ON NEW INEQUALITIES OF SIMPSON'S TYPE FOR CONVEX FUNCTIONS

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Abstract. In this paper, we introduce some inequalities of Simpson's type based on convexity. Some applications for special means of real numbers are also given.

1. Introduction

The following inequality is well known in the literature as Simpson's inequality.

Theorem 1. *Let $f:[a,b]\rightarrow\mathbb{R}$ be a four times continuously differentiable mapping on (a,b) and $\left\Vert f^{(4)}\right\Vert_{\infty}=\sup\limits_{x\in(a,b)}\left\vert f^{(4)}(x)\right\vert<\infty$. Then, the following inequality holds:*

$$\left\vert \frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right\vert \leq \frac{1}{2880}\left\Vert f^{(4)}\right\Vert_{\infty}(b-a)^4.$$

For recent refinements, counterparts, generalizations and new Simpson's type inequalities, see ([1],[2],[6]).

In [2], Dragomir et. al. proved the following some recent developments on Simpson's inequality for which the remainder is expressed in terms of lower derivatives than the fourth.

Theorem 2. *Suppose $f:[a,b]\rightarrow\mathbb{R}$ is a differentiable mapping whose derivative is continuous on (a,b) and $f'\in L[a,b]$. Then the following inequality*

$$\left\vert \frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right\vert \leq \frac{b-a}{3}\left\Vert f'\right\Vert_1 \tag{1.1}$$

holds, where $\left\Vert f'\right\Vert_1=\int_a^b|f'(x)|\,dx$.

The bound of (1.1) for L-Lipschitzian mapping was given in [2] by $\frac{5}{36}L(b-a)$.

Also, the following inequality was obtained in [2].

Theorem 3. *Suppose $f:[a,b]\rightarrow\mathbb{R}$ is an absolutely continuous mapping on $[a,b]$ whose derivative belongs to $L_p[a,b]$. Then the following inequality holds,*

$$\begin{aligned}&\left\vert \frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right\vert \\ &\leq \frac{1}{6}\left[\frac{2^{q+1}+1}{3(q+1)}\right]^{\frac{1}{q}}(b-a)^{\frac{1}{q}}\left\Vert f'\right\Vert_p\end{aligned} \tag{1.2}$$

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where $\frac{1}{p}+\frac{1}{q}=1$.

In [4], some inequalities of Hermite-Hadamard's type for differentiable convex mappings were presented as follows:

Theorem 4. *Let $f: I\subset\mathbb{R}\to\mathbb{R}$ be differentiable mapping on I°, where $a,b\in I$ with $a<b$. If $|f'|$ is convex on $[a,b]$, then the following equality holds,*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-f\left(\frac{a+b}{2}\right)\right|\leq\frac{(b-a)}{4}\left[\frac{|f'(a)|+|f'(b)|}{2}\right]. \tag{1.3}$$

The main aim of this paper is to establish new Simpson's type inequalities for the class of functions whose derivatives in absolute value at certain powers are convex functions.

2. Main Results

In order to prove our main theorems, we need the following Lemma.

Lemma 1. *Let $f: I\subset\mathbb{R}\to\mathbb{R}$ be an absolutely continuous mapping on I° such that $f'\in L_1[a,b]$, where $a,b\in I$ with $a<b$, then the following equality holds:*

$$\begin{aligned}&\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\\ =&\ \frac{b-a}{2}\int_0^1\left[\left(\frac{t}{2}-\frac{1}{3}\right)f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right.\\ &\left.+\left(\frac{1}{3}-\frac{t}{2}\right)f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right]dt.\end{aligned} \tag{2.1}$$

Proof. It suffices to note that

$$\begin{aligned}I_1&=\int_0^1\left(\frac{t}{2}-\frac{1}{3}\right)f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)dt\\ &=\left(\frac{t}{2}-\frac{1}{3}\right)\frac{2}{b-a}\,f\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\Bigg|_0^1\\ &\quad-\frac{1}{b-a}\int_0^1 f\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)dt\\ &=\frac{2}{6(b-a)}f(b)+\frac{2}{3(b-a)}f(\frac{a+b}{2})\\ &\quad-\frac{1}{b-a}\int_0^1 f\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)dt.\end{aligned} \tag{2.2}$$

Setting $x=\frac{1+t}{2}b+\frac{1-t}{2}a$ and $dx=\frac{b-a}{2}dt$, which gives

$$I_1=\frac{2}{6(b-a)}f(b)+\frac{2}{3(b-a)}f(\frac{a+b}{2})-\frac{2}{(b-a)^2}\int_{\frac{a+b}{2}}^b f(x)\,dx \tag{2.3}$$

Similarly, we can show that

$$
\begin{aligned}
(2.4)\qquad I_2 &= \int_0^1 \left(\frac{1}{3}-\frac{t}{2}\right) f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right) dt \\
&= \left(\frac{1}{3}-\frac{t}{2}\right)\frac{2}{a-b}\, f\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\Big|_0^1 \\
&\quad +\frac{1}{a-b}\int_0^1 f\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right) dt \\
&= \frac{2}{6\,(b-a)} f(a)+\frac{2}{3\,(b-a)} f(\frac{a+b}{2})-\frac{2}{(b-a)^2}\int_a^{\frac{a+b}{2}} f\,(x)\,dx.
\end{aligned}
$$

Adding (2.3) and (2.4), we obtain

$$
\frac{b-a}{2}\,(I_1+I_2)=\left[\frac{f(a)+f(b)}{6}+\frac{2}{3}f(\frac{a+b}{2})-\frac{1}{b-a}\int_a^b f\,(x)\,dx\right]
$$

which is required. □

The next theorems give a new refinement of the Simpson inequality for differentiable functions:

Theorem 5. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L_1[a,b]$, where $a,b \in I$ with $a < b$. If $|f'|$ is a convex on $[a,b]$, then the following inequality holds:*

$$
\begin{aligned}
(2.5)\qquad &\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq \frac{5\,(b-a)}{72}\left[|f'(a)|+|f'(b)|\right].
\end{aligned}
$$

Proof. From Lemma 1 and by used convexity of $|f'|$, we get

$$
\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq \frac{b-a}{2}\int_0^1 \left[\left|\frac{t}{2}-\frac{1}{3}\right|\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|\right. \\
&\quad \left. +\left|\frac{1}{3}-\frac{t}{2}\right|\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|\right] dt \\
&\leq \frac{b-a}{2}\int_0^1 \left(\left|\frac{t}{2}-\frac{1}{3}\right|\right. \\
&\quad \left.\times\left[\frac{1+t}{2}\,|f'\,(b)|+\frac{1-t}{2}\,|f'\,(a)|+\frac{1+t}{2}\,|f'\,(a)|+\frac{1-t}{2}\,|f'\,(b)|\right]\right) dt \\
&= \frac{b-a}{2}\int_0^1 \left|\frac{t}{2}-\frac{1}{3}\right|\left[|f'\,(a)|+|f'\,(b)|\right] dt.
\end{aligned}
$$

By simple computation,

$$
\int_0^1 \left|\frac{t}{2}-\frac{1}{3}\right| = \int_0^{\frac{2}{3}}\left(\frac{1}{3}-\frac{t}{2}\right) dt+\int_{\frac{2}{3}}^1 \left(\frac{t}{2}-\frac{1}{3}\right) dt=\frac{5}{36}.
$$

Thus, we get (2.5) which completes the proof. □

Corollary 1. *In Theorem 5, if $f(a)=f(\frac{a+b}{2})=f(b)$, then we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-f\left(\frac{a+b}{2}\right)\right|\leq\frac{5\,(b-a)}{72}\left[|f''(a)|+|f''(b)|\right]. \tag{2.6}$$

Remark 1. *We note that the obtained midpoint inequality (2.6) is better than the inequality (1.3).*

Remark 2. *We note that the obtained midpoint inequality (2.5) is better than the inequality (1.1).*

Theorem 6. *Let $f: I\subset\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I° such that $f'\in L_1[a,b]$, wherea, $b\in I$ with $a<b$. If $|f'|^q$ is a convex on $[a,b]$, $q>1$, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \qquad (2.7)\\
\leq\;&\frac{(b-a)}{12}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}}\\
&\times\left\{\left(\frac{3\,|f'(b)|^q+|f'(a)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{|f'(b)|^q+3\,|f'(a)|^q}{4}\right)^{\frac{1}{q}}\right\}
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 1 and by Hölder's integral inequality, we get

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|\\
\leq\;&\frac{b-a}{2}\int_0^1\left[\left|\frac{t}{2}-\frac{1}{3}\right|\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|\right.\\
&\left.+\left|\frac{1}{3}-\frac{t}{2}\right|\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|\right]dt\\
\leq\;&\frac{b-a}{2}\left\{\left(\int_0^1\left|\frac{t}{2}-\frac{1}{3}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|^q dt\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_0^1\left|\frac{1}{3}-\frac{t}{2}\right|^p\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Since $|f'|^q$ is a convex on $[a,b]$, we know that for $t\in[0,1]$

$$\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|^q\leq\frac{1+t}{2}\,|f'(b)|^q+\frac{1-t}{2}\,|f'(a)|^q,$$

hence

$$\begin{aligned} & \left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \\ \leq\ & \frac{b-a}{2}\left\{\left(\int_0^1\left|\frac{t}{2}-\frac{1}{3}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left[\frac{1+t}{2}\left|f'(b)\right|^q+\frac{1-t}{2}\left|f'(a)\right|^q\right]dt\right)^{\frac{1}{q}}\right. \\ & \left.+\left(\int_0^1\left|\frac{1}{3}-\frac{t}{2}\right|^p\right)^{\frac{1}{p}}\left(\int_0^1\left[\frac{1+t}{2}\left|f'(a)\right|^q+\frac{1-t}{2}\left|f'(b)\right|^q\right]dt\right)^{\frac{1}{q}}\right\} \\ =\ & \frac{b-a}{2}\left(\int_0^1\left|\frac{t}{2}-\frac{1}{3}\right|^p dt\right)^{\frac{1}{p}} \\ & \times\left\{\left(\frac{3\left|f'(b)\right|^q+\left|f'(a)\right|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3\left|f'(a)\right|^q+\left|f'(b)\right|^q}{4}\right)^{\frac{1}{q}}\right\}. \end{aligned}$$

By simple computation,

$$\int_0^1\left|\frac{t}{2}-\frac{1}{3}\right|^p=\int_0^{\frac{2}{3}}\left(\frac{1}{3}-\frac{t}{2}\right)^p dt+\int_{\frac{2}{3}}^1\left(\frac{t}{2}-\frac{1}{3}\right)^p dt=\frac{2(1+2^{p+1})}{6^{p+1}(p+1)}.$$

Thus, we get (2.7) which completes the proof. □

Corollary 2. *In Theorem 6, if $f(a)=f(\frac{a+b}{2})=f(b)$, then we have*

$$\begin{aligned} & \left|\frac{1}{b-a}\int_a^b f(x)dx-f\left(\frac{a+b}{2}\right)\right| \\ \leq\ & \frac{(b-a)}{12}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}} \\ & \times\left\{\left(\frac{3\left|f'(b)\right|^q+\left|f'(a)\right|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{\left|f'(b)\right|^q+3\left|f'(a)\right|^q}{4}\right)^{\frac{1}{q}}\right\}. \end{aligned}$$

Remark 3. *We note that the obtained midpoint inequality (2.7) is better than the inequality (1.2).*

Theorem 7. *Let $f: I\subset\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I° such that $f'\in L_1[a,b]$, wherea, $b\in I$ with $a<b$. If $|f'|^q$ is a convex on $[a,b]$, $q\geq 1$, then the following inequality holds:*

$$\begin{aligned} (2.8)\qquad & \left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \\ \leq\ & \frac{b-a}{72}(5)^{1-\frac{1}{q}} \\ & \times\left\{\left(\frac{61\left|f'(b)\right|^q+29\left|f'(a)\right|^q}{18}\right)^{\frac{1}{q}}+\left(\frac{61\left|f'(a)\right|^q+29\left|f'(b)\right|^q}{18}\right)^{\frac{1}{q}}\right\}. \end{aligned}$$

Proof. From Lemma 1 and by power mean inequality, we get

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\ & \frac{b-a}{2}\int_0^1\left[\left|\frac{t}{2}-\frac{1}{3}\right|\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|\right. \\
&\left.+\left|\frac{1}{3}-\frac{t}{2}\right|\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|\right]dt \\
\leq\ & \frac{b-a}{2}\left\{\left(\int_0^1\left|\frac{t}{2}-\frac{1}{3}\right|dt\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\frac{t}{2}-\frac{1}{3}\right|\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_0^1\left|\frac{1}{3}-\frac{t}{2}\right|\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\frac{1}{3}-\frac{t}{2}\right|\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Since $|f'|^q$ is a convex on $[a,b]$, we know that for $t\in[0,1]$

$$\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|^q\leq\frac{1+t}{2}|f'(b)|^q+\frac{1-t}{2}|f'(a)|^q,$$

hence

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\ & \frac{b-a}{2}\left\{\left(\int_0^1\left|\frac{t}{2}-\frac{1}{3}\right|dt\right)^{1-\frac{1}{q}}\right. \\
&\times\left(\int_0^1\left|\frac{t}{2}-\frac{1}{3}\right|\left[\frac{1+t}{2}|f'(b)|^q+\frac{1-t}{2}|f'(a)|^q\right]dt\right)^{\frac{1}{q}} \\
&\left.+\left(\int_0^1\left|\frac{1}{3}-\frac{t}{2}\right|\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\frac{1}{3}-\frac{t}{2}\right|\left[\frac{1+t}{2}|f'(a)|^q+\frac{1-t}{2}|f'(b)|^q\right]dt\right)^{\frac{1}{q}}\right\} \\
=\ & \frac{b-a}{2}\left(\frac{5}{36}\right)^{1-\frac{1}{q}} \\
&\times\left\{\left(\frac{61}{648}|f'(b)|^q+\frac{29}{648}|f'(a)|^q\right)^{\frac{1}{q}}+\left(\frac{61}{648}|f'(a)|^q+\frac{29}{648}|f'(b)|^q\right)^{\frac{1}{q}}\right\} \\
=\ & \frac{b-a}{72}(5)^{1-\frac{1}{q}} \\
&\times\left\{\left(\frac{61|f'(b)|^q+29|f'(a)|^q}{18}\right)^{\frac{1}{q}}+\left(\frac{61|f'(a)|^q+29|f'(b)|^q}{18}\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

By simple computation,

$$\int_0^1\left|\frac{t}{2}-\frac{1}{3}\right|\left[\frac{1+t}{2}|f'(b)|^q+\frac{1-t}{2}|f'(a)|^q\right]dt=\frac{61}{648}|f'(b)|^q+\frac{29}{648}|f'(a)|^q$$

$$\int_0^1\left|\frac{1}{3}-\frac{t}{2}\right|\left[\frac{1+t}{2}|f'(a)|^q+\frac{1-t}{2}|f'(b)|^q\right]dt=\frac{61}{648}|f'(a)|^q+\frac{29}{648}|f'(b)|^q$$

$$\int_0^1 \left| \frac{t}{2} - \frac{1}{3} \right| = \frac{5}{36}$$

Thus, we get (2.8) which completes the proof. □

Corollary 3. *In Theorem 6, if* $f(a) = f(\frac{a+b}{2}) = f(b)$*, then we have*

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) \right| \\ \leq \; & \frac{b-a}{72} (5)^{1-\frac{1}{q}} \\ & \times \left\{ \left(\frac{61 |f'(b)|^q + 29 |f'(a)|^q}{18} \right)^{\frac{1}{q}} + \left(\frac{61 |f'(a)|^q + 29 |f'(b)|^q}{18} \right)^{\frac{1}{q}} \right\}. \end{aligned}$$

3. Applications to Special Means

We shall consider the following special means:

(a) The arithmetic mean: $A = A(a,b) := \dfrac{a+b}{2}, \; a, b \geq 0,$

(b) The harmonic mean:

$$H = H(a,b) := \frac{2ab}{a+b}, \; a, b > 0,$$

(c) The logarithmic mean:

$$L = L(a,b) := \begin{cases} a & if \quad a = b \\ \frac{b-a}{\ln b - \ln a} & if \quad a \neq b \end{cases}, \quad a, b > 0,$$

(d) The $p-$logarithmic mean

$$L_p = L_p(a,b) := \begin{cases} \left[\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right]^{\frac{1}{p}} & \text{if} \quad a \neq b \\ a & \text{if} \quad a = b \end{cases}, \quad p \in \mathbb{R} \diagdown \{-1, 0\}; \; a, b > 0.$$

It is well known that L_p is monotonic nondecreasing over $p \in \mathbb{R}$ with $L_{-1} := L$ and $L_0 := I$. In particular, we have the following inequalities

$$H \leq L \leq A.$$

Now, using the results of Section 2, some new inequalities is derived for the above means.

Proposition 1. *Let* $a, b \in R$*,* $0 < a < b$ *and* $n \in \mathbb{N}$*,* $n \geq 2$*. Then, we have*

$$\left| \frac{1}{3} A(a^n, b^n) + \frac{2}{3} A^n(a,b) - L_n^n(a,b) \right| \leq n \frac{5(b-a)}{36} A(a^{n-1}, b^{n-1}).$$

Proof. The assertion follows from Theorem 5 applied for $f(x) = x^n, \; x \in [a,b]$ and $n \in \mathbb{N}$. □

Proposition 2. *Let $a, b \in R$, $0 < a < b$. Then, for all $p > 1$, we have*

$$\begin{aligned} &\left|\frac{1}{3}H^{-1}(a,b)+\frac{2}{3}A^{-1}(a,b)-L^{-1}(a,b)\right| \\ \leq\ & \frac{(b-a)}{12a^2b^2}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}}\left\{\left(\frac{3a^{2q}+b^{2q}}{4}\right)^{\frac{1}{q}}+\left(\frac{a^{2q}+3b^{2q}}{4}\right)^{\frac{1}{q}}\right\}. \end{aligned}$$

Proof. The assertion follows from Theorem 6 applied for $f(x) = 1/x$, $x \in [a, b]$. □

Proposition 3. *Let $a, b \in R$, $0 < a < b$ and $n \in \mathbb{N}$, $n \geq 2$. Then, for all $q > 1$, we have*

$$\begin{aligned} &\left|\frac{1}{3}A(a^n,b^n)+\frac{2}{3}A^n(a,b)-L_n^n(a,b)\right| \\ \leq\ & n\frac{5^{1-\frac{1}{q}}(b-a)}{72}\left\{\left(\frac{29a^{(n-1)q}+61b^{(n-1)q}}{18}\right)^{\frac{1}{q}}+\left(\frac{61a^{(n-1)q}+29b^{(n-1)q}}{18}\right)^{\frac{1}{q}}\right\}. \end{aligned}$$

Proof. The assertion follows from Theorem 7 applied for $f(x) = x^n$, $x \in [a, b]$ and $n \in \mathbb{N}$. □

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