ON NEW INEQUALITIES OF SIMPSON'S TYPE FOR FUNCTIONS WHOSE SECOND DERIVATIVES ABSOLUTE VALUES ARE CONVEX.

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Abstract. In this note, we obtain new some inequalities of Simpson's type based on convexity. Some applications for special means of real numbers are also given.

1. Introduction

The following inequality is one of the best-known results in the literature as Simpson's inequality.

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be a four times continuously differentiable mapping on (a,b) and $\left\|f^{(4)}\right\|_{\infty}=\sup\limits_{x\in(a,b)}\left|f^{(4)}(x)\right|<\infty$. Then, the following inequality holds:*

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{1}{2880}\left\|f^{(4)}\right\|_{\infty}(b-a)^4.$$

For recent refinements, counterparts, generalizations and new Simpson's type inequalities, see ([2],[3],[5]).

In [3], Dragomir et. al. proved the following some recent developments on Simpson's inequality for which the remainder is expressed in terms of lower derivatives than the fourth.

Theorem 2. *Suppose $f:[a,b]\to\mathbb{R}$ is a differentiable mapping whose derivative is continuous on (a,b) and $f'\in L[a,b]$. Then the following inequality*

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{b-a}{3}\left\|f'\right\|_1 \tag{1.1}$$

holds, where $\left\|f'\right\|_1=\int_a^b\left|f'(x)\right|dx$.

The bound of (1.1) for L-Lipschitzian mapping was given in [3] by $\frac{5}{36}L(b-a)$.

Also, the following inequality was obtained in [3].

2000 *Mathematics Subject Classification.* 26D15, 26D10.
Key words and phrases. Simpson's inequality, convex function.
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Theorem 3. *Suppose $f:[a,b]\rightarrow\mathbb{R}$ is an absolutely continuous mapping on $[a,b]$ whose derivative belongs to $L_p[a,b]$. Then the following inequality holds,*

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \tag{1.2}$$
$$\leq \frac{1}{6}\left[\frac{2^{q+1}+1}{3(q+1)}\right]^{\frac{1}{q}}(b-a)^{\frac{1}{q}}\left\|f'\right\|_p$$

where $\frac{1}{p}+\frac{1}{q}=1$.

In [1] Alomari et. al. obtained some inequalities for functions whose second derivatives absolute values are quasi-convex connecting with the Hermit-Hadamard inequality on the basis of the following Lemma.

Lemma 1. *Let $f:I\subset\mathbb{R}\rightarrow\mathbb{R}$ be twice differentiable mapping on I° with $f''\in L_1[a,b]$, then*

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx=\frac{(b-a)^2}{2}\int_0^1 t(1-t)f''(ta+(1-t)b)\,dt. \tag{1.3}$$

In [4], Hussain et. al. prove some inequalities related to Hermite-Hadamard's inequality for s-convex functions by used the above lemma.

Theorem 4. *Let $f:I\subset[0,\infty)\rightarrow\mathbb{R}$ be twice differentiable mapping on I° such that $f''\in L_1[a,b]$ where $a,b\in I$ with $a<b$. If $|f''|$ is $s-$convex on $[a,b]$ for some fixed $s\in[0,1]$ and $q\geq1$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{(b-a)^2}{2.6^{\frac{1}{p}}}\left[\frac{|f''(a)|^q+|f''(b)|^q}{(s+2)(s+3)}\right]^{\frac{1}{q}} \tag{1.4}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Remark 1. *If we take $s=1$ in (1.4), then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{(b-a)^2}{12}\left[\frac{|f''(a)|^q+|f''(b)|^q}{2}\right]^{\frac{1}{q}}. \tag{1.5}$$

The main aim of this paper is to establish new Simpson's type inequalities for the class of functions whose derivatives in absolute value at certain powers are convex functions.

2. Main Results

In order to prove our main theorems, we need the following Lemma.

Lemma 2. *Let $f:I\subset\mathbb{R}\rightarrow\mathbb{R}$ be twice differentiable mapping on I° such that $f''\in L_1[a,b]$, where $a,b\in I$ with $a<b$, then the following equality holds:*

$$\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx \tag{2.1}$$
$$=(b-a)^2\int_0^1 k(t)f''(tb+(1-t)a)\,dt$$

where

$$k(t)=\begin{cases}\dfrac{t}{2}\left(\dfrac{1}{3}-t\right), & t\in\left[0,\dfrac{1}{2}\right) \\ (1-t)\left(\dfrac{t}{2}-\dfrac{1}{3}\right), & t\in\left[\dfrac{1}{2},1\right].\end{cases}$$

Proof. By definition of $k(t)$, we have

$$\begin{aligned} I &= \int_0^1 k(t)\, f''(tb+(1-t)a)\,dt \\ &= \int_0^{\frac{1}{2}} \frac{t}{2}\left(\frac{1}{3}-t\right) f''(tb+(1-t)a)\,dt \\ &\quad +\int_{\frac{1}{2}}^1 (1-t)\left(\frac{t}{2}-\frac{1}{3}\right) f''(tb+(1-t)a)\,dt \\ &= I_1+I_2. \end{aligned} \tag{2.2}$$

Integrating by parts twice, we can state:

$$\begin{aligned} I_1 &= -\frac{1}{24(b-a)}f'(\frac{a+b}{2}) \\ &\quad +\frac{1}{b-a}\int_0^{\frac{1}{2}}(t-\frac{1}{6})f'(tb+(1-t)a)\,dt \\ &= -\frac{1}{24(b-a)}f'(\frac{a+b}{2}) \\ &\quad +\frac{1}{(b-a)^2}\left[\frac{1}{3}f(\frac{a+b}{2})+\frac{1}{6}f(a)-\int_0^{\frac{1}{2}} f(tb+(1-t)a)\,dt\right] \end{aligned} \tag{2.3}$$

and similarly,

$$\begin{aligned} I_2 &= \frac{1}{24(b-a)}f'(\frac{a+b}{2}) \\ &\quad +\frac{1}{b-a}\int_{\frac{1}{2}}^1(t-\frac{5}{6})f'(tb+(1-t)a)\,dt \\ &= \frac{1}{24(b-a)}f'(\frac{a+b}{2}) \\ &\quad +\frac{1}{(b-a)^2}\left[\frac{1}{3}f(\frac{a+b}{2})+\frac{1}{6}f(b)-\int_{\frac{1}{2}}^1 f(tb+(1-t)a)\,dt\right]. \end{aligned} \tag{2.4}$$

Adding (2.3) and (2.4),

$$\begin{aligned} I &= I_1+I_2 \\ &= \frac{1}{(b-a)^2}\left[\frac{1}{6}f(a)+\frac{2}{3}f(\frac{a+b}{2})+\frac{1}{6}f(b)-\int_0^1 f(tb+(1-t)a)\,dt\right]. \end{aligned}$$

Using the change of the variable $x=tb+(1-t)a$ for $t\in[0,1]$ and multiplying the both sides by $(b-a)^2$, we obtain (2.1) which completes the proof. □

The next theorems give a new refinement of the Simpson inequality for twice differentiable functions:

Theorem 5. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable mapping on I° such that $f'' \in L_1[a,b]$, where $a, b \in I$ with $a < b$. If $|f''|$ is a convex on $[a,b]$, then the following inequality holds:*

$$\left| \frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a}\int_a^b f(x)dx \right| \leq \frac{(b-a)^2}{162}\left[|f''(a)| + |f''(b)| \right]. \tag{2.5}$$

Proof. From Lemma 2 and by used convexity of $|f''|$, we get

$$\begin{aligned}
&\left| \frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a}\int_a^b f(x)dx \right| \\
\leq\ & (b-a)^2 \int_0^1 |k(t)| \left| f''(tb + (1-t)a) \right| dt \\
\leq\ & (b-a)^2 \left\{ \int_0^{\frac{1}{2}} \left| \frac{t}{2}\left(\frac{1}{3} - t\right) \right| \left[t\left| f''(b) \right| + (1-t)\left| f''(a) \right| \right] dt \right. \\
& \left. + \int_{\frac{1}{2}}^1 \left| (1-t)\left(\frac{t}{2} - \frac{1}{3}\right) \right| \left[t\left| f''(b) \right| + (1-t)\left| f''(a) \right| \right] dt \right\} \\
=\ & (b-a)^2 (J_1 + J_2)
\end{aligned}$$

where

$$J_1 = \int_0^{\frac{1}{2}} \left| \frac{t}{2}\left(\frac{1}{3} - t\right) \right| \left[t\left| f''(b) \right| + (1-t)\left| f''(a) \right| \right] dt$$

and

$$J_2 = \int_{\frac{1}{2}}^1 \left| (1-t)\left(\frac{t}{2} - \frac{1}{3}\right) \right| \left[t\left| f''(b) \right| + (1-t)\left| f''(a) \right| \right] dt.$$

By simple computation,

$$\begin{aligned}
J_1 &= \int_0^{\frac{1}{3}} \frac{t}{2}\left(\frac{1}{3} - t\right) \left[t\left| f''(b) \right| + (1-t)\left| f''(a) \right| \right] dt \\
&\quad + \int_{\frac{1}{3}}^{\frac{1}{2}} \frac{t}{2}\left(t - \frac{1}{3}\right) \left[t\left| f''(b) \right| + (1-t)\left| f''(a) \right| \right] dt \\
&= \frac{59}{3^5 2^7}\left| f''(b) \right| + \frac{133}{3^5 2^7}\left| f''(a) \right|
\end{aligned}$$

and

$$\begin{aligned}
J_2 &= \int_{\frac{1}{2}}^{\frac{2}{3}} (1-t)\left(\frac{1}{3} - \frac{t}{2}\right) \left[t\left| f''(b) \right| + (1-t)\left| f''(a) \right| \right] dt \\
&\quad + \int_{\frac{2}{3}}^1 (1-t)\left(\frac{t}{2} - \frac{1}{3}\right) \left[t\left| f''(b) \right| + (1-t)\left| f''(a) \right| \right] dt \\
&= \frac{133}{3^5 2^7}\left| f''(b) \right| + \frac{59}{3^5 2^7}\left| f''(a) \right|
\end{aligned}$$

which completes the proof. □

Corollary 1. *In Theorem 5, if* $f(a) = f(\frac{a+b}{2}) = f(b)$, *then we have*

$$\left| \frac{1}{b-a} \int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) \right| \leq \frac{(b-a)^2}{162} \left[|f''(a)| + |f''(b)| \right].$$

Remark 2. *We note that the obtained midpoint inequality (2.5) is better than the inequality (1.1).*

A similar results is embodied in the following theorem.

Theorem 6. *Let* $f : I \subset \mathbb{R} \to \mathbb{R}$ *be twice differentiable mapping on* I° *such that* $f'' \in L_1[a,b]$, *where* $a, b \in I$ *with* $a < b$. *If* $|f''|^q$ *is a convex on* $[a,b]$ *and* $q \geq 1$, *then the following inequality holds:*

$$\begin{aligned} & \left| \frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a}\int_a^b f(x)dx \right| \\ \leq \; & (b-a)^2 \left(\frac{1}{162}\right)^{1-\frac{1}{q}} \left\{ \left(\frac{59}{3^5 2^7} |f''(b)|^q + \frac{133}{3^5 2^7} |f''(a)|^q \right)^{\frac{1}{q}} \right. \\ & \left. + \left(\frac{133}{3^5 2^7} |f''(b)|^q + \frac{59}{3^5 2^7} |f''(a)|^q \right)^{\frac{1}{q}} \right\} \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Suppose that $q \geq 1$. From Lemma 2 and using the well known power mean inequality, we have

$$\begin{aligned} & \left| \frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a}\int_a^b f(x)dx \right| \\ \leq \; & (b-a)^2 \int_0^1 |k(t)| \left| f''(tb + (1-t)a) \right| dt \\ = \; & (b-a)^2 \left\{ \int_0^{\frac{1}{2}} \left| \frac{t}{2}\left(\frac{1}{3} - t\right) \right| \left| f''(tb + (1-t)a) \right| dt \right. \\ & \left. + \int_{\frac{1}{2}}^1 \left| (1-t)\left(\frac{t}{2} - \frac{1}{3}\right) \right| \left| f''(tb + (1-t)a) \right| dt \right\} \\ \leq \; & (b-a)^2 \left\{ \left(\int_0^{\frac{1}{2}} \left| \frac{t}{2}\left(\frac{1}{3} - t\right) \right| dt \right)^{1-\frac{1}{q}} \right. \\ & \times \left(\int_0^{\frac{1}{2}} \left| \frac{t}{2}\left(\frac{1}{3} - t\right) \right| \left| f''(tb + (1-t)a) \right|^q dt \right)^{\frac{1}{q}} \\ & + \left(\int_{\frac{1}{2}}^1 \left| (1-t)\left(\frac{t}{2} - \frac{1}{3}\right) \right| dt \right)^{1-\frac{1}{q}} \\ & \left. \times \left(\int_{\frac{1}{2}}^1 \left| (1-t)\left(\frac{t}{2} - \frac{1}{3}\right) \right| \left| f''(tb + (1-t)a) \right|^q dt \right)^{\frac{1}{q}} \right\}. \end{aligned}$$

Since $|f''|^q$ is a convex, therefore we have

$$
\begin{aligned}
&\int_0^{\frac{1}{2}} \left|\frac{t}{2}\left(\frac{1}{3}-t\right)\right| \left|f''\left(tb+(1-t)a\right)\right|^q dt \\
\leq\; & \int_0^{\frac{1}{2}} \left|\frac{t}{2}\left(\frac{1}{3}-t\right)\right| \left[t\left|f''(b)\right|^q+(1-t)\left|f''(a)\right|^q\right] dt \\
=\; & \int_0^{\frac{1}{3}} \left[\frac{t}{2}\left(\frac{1}{3}-t\right)\right] \left[t\left|f''(b)\right|^q+(1-t)\left|f''(a)\right|^q\right] dt \\
& +\int_{\frac{1}{3}}^{\frac{1}{2}} \left[\frac{t}{2}\left(t-\frac{1}{3}\right)\right] \left[t\left|f''(b)\right|^q+(1-t)\left|f''(a)\right|^q\right] dt \\
=\; & \frac{59}{3^5 2^7}\left|f''(b)\right|^q+\frac{133}{3^5 2^7}\left|f''(a)\right|^q
\end{aligned}
\tag{2.6}
$$

and

$$
\begin{aligned}
&\int_{\frac{1}{2}}^{1} \left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right| \left|f''\left(tb+(1-t)a\right)\right|^q dt \\
\leq\; & \int_{\frac{1}{2}}^{1} \left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right| \left(t\left|f''(b)\right|^q+(1-t)\left|f''(a)\right|^q\right) dt \\
=\; & \int_{\frac{1}{2}}^{\frac{2}{3}} (1-t)\left(\frac{1}{3}-\frac{t}{2}\right) \left(t\left|f''(b)\right|^q+(1-t)\left|f''(a)\right|^q\right) dt \\
& +\int_{\frac{2}{3}}^{1} (1-t)\left(\frac{t}{2}-\frac{1}{3}\right) \left(t\left|f''(b)\right|^q+(1-t)\left|f''(a)\right|^q\right) dt \\
=\; & \frac{133}{3^5 2^7}\left|f''(b)\right|^q+\frac{59}{3^5 2^7}\left|f''(a)\right|^q
\end{aligned}
\tag{2.7}
$$

From (2.6) and (2.7), we have

$$
\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\; & (b-a)^2\left\{\left(\int_0^{\frac{1}{2}}\left|\frac{t}{2}\left(\frac{1}{3}-t\right)\right|dt\right)^{1-\frac{1}{q}}\right. \\
& \times\left(\frac{59}{3^5 2^7}\left|f''(b)\right|^q+\frac{133}{3^5 2^7}\left|f''(a)\right|^q\right)^{\frac{1}{q}} \\
& +\left(\int_{\frac{1}{2}}^{1}\left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right|dt\right)^{1-\frac{1}{q}} \\
& \left.\times\left(\frac{133}{3^5 2^7}\left|f''(b)\right|^q+\frac{59}{3^5 2^7}\left|f''(a)\right|^q\right)^{\frac{1}{q}}\right\}
\end{aligned}
$$

$$= (b-a)^2 \left(\frac{1}{162}\right)^{1-\frac{1}{q}} \left\{ \left(\frac{59}{3^5 2^7} |f''(b)|^q + \frac{133}{3^5 2^7} |f''(a)|^q \right)^{\frac{1}{q}} + \left(\frac{133}{3^5 2^7} |f''(b)|^q + \frac{59}{3^5 2^7} |f''(a)|^q \right)^{\frac{1}{q}} \right\}$$

where we use the fact that

$$\int_0^{\frac{1}{2}} \left| \frac{t}{2}\left(\frac{1}{3}-t\right) \right| dt = \int_{\frac{1}{2}}^1 \left| (1-t)\left(\frac{t}{2}-\frac{1}{3}\right) \right| dt = \frac{1}{162}.$$

The proof is complete. □

Remark 3. *In Theorem 6, if $q=1$, then we have the inequality of (2.5).*

Corollary 2. *In Theorem 6, if $f(a) = f(\frac{a+b}{2}) = f(b)$, then we have*

$$\left| \frac{1}{b-a} \int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) \right|$$
$$\leq (b-a)^2 \left(\frac{1}{162}\right)^{1-\frac{1}{q}} \left\{ \left(\frac{59}{3^5 2^7} |f''(b)|^q + \frac{133}{3^5 2^7} |f''(a)|^q \right)^{\frac{1}{q}} + \left(\frac{133}{3^5 2^7} |f''(b)|^q + \frac{59}{3^5 2^7} |f''(a)|^q \right)^{\frac{1}{q}} \right\}.$$

Corollary 3. *In Theorem 6, if $f(a) = f(\frac{a+b}{2}) = f(b)$ and $p = q = 2$, then we have*

$$\left| \frac{1}{b-a} \int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) \right|$$
$$\leq (b-a)^2 \left(\frac{1}{162}\right)^{\frac{1}{2}} \left\{ \left(\frac{59}{3^5 2^7} |f''(b)|^2 + \frac{133}{3^5 2^7} |f''(a)|^2 \right)^{\frac{1}{2}} + \left(\frac{133}{3^5 2^7} |f''(b)|^2 + \frac{59}{3^5 2^7} |f''(a)|^2 \right)^{\frac{1}{2}} \right\}.$$

3. Applications to Special Means

We shall consider the following special means:

(a) The arithmetic mean: $A = A(a,b) := \dfrac{a+b}{2}$, $a, b \geq 0$,

(b) The harmonic mean:

$$H = H(a,b) := \frac{2ab}{a+b}, \ a, b > 0,$$

(c) The logarithmic mean:

$$L = L(a,b) := \begin{cases} a & if \quad a = b \\ \frac{b-a}{\ln b - \ln a} & if \quad a \neq b \end{cases}, \quad a, b > 0,$$

(d) The $p-$logarithmic mean

$$L_p = L_p(a,b) := \begin{cases} \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}} & \text{if} \quad a \neq b \\ a & \text{if} \quad a = b \end{cases}, \quad p \in \mathbb{R} \setminus \{-1, 0\}\,; \ a,b > 0.$$

It is well known that L_p is monotonic nondecreasing over $p \in \mathbb{R}$ with $L_{-1} := L$ and $L_0 := I$. In particular, we have the following inequalities

$$H \leq L \leq A.$$

Now, using the results of Section 2, some new inequalities is derived for the above means.

Proposition 1. *Let $a, b \in R$, $0 < a < b$ and $n \in \mathbb{N}$, $n > 2$. Then, we have*

$$\left|\frac{1}{3}A\left(a^n, b^n\right) + \frac{2}{3}A^n\left(a,b\right) - L_n^n\left(a,b\right)\right| \leq n(n-1)\frac{\left(b-a\right)^2}{162}\left[a^{n-2}+b^{n-2}\right].$$

Proof. The assertion follows from Theorem 5 applied to convex mapping $f(x) = x^n,\ x \in [a,b]$ and $n \in \mathbb{N}$. □

Proposition 2. *Let $a, b \in R$, $0 < a < b$. Then, for all $p > 1$, we have*

$$\begin{aligned} &\left|\frac{1}{3}H^{-1}\left(a,b\right) + \frac{2}{3}A^{-1}\left(a,b\right) - L^{-1}\left(a,b\right)\right| \\ \leq\ & (b-a)^2\left(\frac{1}{162}\right)^{1-\frac{1}{q}}\left\{\left(\frac{59}{3^5 2^7}\left|\frac{2}{b^3}\right|^q + \frac{133}{3^5 2^7}\left|\frac{2}{a^3}\right|^q\right)^{\frac{1}{q}}\right. \\ &\left. + \left(\frac{133}{3^5 2^7}\left|\frac{2}{b^3}\right|^q + \frac{59}{3^5 2^7}\left|\frac{2}{a^3}\right|^q\right)^{\frac{1}{q}}\right\}. \end{aligned}$$

Proof. The assertion follows from Theorem 6 applied to the convex mapping $f(x) = 1/x,\ x \in [a,b]$. □

References

[1] M. Alomari, M. Darus and S.S. Dragomir, New inequalities of Hermite-Hadamard type for functions whose second derivatives absolute values are quasi-convex, *RGMIA Res. Rep. Coll.*, 12 (2009), Supplement, Article 17. [Online:http://www.staff.vu.edu.au/RGMIA/v12(E).asp]

[2] M. Alomari, M. Darus and S.S. Dragomir, New inequalities of Simpson's type for s-convex functions with applications, *RGMIA Res. Rep. Coll.*, 12 (4) (2009), Article 9. [Online:http://www.staff.vu.edu.au/RGMIA/v12n4.asp]

[3] S.S. Dragomir, R.P. Agarwal and P. Cerone, On Simpson's inequality and applications, *J. of Inequal. Appl.*, 5(2000), 533-579.

[4] S. Hussain, M.I. Bhatti and M. Iqbal, Hadamard-type inequalities for s-convex functions I, *Punjab Univ. Jour. of Math.*, Vol.41, pp:51-60, (2009).

[5] B.Z. Liu, An inequality of Simpson type, *Proc. R. Soc. A, 461 (2005), 2155-2158.*

[6] J. Pečarić, F. Proschan and Y.L. Tong, Convex functions, partial ordering and statistical applications, *Academic Press*, New York, 1991.

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