{2}$, then*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \leq \frac{5(b-a)}{24}\left[\left|f'(b)\right|+\left|f'(a)\right|\right]. \tag{2.4}$$

The corresponding version for powers of the absolute value of the first derivative is incorporated in the following result:

Theorem 2. *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|^{p/(p-1)}$ is convex on $[a,b]$, then the following inequality holds:*

$$\begin{aligned}
\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right| &\leq \frac{2^{-1/q}}{\left[(b-a)(p+1)\right]^{1/p}}\left[(b-x)^{\frac{p+1}{p}}\left(\left|f'(x)\right|^q+\left|f'(b)\right|^q\right)^{1/q}\right. \\
&\quad \left. + (x-a)^{\frac{p+1}{p}}\left(\left|f'(a)\right|^q+\left|f'(x)\right|^q\right)^{1/q}\right]
\end{aligned} \tag{2.5}$$

for each $x \in [a,b]$, where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Suppose that $p>1$. From Lemma 1 and using the Hölder inequality, we have

$$\begin{aligned}
&\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \\
&\leq (b-a)\int_0^{\frac{b-x}{b-a}} t\left|f'(ta+(1-t)b)\right|dt \\
&\qquad + (b-a)\int_{\frac{b-x}{b-a}}^1 |t-1|\left|f'(ta+(1-t)b)\right|dt \\
&\leq (b-a)\left(\int_0^{\frac{b-x}{b-a}} t^p dt\right)^{1/p}\left(\int_0^{\frac{b-x}{b-a}}\left|f'(ta+(1-t)b)\right|^q dt\right)^{1/q} \\
&\qquad + (b-a)\left(\int_{\frac{b-x}{b-a}}^1 (1-t)^p\,dt\right)^{1/p}\left(\int_{\frac{b-x}{b-a}}^1\left|f'(ta+(1-t)b)\right|^q dt\right)^{1/q}.
\end{aligned}$$

Since $|f'|$ is convex, by Hermite-Hadamard inequality we have,

$$\int_0^{\frac{b-x}{b-a}}\left|f'(ta+(1-t)b)\right|dt \leq \frac{|f'(x)|+|f'(b)|}{2}, \tag{2.6}$$

and

$$\int_{\frac{b-x}{b-a}}^1\left|f'(ta+(1-t)b)\right|dt \leq \frac{|f'(a)|+|f'(x)|}{2}. \tag{2.7}$$

Therefore,

$$\begin{aligned}
\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \leq \frac{2^{-1/q}}{[(b-a)(p+1)]^{1/p}}\Big[&(b-x)^{\frac{p+1}{p}}\left(|f'(x)|^q+|f'(b)|^q\right)^{1/q} \\
&+(x-a)^{\frac{p+1}{p}}\left(|f'(a)|^q+|f'(x)|^q\right)^{1/q}\Big],
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$. This completes the proof. ∎

Corollary 3. *In Theorem 2. Additionally, if $|f'(x)|\leq M$, $M>0$, then inequality*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \leq M\cdot\frac{(b-x)^{\frac{p+1}{p}}+(x-a)^{\frac{p+1}{p}}}{(p+1)^{\frac{1}{p}}(b-a)^{\frac{1}{p}}} \tag{2.8}$$

holds, where $\frac{1}{p}+\frac{1}{q}=1$.

Corollary 4. *In Theorem 2, choose $x=\frac{a+b}{2}$, then*

$$\begin{aligned}
\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \leq \quad \frac{(b-a)}{4(p+1)^{1/p}}\Bigg[&\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(b)|^q\right)^{1/q} \\
&+\left(|f'(a)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{1/q}\Bigg].
\end{aligned} \tag{2.9}$$

Theorem 3. *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|^{p/(p-1)}$ is concave on $[a,b]$, then the following inequality holds:*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(u)\,du\right| \le \frac{(b-a)}{(p+1)^{1/p}}\left[\left(\frac{b-x}{b-a}\right)^{(p+1)/p}\left|f'\left(\frac{b+x}{2}\right)\right| + \left(\frac{x-a}{b-a}\right)^{(p+1)/p}\left|f'\left(\frac{a+x}{2}\right)\right|\right], \tag{2.10}$$

for each $x \in [a,b]$, where $p > 1$.

Proof. Suppose that $p > 1$. From Lemma 1 and using the Hölder inequality, we have

$$\begin{aligned}
&\left|f(x) - \frac{1}{b-a}\int_a^b f(u)\,du\right| \\
&\le (b-a)\int_0^{\frac{b-x}{b-a}} t\,|f'(ta+(1-t)b)|\,dt \\
&\qquad + (b-a)\int_{\frac{b-x}{b-a}}^1 |t-1|\,|f'(ta+(1-t)b)|\,dt \\
&\le (b-a)\left(\int_0^{\frac{b-x}{b-a}} t^p dt\right)^{1/p}\left(\int_0^{\frac{b-x}{b-a}} |f'(ta+(1-t)b)|^q\,dt\right)^{1/q} \\
&\qquad + (b-a)\left(\int_{\frac{b-x}{b-a}}^1 (1-t)^p\,dt\right)^{1/p}\left(\int_{\frac{b-x}{b-a}}^1 |f'(ta+(1-t)b)|^q\,dt\right)^{1/q}.
\end{aligned}$$

Since $|f'|^q$ is concave on $[a,b]$, by Hermite-Hadamard's inequality we get

$$\int_0^{\frac{b-x}{b-a}} |f'(ta+(1-t)b)|^q\,dt \le \left|f'\left(\frac{b+x}{2}\right)\right|^q \tag{2.11}$$

and

$$\int_{\frac{b-x}{b-a}}^1 |f'(ta+(1-t)b)|^q\,dt \le \left|f'\left(\frac{a+x}{2}\right)\right|^q \tag{2.12}$$

Therefore,

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(u)\,du\right| \le \frac{(b-a)}{(p+1)^{1/p}}\left[\left(\frac{b-x}{b-a}\right)^{(p+1)/p}\left|f'\left(\frac{b+x}{2}\right)\right| + \left(\frac{x-a}{b-a}\right)^{(p+1)/p}\left|f'\left(\frac{a+x}{2}\right)\right|\right]$$

This completes the proof. ∎

Corollary 5. *In Theorem 3, choose* $x = \frac{a+b}{2}$*, then*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \leq \frac{(b-a)}{\left(2^{p+1}(p+1)\right)^{1/p}}\left[\left|f'\left(\frac{a+3b}{4}\right)\right|+\left|f'\left(\frac{3a+b}{4}\right)\right|\right], \tag{2.13}$$

for each $x \in [a,b]$*, where* $p > 1$*.*

The following result refines the above inequality (2.10).

Theorem 4. *Let* $f : I \subset [0,\infty) \to \mathbb{R}$ *be a differentiable mapping on* I° *such that* $f' \in L[a,b]$*, where* $a, b \in I$ *with* $a < b$*. If* $|f'|^{p/(p-1)}$ *is concave on* $[a,b]$*, then the following inequality holds:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \leq \frac{(b-x)^2}{(b-a)(p+1)^{1/p}}\left|f'\left(\frac{b+x}{2}\right)\right| + \frac{(x-a)^2}{(b-a)(p+1)^{1/p}}\left|f'\left(\frac{a+x}{2}\right)\right| \tag{2.14}$$

for each $x \in [a,b]$*, where* $p > 1$*.*

Proof. Suppose that $p > 1$. From Lemma 1 and using the Hölder inequality, we have

$$\begin{aligned}
&\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \\
&\leq (b-a)\int_0^{\frac{b-x}{b-a}} t\left|f'(ta+(1-t)b)\right|dt \\
&\qquad + (b-a)\int_{\frac{b-x}{b-a}}^1 |t-1|\left|f'(ta+(1-t)b)\right|dt \\
&\leq (b-a)\left(\int_0^{\frac{b-x}{b-a}} t^p dt\right)^{1/p}\left(\int_0^{\frac{b-x}{b-a}} \left|f'(ta+(1-t)b)\right|^q dt\right)^{1/q} \\
&\qquad + (b-a)\left(\int_{\frac{b-x}{b-a}}^1 (1-t)^p\,dt\right)^{1/p}\left(\int_{\frac{b-x}{b-a}}^1 \left|f'(ta+(1-t)b)\right|^q dt\right)^{1/q}.
\end{aligned}$$

Since $|f'|^q$ is concave on $[a,b]$, we can use the Jensen's integral inequality to obtain

$$\begin{aligned}
\int_0^{\frac{b-x}{b-a}} \left|f'(ta+(1-t)b)\right|^q dt &\leq \int_0^{\frac{b-x}{b-a}} t^0\left|f'(ta+(1-t)b)\right|^q dt \\
&\leq \left(\int_0^{\frac{b-x}{b-a}} t^0 dt\right)\left|f'\left(\frac{1}{\int_0^{\frac{b-x}{b-a}} t^0 dt}\int_0^{\frac{b-x}{b-a}} (ta+(1-t)b)\,dt\right)\right|^q \\
&= \frac{b-x}{b-a}\left|f'\left(\frac{b+x}{2}\right)\right|^q
\end{aligned} \tag{2.15}$$

and

$$\int_{\frac{b-x}{b-a}}^{1} \left|f'\left(ta+(1-t)b\right)\right|^{q} dt \leq \int_{\frac{b-x}{b-a}}^{1} t^{0} \left|f'\left(ta+(1-t)b\right)\right|^{q} dt$$

$$\leq \left(\int_{\frac{b-x}{b-a}}^{1} t^{0} dt\right) \left|f'\left(\frac{1}{\int_{\frac{b-x}{b-a}}^{1} t^{0} dt} \int_{\frac{b-x}{b-a}}^{1} \left(ta+(1-t)b\right) dt\right)\right|^{q}$$

$$= \frac{x-a}{b-a} \left|f'\left(\frac{a+x}{2}\right)\right|^{q} \tag{2.16}$$

Therefore,

$$\left|f(x) - \frac{1}{b-a}\int_{a}^{b} f(u)\,du\right| \leq \frac{(b-x)^{2}}{(b-a)(p+1)^{1/p}} \left|f'\left(\frac{b+x}{2}\right)\right| + \frac{(x-a)^{2}}{(b-a)(p+1)^{1/p}} \left|f'\left(\frac{a+x}{2}\right)\right|$$

This completes the proof. ∎

Corollary 6. *In Theorem 4, choose* $x = \frac{a+b}{2}$*, then*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_{a}^{b} f(u)\,du\right| \leq \frac{(b-a)}{4(p+1)^{1/p}} \left[\left|f'\left(\frac{a+3b}{4}\right)\right| + \left|f'\left(\frac{3a+b}{4}\right)\right|\right], \tag{2.17}$$

for each $x \in [a,b]$*, where* $p > 1$*.*

A different approach for powers of the absolute value of the first derivative leads to the following result:

Theorem 5. *Let* $f : I \subset [0,\infty) \to \mathbb{R}$ *be a differentiable mapping on* I° *such that* $f' \in L[a,b]$*, where* $a, b \in I$ *with* $a < b$*. If* $|f'|^{q}$ *is convex on* $[a,b]$*,* $q \geq 1$*, and* $|f'(x)| \leq M$*,* $x \in [a,b]$*, then the following inequality holds:*

$$\left|f(x) - \frac{1}{b-a}\int_{a}^{b} f(u)\,du\right| \tag{2.18}$$

$$\leq (b-a)\left(\frac{b-x}{b-a}\right)^{2(1-1/q)} \left(\frac{1}{2}\left(\frac{b-x}{b-a}\right)^{2} |f'(a)|^{q} + \frac{(b-x)^{2}(b-3a+2x)}{6(b-a)^{3}} |f'(b)|^{q}\right)^{1/q}$$

$$+(b-a)\left(\frac{x-a}{b-a}\right)^{2(1-1/q)} \left(\left[\frac{1}{6} + \frac{(b-x)^{2}(3a-2x-b)}{6(b-a)^{3}}\right] |f'(a)|^{q} + \frac{1}{2}\left(\frac{x-a}{b-a}\right)^{2} |f'(b)|^{q}\right)^{1/q},$$

for each $x \in [a,b]$*.*

Proof. Suppose that $q \geq 1$. From Lemma 1 and using the well known power mean inequality, we have

$$\begin{aligned}
&\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \\
&\leq (b-a)\int_0^{\frac{b-x}{b-a}} t\left|f'(ta+(1-t)b)\right|dt \\
&\qquad + (b-a)\int_{\frac{b-x}{b-a}}^1 |t-1|\left|f'(ta+(1-t)b)\right|dt \\
&\leq (b-a)\left(\int_0^{\frac{b-x}{b-a}} tdt\right)^{1-1/q}\left(\int_0^{\frac{b-x}{b-a}} t\left|f'(ta+(1-t)b)\right|^q dt\right)^{1/q} \\
&\qquad + (b-a)\left(\int_{\frac{b-x}{b-a}}^1 (1-t)\,dt\right)^{1-1/q}\left(\int_{\frac{b-x}{b-a}}^1 (1-t)\left|f'(ta+(1-t)b)\right|^q dt\right)^{1/q}.
\end{aligned}$$

Since $|f'|^q$ is convex, we have

$$\begin{aligned}
\int_0^{\frac{b-x}{b-a}} t\left|f'(ta+(1-t)b)\right|^q dt &\leq \int_0^{\frac{b-x}{b-a}} t\cdot\left[t\left|f'(a)\right|^q+(1-t)\left|f'(b)\right|^q\right]dt \\
&= \frac{1}{2}\left(\frac{b-x}{b-a}\right)^2\left|f'(a)\right|^q + \frac{(b-x)^2(b-3a+2x)}{6(b-a)^3}\left|f'(b)\right|^q
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\frac{b-x}{b-a}}^1 (1-t)\left|f'(ta+(1-t)b)\right|^q dt \\
&\leq \int_{\frac{b-x}{b-a}}^1 (1-t)\cdot\left[t\left|f'(a)\right|^q+(1-t)\left|f'(b)\right|^q\right]dt \\
&= \left[\frac{1}{6}+\frac{(b-x)^2(3a-2x-b)}{6(b-a)^3}\right]\left|f'(a)\right|^q + \frac{1}{2}\left(\frac{x-a}{b-a}\right)^2\left|f'(b)\right|^q
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \\
&\leq (b-a)\left(\frac{b-x}{b-a}\right)^{2(1-1/q)}\left(\frac{1}{2}\left(\frac{b-x}{b-a}\right)^2\left|f'(a)\right|^q+\frac{(b-x)^2(b-3a+2x)}{6(b-a)^3}\left|f'(b)\right|^q\right)^{1/q} \\
&\quad +(b-a)\left(\frac{x-a}{b-a}\right)^{2(1-1/q)}\left(\left[\frac{1}{6}+\frac{(b-x)^2(3a-2x-b)}{6(b-a)^3}\right]\left|f'(a)\right|^q+\frac{1}{2}\left(\frac{x-a}{b-a}\right)^2\left|f'(b)\right|^q\right)^{1/q},
\end{aligned}$$

which is required. ∎

Corollary 7. *In Theorem 5, choose $x=\frac{a+b}{2}$, then*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(u)\,du\right|\leq\left(\frac{1}{2}\right)^{2(1-1/q)}\left[\left(\frac{1}{8}\left|f'(a)\right|^q+\frac{1}{12}\left|f'(b)\right|^q\right)^{1/q}\right.$$

$$\left.+\left(\frac{1}{12}\left|f'(a)\right|^q+\frac{1}{8}\left|f'(b)\right|^q\right)^{1/q}\right]. \tag{2.19}$$

For instance, if $q=1$, then (2.19) becomes

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(u)\,du\right|\leq\frac{5}{24}\left(\left|f'(a)\right|+\left|f'(b)\right|\right).$$

which improves the inequality (2.4).

Theorem 6. *Let $f:I\subset[0,\infty)\to\mathbb{R}$ be a differentiable mapping on I° such that $f'\in L[a,b]$, where $a,b\in I$ with $a<b$. If $|f'|^q$ is concave on $[a,b]$, $q\geq1$, and $|f'(x)|\leq M$, $x\in[a,b]$, then the following inequality holds:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right|\leq 2^{-1/q}(b-a)\left[\left(\frac{b-x}{b-a}\right)^2\left|f'\left(\frac{b+2x}{3}\right)\right|\right.$$
$$\left.+\left(\frac{x-a}{b-a}\right)^2\left|f'\left(\frac{a+2x}{3}\right)\right|\right], \tag{2.20}$$

for each $x\in[a,b]$.

Proof. First, we note that by concavity of $|f'|^q$ and the power-mean inequality, we have

$$\left|f'(\alpha x+(1-\alpha)y)\right|^q\geq\alpha\left|f'(x)\right|^q+(1-\alpha)\left|f'(y)\right|^q.$$

Hence,

$$\left|f'(\alpha x+(1-\alpha)y)\right|\geq\alpha\left|f'(x)\right|+(1-\alpha)\left|f'(y)\right|.$$

so, $|f'|$ is also concave.

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right|$$
$$\leq(b-a)\int_0^{\frac{b-x}{b-a}}t\left|f'(ta+(1-t)b)\right|dt$$
$$+(b-a)\int_{\frac{b-x}{b-a}}^1|t-1|\left|f'(ta+(1-t)b)\right|dt$$
$$\leq(b-a)\left(\int_0^{\frac{b-x}{b-a}}tdt\right)^{1-1/q}\left(\int_0^{\frac{b-x}{b-a}}t\left|f'(ta+(1-t)b)\right|^q dt\right)^{1/q}$$
$$+(b-a)\left(\int_{\frac{b-x}{b-a}}^1(1-t)\,dt\right)^{1-1/q}\left(\int_{\frac{b-x}{b-a}}^1(1-t)\left|f'(ta+(1-t)b)\right|^q dt\right)^{1/q}.$$

Accordingly, by Lemma 1 and the Jensen integral inequality, we have

$$\int_0^{\frac{b-x}{b-a}} t\left|f'\left(ta+(1-t)b\right)\right|^q dt \le \left(\int_0^{\frac{b-x}{b-a}} tdt\right)\left|f'\left(\frac{\int_0^{\frac{b-x}{b-a}} t\left(ta+(1-t)b\right)dt}{\int_0^{\frac{b-x}{b-a}} tdt}\right)\right|^q$$
$$= \frac{1}{2}\left(\frac{b-x}{b-a}\right)^2\left|f'\left(\frac{b+2x}{3}\right)\right|^q$$

and

$$\int_{\frac{b-x}{b-a}}^1 (1-t)\left|f'\left(ta+(1-t)b\right)\right|^q dt \le \left(\int_{\frac{b-x}{b-a}}^1 (1-t)\,dt\right)\left|f'\left(\frac{\int_{\frac{b-x}{b-a}}^1 (1-t)\left(ta+(1-t)b\right)dt}{\int_{\frac{b-x}{b-a}}^1 (1-t)\,dt}\right)\right|^q$$
$$= \frac{1}{2}\left(\frac{x-a}{b-a}\right)^2\left|f'\left(\frac{a+2x}{3}\right)\right|^q$$

Therefore,

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \le 2^{-1/q}(b-a)\left[\left(\frac{b-x}{b-a}\right)^2\left|f'\left(\frac{b+2x}{3}\right)\right| + \left(\frac{x-a}{b-a}\right)^2\left|f'\left(\frac{a+2x}{3}\right)\right|\right],$$

which completes the proof. ∎

Corollary 8. *In Theorem 6, choose $x=\frac{a+b}{2}$, we get*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \le \frac{2^{-1/q}}{4}(b-a)\left[\left|f'\left(\frac{a+2b}{3}\right)\right|+\left|f'\left(\frac{2a+b}{3}\right)\right|\right]. \tag{2.21}$$

For instance if $q=1$, then

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \le \frac{(b-a)}{8}\left[\left|f'\left(\frac{a+2b}{3}\right)\right|+\left|f'\left(\frac{2a+b}{3}\right)\right|\right]. \tag{2.22}$$

3. Applications to Special Means

We shall consider the means for arbitrary real numbers α,β $(\alpha\neq\beta)$. We take

(1) The arithmetic mean :

$$A := A(\alpha,\beta) = \frac{\alpha+\beta}{2}, \quad \alpha,\beta\in\mathbb{R}_+.$$

(2) The geometric mean :

$$G := G(\alpha,\beta) = \sqrt{\alpha\beta}, \quad \alpha,\beta\in\mathbb{R}_+$$

(3) The harmonic mean :

$$H := H(\alpha, \beta) = \frac{2}{\frac{1}{\alpha} + \frac{1}{\beta}}, \quad \alpha, \beta \in \mathbb{R}_+ - \{0\}.$$

(4) The identric mean:

$$I(\alpha, \beta) = \begin{cases} \frac{1}{e}\left(\frac{\beta^\beta}{\alpha^\alpha}\right)^{\frac{1}{\beta-\alpha}}, & \alpha \neq \beta \\ \alpha, & \alpha = \beta \end{cases}, \quad \alpha, \beta > 0$$

(5) The logarithmic mean :

$$L := L(\alpha, \beta) = \frac{\alpha - \beta}{\ln|\alpha| - \ln|\beta|}, \; |\alpha| \neq |\beta|, \; \alpha, \beta \neq 0, \; \alpha, \beta \in \mathbb{R}_+.$$

(6) The generalized log-mean:

$$L_p := L_p(\alpha, \beta) = \left[\frac{\beta^{p+1} - \alpha^{p+1}}{(p+1)(\beta - \alpha)}\right]^{\frac{1}{n}}, \; p \in \mathbb{R}\backslash\{-1, 0\}, \; \alpha, \beta > 0.$$

It is well known that L_p is monotonic nondecreasing over $p \in \mathbb{R}$, with $L_{-1} := L$ and $L_0 := I$. In particular, we have the following inequality $L \leq A$.

Now, using the results of Section 2, we give some applications to special means of real numbers. In the following we obtain some error estimates for some special means.

(1) Consider $f : [a, b] \to \mathbb{R}$, $(0 < a < b)$, $f(x) = x^r$, $r \in \mathbb{R}\backslash\{-1, 0\}$. Then,

$$\frac{1}{b-a}\int_a^b f(x)\, dx = L_r^r(a, b),$$

(a) Using the inequality (2.3), we get

$$|x^r - L_r^r| \leq (b-a)\,\mu_r(a, b)\left[\frac{1}{3} + \frac{(b-x)^3 + (x-a)^3}{3(b-a)^3}\right],$$

where,

$$\mu_r(a, b) = \begin{cases} rb^{r-1}, & r \geq 1 \\ |p|\, a^{r-1}, & r \in (-\infty, 0) \cup (0, 1) \setminus \{-1\} \end{cases}$$

For instance, if we choose

(i) $x = A$, then we get

$$|A^r - L_r^r| \leq \frac{5(b-a)}{12}\mu_r(a, b),$$

(ii) $x = G$, then we get

$$|G^r - L_r^r| \leq (b-a)\,\mu_r(a, b)\left[\frac{1}{3} + \frac{(b-G)^3 + (G-a)^3}{3(b-a)^3}\right],$$

(iii) $x=H$, then we get

$$|H^r-L_r^r|\leq (b-a)\,\mu_r\,(a,b)\left[\frac{1}{3}+\frac{(b-H)^3+(H-a)^3}{3\,(b-a)^3}\right],$$

(iv) $x=I$, then we get

$$|I^r-L_r^r|\leq (b-a)\,\mu_r\,(a,b)\left[\frac{1}{3}+\frac{(b-I)^3+(I-a)^3}{3\,(b-a)^3}\right],$$

(v) $x=L$, then we get

$$|L^r-L_r^r|\leq (b-a)\,\mu_r\,(a,b)\left[\frac{1}{3}+\frac{(b-L)^3+(L-a)^3}{3\,(b-a)^3}\right].$$

(b) Using the inequality (2.8), we get

$$|x^r-L_r^r|\leq \mu_r\,(a,b)\,\frac{(b-x)^{\frac{p+1}{p}}+(x-a)^{\frac{p+1}{p}}}{(p+1)^{\frac{1}{p}}\,(b-a)^{\frac{1}{p}}},$$

where, $p>1$. For instance, if we choose

(i) $x=A$, then we get

$$|A^r-L_r^r|\leq \mu_r\,(a,b)\,\frac{(b-A)^{\frac{p+1}{p}}+(A-a)^{\frac{p+1}{p}}}{(p+1)^{\frac{1}{p}}\,(b-a)^{\frac{1}{p}}},$$

(ii) $x=G$, then we get

$$|G^r-L_r^r|\leq \mu_r\,(a,b)\,\frac{(b-G)^{\frac{p+1}{p}}+(G-a)^{\frac{p+1}{p}}}{(p+1)^{\frac{1}{p}}\,(b-a)^{\frac{1}{p}}},$$

(iii) $x=H$, then we get

$$|H^r-L_r^r|\leq \mu_r\,(a,b)\,\frac{(b-H)^{\frac{p+1}{p}}+(H-a)^{\frac{p+1}{p}}}{(p+1)^{\frac{1}{p}}\,(b-a)^{\frac{1}{p}}},$$

(iv) $x=I$, then we get

$$|I^r-L_r^r|\leq \mu_r\,(a,b)\,\frac{(b-I)^{\frac{p+1}{p}}+(I-a)^{\frac{p+1}{p}}}{(p+1)^{\frac{1}{p}}\,(b-a)^{\frac{1}{p}}},$$

(v) $x=L$, then we get

$$|L^r-L_r^r|\leq \mu_r\,(a,b)\,\frac{(b-L)^{\frac{p+1}{p}}+(L-a)^{\frac{p+1}{p}}}{(p+1)^{\frac{1}{p}}\,(b-a)^{\frac{1}{p}}}.$$

(2) Consider $f:[a,b]\subseteq(0,\infty)\to\mathbb{R}$, $(0<a<b)$, $f(x)=\ln x$, then,

$$\frac{1}{b-a}\int\limits_a^b f\,(x)\,dx=\ln I\,(a,b):=\ln I,$$

(a) Using the inequality (2.10), we get

$$|\ln x - \ln I| \leq \frac{2}{(b-a)^{1/p}(p+1)^{1/p}} \left[\frac{(b-x)^{(p+1)/p}}{b+x} + \frac{(x-a)^{(p+1)/p}}{x+a} \right].$$

where, $x \neq I$ and $p > 1$. For instance, if we choose

(i) $x = A$, then we get

$$|\ln A - \ln I| \leq \frac{2}{(b-a)^{1/p}(p+1)^{1/p}} \left[\frac{(b-A)^{(p+1)/p}}{b+A} + \frac{(A-a)^{(p+1)/p}}{A+a} \right],$$

(ii) $x = G$, then we get

$$|\ln G - \ln I| \leq \frac{2}{(b-a)^{1/p}(p+1)^{1/p}} \left[\frac{(b-G)^{(p+1)/p}}{b+G} + \frac{(G-a)^{(p+1)/p}}{G+a} \right],$$

(iii) $x = H$, then we get

$$|\ln H - \ln I| \leq \frac{2}{(b-a)^{1/p}(p+1)^{1/p}} \left[\frac{(b-H)^{(p+1)/p}}{b+H} + \frac{(H-a)^{(p+1)/p}}{H+a} \right],$$

(iv) $x = L$, then we get

$$|\ln L - \ln I| \leq \frac{2}{(b-a)^{1/p}(p+1)^{1/p}} \left[\frac{(b-L)^{(p+1)/p}}{b+L} + \frac{(L-a)^{(p+1)/p}}{L+a} \right].$$

(b) Using the inequality (2.14), we get

$$|\ln x - \ln I| \leq \frac{3 \cdot 2^{-1/q}}{(b-a)} \left[\frac{(b-x)^2}{b+2x} + \frac{(x-a)^2}{a+2x} \right],$$

where, $x \neq I$ and $q \geq 1$. For instance, if we choose

(i) $x = A$, then we get

$$|\ln A - \ln I| \leq \frac{3 \cdot 2^{-1/q}}{(b-a)} \left[\frac{(b-A)^2}{b+2A} + \frac{(A-a)^2}{a+2A} \right],$$

(ii) $x = G$, then we get

$$|\ln G - \ln I| \leq \frac{3 \cdot 2^{-1/q}}{(b-a)} \left[\frac{(b-G)^2}{b+2G} + \frac{(G-a)^2}{a+2G} \right],$$

(iii) $x = H$, then we get

$$|\ln H - \ln I| \leq \frac{3 \cdot 2^{-1/q}}{(b-a)} \left[\frac{(b-H)^2}{b+2H} + \frac{(H-a)^2}{a+2H} \right],$$

(iv) $x = L$, then we get

$$|\ln L - \ln I| \leq \frac{3 \cdot 2^{-1/q}}{(b-a)} \left[\frac{(b-L)^2}{b+2L} + \frac{(L-a)^2}{a+2L} \right].$$

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