

ON NEW INEQUALITIES OF SIMPSON'S TYPE FOR QUASI-CONVEX FUNCTIONS WITH APPLICATIONS.

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ABSTRACT. In this paper, we introduce some inequalities of Simpson's type based on quasi-convexity. Some applications for special means of real numbers are also given.

1. INTRODUCTION

The following inequality is well known in the literature as Simpson's inequality.

Theorem 1. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a four times continuously differentiable mapping on (a, b) and $\|f^{(4)}\|_\infty = \sup_{x \in (a, b)} |f^{(4)}(x)| < \infty$. Then, the following inequality holds:*

$$\left| \frac{1}{3} \left[\frac{f(a) + f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{1}{2880} \|f^{(4)}\|_\infty (b-a)^4.$$

For recent refinements, counterparts, generalizations and new Simpson's type inequalities, see ([1],[2],[4]).

In [2], Dragomir, Agarwal and Cerone proved the following some recent developments on Simpson's inequality for which the remainder is expressed in terms of lower derivatives than the fourth.

Theorem 2. *Suppose $f : [a, b] \rightarrow \mathbb{R}$ is a differentiable mapping whose derivative is continuous on (a, b) and $f' \in L[a, b]$. Then the following inequality*

$$(1.1) \quad \left| \frac{1}{3} \left[\frac{f(a) + f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{3} \|f'\|_1$$

holds, where $\|f'\|_1 = \int_a^b |f'(x)| dx$.

The bound of (1.1) for L-Lipschitzian mapping was given in [2] by $\frac{5}{36}L(b-a)$.

Also, the following inequality was obtained in [2].

Theorem 3. *Suppose $f : [a, b] \rightarrow \mathbb{R}$ is an absolutely continuous mapping on $[a, b]$ whose derivative belongs to $L_p[a, b]$. Then the following inequality holds,*

$$(1.2) \quad \begin{aligned} & \left| \frac{1}{3} \left[\frac{f(a) + f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{1}{6} \left[\frac{2^{q+1} + 1}{3(q+1)} \right]^{\frac{1}{q}} (b-a)^{\frac{1}{q}} \|f'\|_p \end{aligned}$$

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where $\frac{1}{p} + \frac{1}{q} = 1$.

We recall that the notion of quasi-convex functions genelized the notion of convex functions. More precisely, a function $f : [a, b] \rightarrow \mathbb{R}$ is said to be *quasi-convex* on $[a, b]$ if

$$f(tx + (1-t)y) \leq \max\{f(x), f(y)\}, \quad \forall x, y \in [a, b].$$

Any convex function is a quasi-convex function but the reverse are not true. Because there exist quasi-convex functions which are not convex, (see for example [3])

The main aim of this paper is to establish new Simpson's type inequalities for the class of functions whose derivatives in absolute value at certain powers are quasi-convex functions.

2. SIMPSON'S TYPE INEQUALITIES FOR QUASI-CONVEX

In order to prove our main theorems, we need the following lemma, see [1].

Lemma 1. *Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be an absolutely continuous mapping on I° where $a, b \in I$ with $a < b$. Then the following equality holds:*

$$(2.1) \quad \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ = (b-a) \int_0^1 p(t) f'(tb + (1-t)a) dt$$

where

$$p(t) = \begin{cases} t - \frac{1}{6}, & t \in [0, \frac{1}{2}), \\ t - \frac{5}{6}, & t \in [\frac{1}{2}, 1]. \end{cases}$$

A simple proof of this equality can be also done by integrating by parts in the right hand side. The details are left to the interested reader.

The next theorem gives a new result of the Simpson inequality for quasi-convex functions.

Theorem 4. *Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° , such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'|$ is quasi-convex on $[a, b]$, then the following inequality holds:*

$$(2.2) \quad \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ \leq \frac{5(b-a)}{36} \max\{|f'(a)|, |f'(b)|\}.$$

Proof. From Lemma 1, and since $|f'|$ is quasi-convex, we have

$$\left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ = (b-a) \left| \int_0^1 p(t) f'(tb + (1-t)a) dt \right| \\ \leq (b-a) \int_0^{1/2} \left| t - \frac{1}{6} \right| |f'(tb + (1-t)a)| dt \\ + (b-a) \int_{1/2}^1 \left| t - \frac{5}{6} \right| |f'(tb + (1-t)a)| dt$$

$$\begin{aligned}
&\leq (b-a) \int_0^{1/2} \left| t - \frac{1}{6} \right| \max \{ |f'(a)|, |f'(b)| \} dt \\
&\quad + (b-a) \int_{1/2}^1 \left| t - \frac{5}{6} \right| \max \{ |f'(a)|, |f'(b)| \} dt \\
&= (b-a) \int_0^{1/6} \left(\frac{1}{6} - t \right) \max \{ |f'(a)|, |f'(b)| \} dt \\
&\quad + (b-a) \int_{1/6}^{1/2} \left(t - \frac{1}{6} \right) \max \{ |f'(a)|, |f'(b)| \} dt \\
&\quad + (b-a) \int_{1/2}^{5/6} \left(\frac{5}{6} - t \right) \max \{ |f'(a)|, |f'(b)| \} dt \\
&\quad + (b-a) \int_{5/6}^1 \left(t - \frac{5}{6} \right) \max \{ |f'(a)|, |f'(b)| \} dt \\
&= \frac{5(b-a)}{36} \max \{ |f'(a)|, |f'(b)| \}
\end{aligned}$$

which completes the proof. \square

Corollary 1. *In Theorem 4, if $f(a) = f(\frac{a+b}{2}) = f(b)$, then we have*

$$\left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right| \leq \frac{5(b-a)}{36} \max \{ |f'(a)|, |f'(b)| \}.$$

A similar results is embodied in the following theorem.

Theorem 5. *Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° , such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is quasi-convex on $[a, b]$ and $q > 1$, then the following inequality holds:*

$$\begin{aligned}
(2.3) \quad &\left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\
&\leq \frac{1}{6} (b-a) \left(\frac{1+2^{p+1}}{3(p+1)} \right)^{\frac{1}{p}} \left(\max \{ |f'(a)|^q, |f'(b)|^q \} \right)^{\frac{1}{q}}
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. From Lemma 1, using the well known Hölder integral inequality, we have

$$\begin{aligned}
&\left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\
&= (b-a) \left| \int_0^1 p(t) f'(tb + (1-t)a) dt \right| \\
&\leq (b-a) \int_0^{1/2} \left| t - \frac{1}{6} \right| |f'(tb + (1-t)a)| dt \\
&\quad + (b-a) \int_{1/2}^1 \left| t - \frac{5}{6} \right| |f'(tb + (1-t)a)| dt
\end{aligned}$$

$$\begin{aligned}
&\leq (b-a) \left(\int_0^{1/2} \left| t - \frac{1}{6} \right|^p dt \right)^{\frac{1}{p}} \left(\int_0^{1/2} |f'(tb + (1-t)a)|^q dt \right)^{\frac{1}{q}} \\
&\quad + (b-a) \left(\int_{1/2}^1 \left| t - \frac{5}{6} \right|^p dt \right)^{\frac{1}{p}} \left(\int_{1/2}^1 |f'(tb + (1-t)a)|^q dt \right)^{\frac{1}{q}} \\
&= (b-a) \left(\int_0^{1/6} \left(\frac{1}{6} - t \right)^p dt + \int_{1/6}^{1/2} \left(t - \frac{1}{6} \right)^p dt \right)^{\frac{1}{p}} \\
&\quad \times \left(\int_0^{1/2} |f'(tb + (1-t)a)|^q dt \right)^{\frac{1}{q}} \\
&\quad + (b-a) \left(\int_{1/2}^{5/6} \left(\frac{5}{6} - t \right)^p dt + \int_{5/6}^1 \left(t - \frac{5}{6} \right)^p dt \right)^{\frac{1}{p}} \\
&\quad \times \left(\int_{1/2}^1 |f'(tb + (1-t)a)|^q dt \right)^{\frac{1}{q}}.
\end{aligned}$$

Since $|f'|^q$ is quasi-convex, we have

$$|f'(tb + (1-t)a)|^q \leq \max \{ |f'(b)|^q, |f'(a)|^q \}$$

hence

$$\begin{aligned}
&\left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\
&\leq 2(b-a) \left(\frac{1+2^{p+1}}{6^{p+1}(p+1)} \right)^{\frac{1}{p}} \left(\frac{\max \{ |f'(a)|^q, |f'(b)|^q \}}{2} \right)^{\frac{1}{q}} \\
&\leq 2^{\frac{1}{p}}(b-a) \left(\frac{1+2^{p+1}}{6^{p+1}(p+1)} \right)^{\frac{1}{p}} (\max \{ |f'(a)|^q, |f'(b)|^q \})^{\frac{1}{q}}
\end{aligned}$$

where we use the fact that

$$\begin{aligned}
&\int_0^{1/6} \left(\frac{1}{6} - t \right)^p dt + \int_{1/6}^{1/2} \left(t - \frac{1}{6} \right)^p dt \\
&= \int_{1/2}^{5/6} \left(\frac{5}{6} - t \right)^p dt + \int_{5/6}^1 \left(t - \frac{5}{6} \right)^p dt \\
&= \frac{1+2^{p+1}}{6^{p+1}(p+1)}
\end{aligned}$$

which completes the proof. \square

Corollary 2. *In Theorem 5, if $f(a) = f(\frac{a+b}{2}) = f(b)$, then we have*

$$\begin{aligned}
&\left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right| \\
&\leq \frac{1}{6}(b-a) \left(\frac{1+2^{p+1}}{3(p+1)} \right)^{\frac{1}{p}} (\max \{ |f'(a)|^q, |f'(b)|^q \})^{\frac{1}{q}}.
\end{aligned}$$

Corollary 3. *In Theorem 5, if $f(a) = f\left(\frac{a+b}{2}\right) = f(b)$ and $p = 2$, then we have*

$$\left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right| \leq \frac{(b-a)}{6} \sqrt{\max\{|f'(a)|^2, |f'(b)|^2\}}.$$

A more general inequality is given using Lemma 1, as follows.

Theorem 6. *Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° , such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is quasi-convex on $[a, b]$ and $q \geq 1$, then the following inequality holds:*

$$(2.4) \quad \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{5(b-a)}{36} (\max\{|f'(a)|^q, |f'(b)|^q\})^{\frac{1}{q}}.$$

Proof. Suppose that $q \geq 1$. From Lemma 1 and using the well known power mean inequality, we have

$$\begin{aligned} & \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ &= (b-a) \left| \int_0^1 p(t) f'(tb + (1-t)a) dt \right| \\ &\leq (b-a) \int_0^{1/2} \left| t - \frac{1}{6} \right| |f'(tb + (1-t)a)| dt \\ &\quad + (b-a) \int_{1/2}^1 \left| t - \frac{5}{6} \right| |f'(tb + (1-t)a)| dt \\ &\leq (b-a) \left(\int_0^{1/2} \left| t - \frac{1}{6} \right| dt \right)^{1-\frac{1}{q}} \left(\int_0^{1/2} \left| t - \frac{1}{6} \right| |f'(tb + (1-t)a)|^q dt \right)^{\frac{1}{q}} \\ &\quad + (b-a) \left(\int_{1/2}^1 \left| t - \frac{5}{6} \right| dt \right)^{1-\frac{1}{q}} \left(\int_{1/2}^1 \left| t - \frac{5}{6} \right| |f'(tb + (1-t)a)|^q dt \right)^{\frac{1}{q}}. \end{aligned}$$

Since $|f'|^q$ is quasi-convex, we have

$$\begin{aligned} & \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ &\leq 2(b-a) \left(\frac{5}{72} \right)^{1-\frac{1}{q}} \left(\frac{5}{72} \max\{|f'(a)|^q, |f'(b)|^q\} \right)^{\frac{1}{q}} \\ &= \frac{5(b-a)}{36} (\max\{|f'(a)|^q, |f'(b)|^q\})^{\frac{1}{q}} \end{aligned}$$

Also, we note that

$$\int_0^{1/2} \left| t - \frac{1}{6} \right| dt = \int_{1/2}^1 \left| t - \frac{5}{6} \right| dt = \frac{5}{72}.$$

Therefore, the proof is completed. \square

Remark 1. *In Theorem 6, if $q = 1$, then we have the inequality of (2.2).*

Corollary 4. *In Theorem 6, if $f(a) = f(\frac{a+b}{2}) = f(b)$, then we have*

$$\left| \frac{1}{b-a} \int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) \right| \leq \frac{5(b-a)}{36} (\max\{|f'(a)|^q, |f'(b)|^q\})^{\frac{1}{q}}.$$

3. APPLICATIONS TO SPECIAL MEANS

We now consider the applications of above Theorems to the following special means:

- (a) The arithmetic mean: $A = A(a, b) := \frac{a+b}{2}$, $a, b \geq 0$,
 (b) The harmonic mean:

$$H = H(a, b) := \frac{2ab}{a+b}, \quad a, b > 0,$$

- (c) The logarithmic mean:

$$L = L(a, b) := \begin{cases} a & \text{if } a = b \\ \frac{b-a}{\ln b - \ln a} & \text{if } a \neq b \end{cases}, \quad a, b > 0,$$

- (d) The p -logarithmic mean:

$$L_p = L_p(a, b) := \begin{cases} \left[\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right]^{\frac{1}{p}} & \text{if } a \neq b \\ a & \text{if } a = b \end{cases}, \quad p \in \mathbb{R} \setminus \{-1, 0\}; \quad a, b > 0.$$

It is well known that L_p is monotonic nondecreasing over $p \in \mathbb{R}$ with $L_{-1} := L$ and $L_0 := I$. In particular, we have the following inequalities

$$H \leq L \leq A.$$

Now, using the results of Section 2, some new inequalities is derived for the above means.

Proposition 1. *Let $a, b \in \mathbb{R}$, $0 < a < b$ and $n \in \mathbb{N}$, $n \geq 2$. Then, we have*

$$\left| \frac{1}{3}A(a^n, b^n) + \frac{2}{3}A^n(a, b) - L_n^n(a, b) \right| \leq n \frac{5(b-a)}{36} \max\{a^{n-1}, b^{n-1}\}.$$

Proof. The assertion follows from Theorem 4 applied to the quasi-convex mapping $f(x) = x^n$, $x \in [a, b]$ and $n \in \mathbb{N}$. \square

Proposition 2. *Let $a, b \in \mathbb{R}$, $0 < a < b$. Then, for all $p > 1$, we have*

$$\begin{aligned} & \left| \frac{1}{3}H^{-1}(a, b) + \frac{2}{3}A^{-1}(a, b) - L^{-1}(a, b) \right| \\ & \leq \frac{1}{6}(b-a) \left(\frac{1+2^{p+1}}{3(p+1)} \right)^{\frac{1}{p}} (\max\{a^{-2q}, b^{-2q}\})^{\frac{1}{q}}. \end{aligned}$$

Proof. The assertion follows from Theorem 5 applied to the quasi-convex mapping $f(x) = 1/x$, $x \in [a, b]$. \square

Proposition 3. *Let $a, b \in \mathbb{R}$, $0 < a < b$ and $n \in \mathbb{N}$, $n \geq 2$. Then, for all $q > 1$, we have*

$$\left| \frac{1}{3}A(a^n, b^n) + \frac{2}{3}A^n(a, b) - L_n^n(a, b) \right| \leq n \frac{5(b-a)}{36} \left(\max \{ a^{q(n-1)}, b^{q(n-1)} \} \right)^{\frac{1}{q}}.$$

Proof. The assertion follows from Theorem 6 applied to the quasi-convex mapping $f(x) = x^n$, $x \in [a, b]$ and $n \in \mathbb{N}$. \square

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