NEW HERMITE-HADAMARD-TYPE INEQUALITIES FOR CONVEX FUNCTIONS

KUEI-LIN TSENG, SHIOW-RU HWANG, AND SEVER S. DRAGOMIR

ABSTRACT. In this paper we establish some new Hermite-Hadamard- type inequalities for convex functions and give several applications of interest.

1. INTRODUCTION

Throughout this paper, let $f : [a, b] \to \mathbb{R}$ be convex, $a \le x < y \le y' < x' \le b$, $x + x' = y + y'$ and $\Omega = [x, y] \cup [y', x']$. We define the following functions on $[0, 1]$ that are associated with the well known Hermite-Hadamard inequality [1]

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(s)\,ds \le \frac{f(a)+f(b)}{2}, \tag{1.1}$$

namely

$$H(t) = \frac{1}{b-a}\int_a^b f\left(ts + (1-t)\frac{a+b}{2}\right)ds;$$

$$H_1(t) = \frac{1}{2(y-x)}\int_x^y \left[f(ts + (1-t)y) + f(t(y+y'-s) + (1-t)y')\right]ds;$$

$$H_2(t) = \frac{1}{2(y-x)}\int_x^y \left[f(ts + (1-t)y') + f(t(y+y'-s) + (1-t)y)\right]ds;$$

$$F(t) = \frac{1}{(b-a)^2}\int_a^b\int_a^b f(ts + (1-t)u)\,dsdu;$$

$$F_1(t) = \frac{1}{4(y-x)^2}\int_\Omega\int_\Omega f(ts + (1-t)u)\,dsdu;$$

$$P(t) = \frac{1}{2(b-a)}\int_a^b \left[f\left(\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)s\right) + f\left(\left(\frac{1+t}{2}\right)b + \left(\frac{1-t}{2}\right)s\right)\right]ds;$$

$$P_1(t) = \frac{1}{2(y-x)}\int_x^y \left[f(tx + (1-t)s) + f(tx' + (1-t)(x+x'-s))\right]ds;$$

$$G(t) = \frac{1}{2}\left[f\left(ta + (1-t)\frac{a+b}{2}\right) + f\left(tb + (1-t)\frac{a+b}{2}\right)\right];$$

$$G_1(t) = \frac{1}{2}\left[f(tx + (1-t)y) + f(tx' + (1-t)y')\right];$$

2000 *Mathematics Subject Classification.* Primary 26D15. Secondary 26D10.
Key words and phrases. Hermite-Hadamard inequality, Convex function.

$$G_2(t) = \frac{1}{2}\left[f\left(tx + (1-t)y'\right) + f\left(tx' + (1-t)y\right)\right];$$

$$L(t) = \frac{1}{2(b-a)}\int_a^b \left[f\left(ta + (1-t)s\right) + f\left(tb + (1-t)s\right)\right] ds$$

and

$$L_1(t) = \frac{1}{4(y-x)}\int_\Omega \left[f\left(tx + (1-t)s\right) + f\left(tx' + (1-t)s\right)\right] ds.$$

Remark 1. *We note that* $\Omega = [a,b]$ *and* $H(t) = H_1(t) = H_2(t)$, $F(t) = F_1(t)$, $P(t) = P_1(t)$, $G(t) = G_1(t) = G_2(t)$, $L(t) = L_1(t)$ *on* $[0,1]$ *as* $x = a$, $y = y' = \frac{a+b}{2}$ *and* $x' = b$.

For some results which generalize, improve, and extend this famous integral inequality (1.1) see [2] – [16].

In [2], Dragomir established the following Hermite-Hadamard-type inequalities related to the functions H, F which refine the first inequality of (1.1).

Theorem A. *Let* f, H *be defined as above. Then* H *is convex, increasing on* $[0,1]$, *and for all* $t \in [0,1]$, *we have*

$$f\left(\frac{a+b}{2}\right) = H(0) \le H(t) \le H(1) = \frac{1}{b-a}\int_a^b f(s)\,ds. \tag{1.2}$$

and

Theorem B. *Let* f, F *be defined as above. Then*

(1) F *is convex on* $[0,1]$, *symmetric about* $\frac{1}{2}$, F *is decreasing on* $\left[0, \frac{1}{2}\right]$ *and increasing on* $\left[\frac{1}{2}, 1\right]$, *and we have*

$$\sup_{t\in[0,1]} F(t) = F(0) = F(1) = \frac{1}{b-a}\int_a^b f(s)\,ds$$

and

$$\inf_{t\in[0,1]} F(t) = F\left(\frac{1}{2}\right) = \frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{s+u}{2}\right) dsdu;$$

(2) *We have:*

$$f\left(\frac{a+b}{2}\right) \le F\left(\frac{1}{2}\right); \quad H(t) \le F(t), \qquad t \in [0,1]. \tag{1.3}$$

In [12], Yang and Hong established the following Hermite-Hadamard-type inequality related to the function P and which refines the second inequality of (1.1).

Theorem C. *Let* f, P *be defined as above. Then* P *is convex, increasing on* $[0,1]$, *and for all* $t \in [0,1]$, *we have*

$$\frac{1}{b-a}\int_a^b f(s)\,ds = P(0) \le P(t) \le P(1) = \frac{f(a) + f(b)}{2}. \tag{1.4}$$

In [7], Dragomir *et al.* established the following Hermite-Hadamard-type inequalities related to the functions H, G, L.

Theorem D. *Let* f, H *be defined as above. Then:*

(1) *The inequality*

$$f\left(\frac{a+b}{2}\right) \leq \frac{2}{b-a}\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} f(s)\,ds \leq \int_0^1 H(t)\,dt \leq \frac{1}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{1}{b-a}\int_a^b f(s)\,ds\right] \tag{1.5}$$

holds.

(2) *If f is differentiable on $[a,b]$, then we have the inequalities*

$$0 \leq \frac{1}{b-a}\int_a^b f(s)\,ds - H(t) \leq (1-t)\left[\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(s)\,ds\right] \tag{1.6}$$

and

$$0 \leq \frac{f(a)+f(b)}{2} - H(t) \leq \frac{(f'(b)-f'(a))(b-a)}{4} \tag{1.7}$$

for all $t \in [0,1]$.

Theorem E. *Let f, H, G be defined as above. Then:*

(1) *G is convex and increasing on $[0,1]$;*

(2) *We have*

$$\inf_{t\in[0,1]} G(t) = G(0) = f\left(\frac{a+b}{2}\right)$$

and

$$\sup_{t\in[0,1]} G(t) = G(1) = \frac{f(a)+f(b)}{2};$$

(3) *The inequality*

$$H(t) \leq G(t) \tag{1.8}$$

holds for all $t \in [0,1]$.

(4) *The inequality*

$$\frac{2}{b-a}\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} f(s)\,ds \leq \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right]\int_0^1 G(t)\,dt \leq \frac{1}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right]. \tag{1.9}$$

(5) *If f is differentiable on $[a,b]$, then we have the inequality*

$$0 \leq H(t) - f\left(\frac{a+b}{2}\right) \leq G(t) - H(t) \tag{1.10}$$

for all $t \in [0,1]$.

Theorem F. *Let f, H, G, L be defined as above. Then:*

(1) *L is convex on $[0,1]$.*

(2) *We have the inequality:*

$$G(t) \le L(t) \le \frac{1-t}{b-a}\int_a^b f(s)\,ds + t\cdot\frac{f(a)+f(b)}{2} \le \frac{f(a)+f(b)}{2} \tag{1.11}$$

for all $t \in [0,1]$ *and*

$$\sup_{t\in[0,1]} L(t) = \frac{f(a)+f(b)}{2}.$$

(3) *One has the inequalities:*

$$H(1-t) \le L(t) \tag{1.12}$$

and

$$\frac{H(t)+H(1-t)}{2} \le L(t) \tag{1.13}$$

for all $t \in [0,1]$.

In [11], Tseng *et al.* established the following Hermite-Hadamard-type inequalities related to the functions H, P, L, G.

Theorem G. *Let* f, H, P *be defined as above. Then we have the following results:*

(1) *The inequality*

$$\begin{aligned} \frac{1}{b-a}\int_a^b f(s)\,ds &\le \frac{2}{b-a}\int_{\left[a,\frac{3a+b}{4}\right]\cup\left[\frac{a+3b}{4},b\right]} f(s)\,ds \\ &\le \int_0^1 P(t)\,dt \\ &\le \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f(s)\,ds + \frac{f(a)+f(b)}{2}\right] \end{aligned} \tag{1.14}$$

holds.

(2) *The inequalities*

$$L(t) \le P(t) \le \frac{1-t}{b-a}\int_a^b f(s)\,ds + t\cdot\frac{f(a)+f(b)}{2} \le \frac{f(a)+f(b)}{2} \tag{1.15}$$

and

$$0 \le P(t) - G(t) \le \frac{f(a)+f(b)}{2} - P(t) \tag{1.16}$$

hold for all $t \in [0,1]$.

(3) *If* f *is differentiable on* $[a,b]$, *then we have the inequalities*

$$0 \le t\left[\frac{1}{b-a}\int_a^b f(s)\,ds - f\left(\frac{a+b}{2}\right)\right] \le P(t) - \frac{1}{b-a}\int_a^b f(s)\,ds, \tag{1.17}$$

$$0 \le P(t) - f\left(\frac{a+b}{2}\right) \le \frac{(f'(b)-f'(a))(b-a)}{4} \tag{1.18}$$

and

$$0 \le P(t) - H(t) \le \frac{(f'(b)-f'(a))(b-a)}{4} \tag{1.19}$$

for all $t \in [0,1]$.

In [5], Dragomir established the following Hermite-Hadamard-type inequality related to the functions H, F, L.

Theorem H. *Let f, F, H, L be defined as above. Then we have the inequalities*

$$0 \leq F(t) - H(t) \leq L(1-t) - F(t) \tag{1.20}$$

for all $t \in [0,1]$.

In this paper, we shall establish some new Hermite-Hadamard-type inequalities which generalize Theorems A – H and give several applications.

2. Main Results

In order to prove our main results, we need the following lemmas:

Lemma 1 (see [9])**.** *Let f be defined as above and let $a \leq A \leq C \leq D \leq B \leq b$ with $A + B = C + D$. Then*

$$f(C) + f(D) \leq f(A) + f(B).$$

The assumptions in Lemma 1 can be weakened as in the following lemma:

Lemma 2. *Let f be defined as above and let $a \leq A \leq C \leq B \leq b$ and $a \leq A \leq D \leq B \leq b$ with $A + B = C + D$. Then*

$$f(C) + f(D) \leq f(A) + f(B).$$

Now, we are ready to state and prove our new results.

Theorem 1. *Let $x, y, y', x', \Omega, f, H_1, H_2$ be defined as above. Then:*

1. *H_1 and H_2 are convex on $[0,1]$.*
2. *H_1 is increasing on $[0,1]$ and the following inequalities*

$$\frac{f(y) + f(y')}{2} = H_1(0) \leq H_1(t) \leq H_1(1) = \frac{1}{2(y-x)} \int_\Omega f(s)\, ds\ , \tag{2.1}$$

$$\begin{aligned} H_1(t) &\leq t \cdot \frac{1}{2(y-x)} \int_\Omega f(s)\, ds + (1-t) \cdot \frac{f(y) + f(y')}{2} \\ &\leq \frac{1}{2(y-x)} \int_\Omega f(s)\, ds \leq \frac{f(x) + f(x')}{2} \end{aligned} \tag{2.2}$$

$$\begin{aligned} f\left(\frac{y+y'}{2}\right) &\leq H_2(t) \\ &\leq t \cdot \frac{1}{2(y-x)} \int_\Omega f(s)\, ds + (1-t) \cdot \frac{f(y) + f(y')}{2} \\ &\leq \frac{1}{2(y-x)} \int_\Omega f(s)\, ds \end{aligned} \tag{2.3}$$

and

$$H_2(t) \leq H_1(t) \tag{2.4}$$

hold for all $t \in [0,1]$.

(3) *The inequalities*

$$\begin{aligned} \frac{f(y)+f(y')}{2} &\leq \frac{1}{y-x}\int_{\left[\frac{x+y}{2},y\right]\cup\left[y',\frac{x'+y'}{2}\right]} f(s)\,ds \\ &\leq \int_0^1 H_1(t)\,dt \\ &\leq \frac{1}{2}\left[\frac{f(y)+f(y')}{2}+\frac{1}{2(y-x)}\int_\Omega f(s)\,ds\right] \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} f\left(\frac{y+y'}{2}\right) &\leq \frac{1}{y-x}\int_{\frac{x+y'}{2}}^{\frac{y+x'}{2}} f(s)\,ds \\ &\leq \int_0^1 H_2(t)\,dt \\ &\leq \frac{1}{2}\left[\frac{f(y)+f(y')}{2}+\frac{1}{2(y-x)}\int_\Omega f(s)\,ds\right] \end{aligned} \tag{2.6}$$

hold.

(4) *If f is differentiable on $[a,b]$, then the inequalities*

$$\begin{aligned} 0 &\leq \frac{1}{2(y-x)}\int_\Omega f(s)\,ds - H_1(t) \\ &\leq (1-t)\left[\frac{f(x)+f(x')}{2}-\frac{1}{2(y-x)}\int_\Omega f(s)\,ds\right], \end{aligned} \tag{2.7}$$

$$0 \leq \frac{f(x)+f(x')}{2} - H_1(t) \leq (y-x)\frac{f'(x')-f'(x)}{2}, \tag{2.8}$$

and

$$0 \leq H_1(t) - \frac{f(y)+f(y')}{2} \leq (y-x)\frac{f'(x')-f'(x)}{2} \tag{2.9}$$

hold for all $t\in[0,1]$.

Proof. (1) It is easily observed from the convexity of f that H_1 and H_2 are convex on $[0,1]$.

(2) Let $t_1 < t_2$ in $[0,1]$. By Lemma 2, the following inequality holds for all $s\in[x,y]$

$$\begin{aligned} &f(t_1 s+(1-t_1)y)+f(t_1(y+y'-s)+(1-t_1)y') \\ &\qquad \leq f(t_2 s+(1-t_2)y)+f(t_2(y+y'-s)+(1-t_2)y'). \end{aligned}$$

Integrating the above inequality over s on $[x,y]$, and dividing both sides by $2(y-x)$, we have

$$H_1(t_1) \leq H_1(t_2).$$

Thus, H_1 is increasing on $[0,1]$ and (2.1) holds. Using the convexity of f, the inequality (2.1) and the substitution rule for integration, we obtain the first and second inequalities of (2.2) and the inequality (2.3). Using simple techniques of integration, we have the following identitiy

$$\frac{1}{2(y-x)}\int_\Omega f(s)\,ds = \frac{1}{2(y-x)}\int_x^y [f(s)+f(y+y'-s)]\,ds.$$

By Lemma 2, the inequality

$$f(s)+f(y+y'-s)\le f(x)+f(x')$$

holds for all $s\in[x,y]$. Integrating the above inequality over s on $[x,y]$, dividing both sides by $2(y-x)$ and using the above identities, we derive the last inequality of (2.2).

Again, using Lemma 2, the inequality

$$f(ts+(1-t)y')+f(t(y+y'-s)+(1-t)y)$$
$$\le f(ts+(1-t)y)+f(t(y+y'-s)+(1-t)y')$$

holds for all $t\in[0,1]$ and $s\in[x,y]$. Integrating this inequality over s on $[x,y]$, dividing both sides by $2(y-x)$ and using the definitions of H_1 and H_2, we derive (2.4).

(3) Using simple techniques of integration, we have the following identities

$$\frac{f(y)+f(y')}{2}=\frac{1}{y-x}\int_x^y\int_0^{\frac{1}{2}}[f(y)+f(y')]\,dtds,$$

$$\frac{1}{y-x}\int_{\left[\frac{x+y}{2},y\right]\cup\left[y',\frac{x'+y'}{2}\right]}f(s)\,ds$$
$$=\frac{1}{y-x}\int_x^y\int_0^{\frac{1}{2}}\left[f\left(\frac{s+y}{2}\right)+f\left(\frac{y+2y'-s}{2}\right)\right]dtds,$$

$$\int_0^1 H_1(t)\,dt=\frac{1}{(y-x)}\int_x^y\int_0^{\frac{1}{2}}\frac{1}{2}[f(ty+(1-t)s)+f(ts+(1-t)y)]\,dtds$$
$$+\frac{1}{(y-x)}\int_x^y\int_0^{\frac{1}{2}}\frac{1}{2}[f(t(y+y'-s)+(1-t)y')$$
$$+f(ty'+(1-t)(y+y'-s))]\,dtds$$

and

$$\frac{1}{2}\left[\frac{f(y)+f(y')}{2}+\frac{1}{2(y-x)}\int_\Omega f(s)\,ds\right]$$
$$=\frac{1}{(y-x)}\int_x^y\int_0^{\frac{1}{2}}\frac{1}{2}[f(s)+f(y)]\,dtds$$
$$+\frac{1}{(y-x)}\int_x^y\int_0^{\frac{1}{2}}\frac{1}{2}[f(y')+f(y+y'-s)]\,dtds.$$

By Lemma 2, the following inequalities hold for all $t\in\left[0,\frac{1}{2}\right]$ and $s\in[x,y]$

$$f(y)+f(y')\le f\left(\frac{s+y}{2}\right)+f\left(\frac{y+2y'-s}{2}\right),$$

$$f\left(\frac{s+y}{2}\right)\le\frac{1}{2}[f(ty+(1-t)s)+f(ts+(1-t)y)],$$

$$f\left(\frac{y+2y'-s}{2}\right)\le\frac{1}{2}[f(t(y+\ y'-s)+(1-t)y')+f(ty'+(1-t)(y+y'-s))],$$

$$\frac{1}{2}\left[f\left(ty+\left(1-t\right)s\right)+f\left(ts+\left(1-t\right)y\right)\right]\leq\frac{f\left(s\right)+f\left(y\right)}{2}$$

and

$$\frac{1}{2}\left[f\left(t\left(y+y'-s\right)+\left(1-t\right)y'\right)+f\left(ty'+\left(1-t\right)\left(y+y'-s\right)\right)\right] \leq\frac{f\left(y'\right)+f\left(y+y'-s\right)}{2}.$$

Integrating the above inequalities over t on $\left[0,\frac{1}{2}\right]$, over s on $[x,y]$, dividing both sides by $(y-x)$ and using the above identities, we derive (2.5).

Again, using simple techniques of integration, we have the following identities

$$f\left(\frac{y+y'}{2}\right)=\frac{1}{y-x}\int_x^y\int_0^{\frac{1}{2}}2f\left(\frac{y+y'}{2}\right)dtds,$$

$$\frac{1}{y-x}\int_{\frac{x+y'}{2}}^{\frac{y+x'}{2}}f\left(s\right)ds=\frac{1}{y-x}\int_x^y\int_0^{\frac{1}{2}}\left[f\left(\frac{s+y'}{2}\right)+f\left(\frac{y'+2y-s}{2}\right)\right]dtds,$$

$$\begin{aligned}\int_0^1 H_2\left(t\right)dt &= \frac{1}{(y-x)}\int_x^y\int_0^{\frac{1}{2}}\frac{1}{2}\left[f\left(ty'+\left(1-t\right)s\right)+f\left(ts+\left(1-t\right)y'\right)\right]dtds\\ &\quad+\frac{1}{(y-x)}\int_x^y\int_0^{\frac{1}{2}}\frac{1}{2}\left[f\left(t\left(y+y'-s\right)+\left(1-t\right)y\right)\right.\\ &\qquad\left.+f\left(ty+\left(1-t\right)\left(y+y'-s\right)\right)\right]dtds\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{2}\left[\frac{f\left(y\right)+f\left(y'\right)}{2}+\frac{1}{2\left(y-x\right)}\int_\Omega f\left(s\right)ds\right]\\ &=\frac{1}{(y-x)}\int_x^y\int_0^{\frac{1}{2}}\frac{1}{2}\left[f\left(s\right)+f\left(y'\right)\right]dtds\\ &\quad+\frac{1}{(y-x)}\int_x^y\int_0^{\frac{1}{2}}\frac{1}{2}\left[f\left(y\right)+f\left(y+y'-s\right)\right]dtds.\end{aligned}$$

By Lemma 2, the following inequalities hold for all $t\in\left[0,\frac{1}{2}\right]$ and $s\in\left[x,y\right]$:

$$2f\left(\frac{y+y'}{2}\right)\leq f\left(\frac{s+y'}{2}\right)+f\left(\frac{y'+2y-s}{2}\right),$$

$$f\left(\frac{s+y'}{2}\right)\leq\frac{1}{2}\left[f\left(ty'+\left(1-t\right)s\right)+f\left(ts+\left(1-t\right)y'\right)\right],$$

$$f\left(\frac{y'+2y-s}{2}\right)\leq\frac{1}{2}\left[f\left(t\left(y+\ y'-s\right)+\left(1-t\right)y\right)+f\left(ty+\left(1-t\right)\left(y+y'-s\right)\right)\right],$$

$$\frac{1}{2}\left[f\left(ty'+\left(1-t\right)s\right)+f\left(ts+\left(1-t\right)y'\right)\right]\leq\frac{f\left(s\right)+f\left(y'\right)}{2}$$

and

$$\frac{1}{2}\left[f\left(t\left(y+y'-s\right)+\left(1-t\right)y\right)+f\left(ty+\left(1-t\right)\left(y+y'-s\right)\right)\right] \leq\frac{f\left(y\right)+f\left(y+y'-s\right)}{2}.$$

Integrating the above inequalities over t on $\left[0, \frac{1}{2}\right]$, over s on $[x, y]$, dividing both sides by $(y-x)$ and using the above identities, we derive (2.6).

(4) By integration by parts, we have the following identity

$$\frac{1}{2(y-x)} \int_x^y \left[(s-y) f'(s) + (y-s) f'(y+y'-s)\right] ds = \frac{f(x)+f(x')}{2} - \frac{1}{2(y-x)} \int_\Omega f(s)\, ds.$$

Now, using the convexity of f, the inequalities

$$f(s) - f(ts + (1-t)y) \le (1-t)(s-y) f'(s)$$

and

$$f(y+y'-s) - f(t(y+y'-s) + (1-t)y') \le (1-t)(y-s) f'(y+y'-s)$$

hold for all $t \in [0,1]$ and $s \in [x,y]$. Integrating the above inequalities over s on $[x,y]$, dividing both sides by $2(y-x)$ and using the above identity and (2.1), we derive (2.7).

On the other hand, we have

$$\frac{f(x) - f(y)}{2} \le \frac{1}{2}(x-y) f'(x)$$

and

$$\frac{f(x') - f(y')}{2} \le \frac{1}{2}(x'-y') f'(x')$$

and taking their sum:

$$\frac{f(x)+f(x')}{2} - \frac{f(y)+f(y')}{2} \le \frac{1}{2}(x-y) f'(x) + \frac{1}{2}(x'-y') f'(x') = (y-x)\frac{f'(x') - f'(x)}{2}. \tag{2.10}$$

Finally, (2.8) and (2.9) follow from (2.1), (2.2)and (2.10).

This completes the proof. □

Remark 2. *Let $x = a$, $y = y' = \frac{a+b}{2}$ and $x' = b$ in Theorem 1. Then $H_1(t) = H_2(t) = H(t)$ $(t \in [0,1])$ and Theorem 1 reduces to Theorems A and D.*

Theorem 2. *Let $x, y, y', x', \Omega, f, H_1, P_1$ be defined as above. Then we have the following results:*

(1) *P_1 is convex on $[0,1]$.*

(2) *P_1 is increasing on $[0,1]$ and the following inequalities*

$$\frac{1}{2(y-x)} \int_\Omega f(s)\, ds = P_1(0) \le P_1(t) \le P_1(1) = \frac{f(x)+f(x')}{2} \tag{2.11}$$

and

$$P_1(t) \le (1-t) \cdot \frac{1}{2(y-x)} \int_\Omega f(s)\, ds + t \cdot \frac{f(x)+f(x')}{2} \le \frac{f(x)+f(x')}{2} \tag{2.12}$$

hold for all $t \in [0,1]$.

(3) *The inequalities*

$$\frac{1}{2(y-x)}\int_{\Omega} f(s)\,ds \le \frac{1}{y-x}\int_{\left[x,\frac{x+y}{2}\right]\cup\left[\frac{x'+y'}{2},x'\right]} f(s)\,ds \le \int_0^1 P_1(t)\,dt \le \frac{1}{2}\left[\frac{f(x)+f(x')}{2}+\frac{1}{2(y-x)}\int_{\Omega} f(s)\,ds\right] \tag{2.13}$$

and

$$H_1(t) \le P_1(t) \qquad (t\in[0,1]) \tag{2.14}$$

hold.

(4) *If f is differentiable on $[a,b]$, then the inequalities*

$$0 \le t\left[\frac{1}{2(y-x)}\int_{\Omega} f(s)\,ds - \frac{f(y)+f(y')}{2}\right] \le P_1(t) - \frac{1}{2(y-x)}\int_{\Omega} f(s)\,ds, \tag{2.15}$$

$$0 \le P_1(t) - \frac{f(y)+f(y')}{2} \le (y-x)\frac{f'(x')-f'(x)}{2}, \tag{2.16}$$

$$0 \le \frac{f(x)+f(x')}{2} - P_1(t) \le (y-x)\frac{f'(x')-f'(x)}{2} \tag{2.17}$$

and

$$0 \le P_1(t) - H_1(t) \le (y-x)\frac{f'(x')-f'(x)}{2} \tag{2.18}$$

hold for all $t\in[0,1]$.

Proof. (1) It is easily observed from the convexity of f that P_1 is convex on $[0,1]$.

Let $t_1 < t_2$ in $[0,1]$. By Lemma 2, the following inequality holds for all $s\in[x,y]$

$$f(t_1x+(1-t_1)s)+f(t_1x'+(1-t_1)(x+x'-s)) \le f(t_2x+(1-t_2)s)+f(t_2x'+(1-t_2)(x+x'-s)).$$

Integrating the above inequality over s on $[x,y]$, and dividing both sides by $2(y-x)$, we have

$$P_1(t_1) \le P_1(t_2).$$

Thus, P_1 is increasing on $[0,1]$ and (2.11) holds.

Using the convexity of f, the inequality (2.11) and the substitution rule for integration, the inequality (2.12) holds.

(3) Using simple techniques of integration, we have the following identities

$$\frac{1}{2(y-x)}\int_{\Omega} f(s)\,ds = \frac{1}{y-x}\int_x^y\int_0^{\frac{1}{2}}[f(s)+f(x+x'-s)]\,dtds,$$

$$\frac{1}{y-x}\int_{\left[x,\frac{x+y}{2}\right]\cup\left[\frac{x'+y'}{2},x'\right]} f(s)\,ds = \frac{1}{y-x}\int_x^y\int_0^{\frac{1}{2}}\left[f\left(\frac{x+s}{2}\right)+f\left(\frac{x+2x'-s}{2}\right)\right]dtds,$$

$$\int_0^1 P_1(t)\,dt = \frac{1}{(y-x)}\int_x^y \int_0^{\frac{1}{2}} \frac{1}{2}\left[f(ts+(1-t)x)+f(tx+(1-t)s)\right]dtds$$
$$+\frac{1}{(y-x)}\int_x^y \int_0^{\frac{1}{2}} \frac{1}{2}\left[f(tx'+(1-t)(x+x'-s))\right.$$
$$\left.+f(t(x+x'-s)+(1-t)x')\right]dtds$$

and

$$\frac{1}{2}\left[\frac{f(x)+f(x')}{2}+\frac{1}{2(y-x)}\int_\Omega f(s)\,ds\right]$$
$$=\frac{1}{(y-x)}\int_x^y \int_0^{\frac{1}{2}} \frac{1}{2}\left[f(x)+f(s)\right]dtds$$
$$+\frac{1}{(y-x)}\int_x^y \int_0^{\frac{1}{2}} \frac{1}{2}\left[f(x+x'-s)+f(x')\right]dtds.$$

By Lemma 2, the following inequalities hold for all $t\in\left[0,\frac{1}{2}\right]$ and $s\in[x,y]$

$$f(s)+f(x+x'-s)\le f\left(\frac{x+s}{2}\right)+f\left(\frac{x+2x'-s}{2}\right),$$
$$f\left(\frac{x+s}{2}\right)\le\frac{1}{2}\left[f(ts+(1-t)x)+f(tx+(1-t)s)\right],$$
$$f\left(\frac{x+2x'-s}{2}\right)\le\frac{1}{2}\left[f(tx'+(1-t)(x+x'-s))+f(t(x+x'-s)+(1-t)x')\right],$$
$$\frac{1}{2}\left[f(ts+(1-t)x)+f(tx+(1-t)s)\right]\le\frac{f(x)+f(s)}{2}$$

and

$$\frac{1}{2}\left[f(tx'+(1-t)(x+x'-s))+f(t(x+x'-s)+(1-t)x')\right]$$
$$\le\frac{f(x+x'-s)+f(x')}{2}.$$

Integrating the above inequalities over t on $\left[0,\frac{1}{2}\right]$, over s on $[x,y]$, dividing both sides by $(y-x)$ and the above identities, we derive (2.13).

Finally, (2.14) follows from (2.1) and (2.11).

(4) Integrating by parts, we have the following identitiy

$$\frac{1}{2(y-x)}\int_x^y\left[(x-s)f'(s)+(s-x)f'(x+x'-s)\right]ds$$
$$=\frac{1}{2(y-x)}\int_\Omega f(s)\,ds-\frac{f(y)+f(y')}{2}.$$

Now, using the convexity of f, the inequalities

$$f(tx+(1-t)s)-f(s)\ge t(x-s)f'(s)$$

and

$$f(tx'+(1-t)(x+x'-s))-f(x+x'-s)\ge t(s-x)f'(x+x'-s)$$

hold for all $t\in[0,1]$ and $s\in[x,y]$. Integrating the above inequalities over s on $[x,y]$, dividing both sides by $2(y-x)$ and using the above identity and (2.1), we derive (2.15).

Finally, $(2.16)-(2.18)$ follow from (2.1), (2.10), (2.11) and (2.14).

This completes the proof. □

Theorem 3. *Let $x, y, y', x', \Omega, f, H_1, H_2, F_1$ be defined as above. Then we have the following results:*

(1) *F_1 is convex on $[0,1]$ and symmetric about $\frac{1}{2}$.*

(2) *F_1 is decreasing on $\left[0, \frac{1}{2}\right]$ and increasing on $\left[\frac{1}{2}, 1\right]$,*

$$\sup_{t\in[0,1]} F_1(t) = F_1(0) = F_1(1) = \frac{1}{2(y-x)} \int_{\Omega} f(s)\, ds \tag{2.19}$$

and

$$\inf_{t\in[0,1]} F_1(t) = F_1\left(\frac{1}{2}\right) = \frac{1}{4(y-x)^2} \int_{\Omega}\int_{\Omega} f\left(\frac{s+u}{2}\right) dsdu. \tag{2.20}$$

(3) *We have:*

$$\frac{H_1(t) + H_2(t)}{2} \leq F_1(t) \tag{2.21}$$

and

$$\frac{f(y) + 2f\left(\frac{y+y'}{2}\right) + f(y')}{4} \leq F_1\left(\frac{1}{2}\right) \tag{2.22}$$

for all $t \in [0,1]$.

Proof. (1) It is easily observed from the convexity of f that F_1 is convex on $[0,1]$.

By changing variables, we have

$$F_1(t) = F_1(1-t), \qquad t \in [0,1]$$

from which we get that F_1 is symmetric about $\frac{1}{2}$.

(2) Let $t_1 < t_2$ in $\left[0, \frac{1}{2}\right]$. Using the symmetry of F_1, we have

$$F_1(t_1) = \frac{1}{2}\left[F_1(t_1) + F_1(1-t_1)\right], \tag{2.23}$$

$$F_1(t_2) = \frac{1}{2}\left[F_1(t_2) + F_1(1-t_2)\right] \tag{2.24}$$

and, by Lemma 2, we obtain

$$\frac{1}{2}\left[F_1(t_2) + F_1(1-t_2)\right] \leq \frac{1}{2}\left[F_1(t_1) + F_1(1-t_1)\right]. \tag{2.25}$$

From $(2.23)-(2.25)$, we obtain that F_1 is decreasing on $\left[0, \frac{1}{2}\right]$. Since F_1 is symmetric about $\frac{1}{2}$ and F_1 is decreasing on $\left[0, \frac{1}{2}\right]$, we get that F_1 is increasing on $\left[\frac{1}{2}, 1\right]$. Using the symmetry and monotonicity of F_1, we derive (2.19) and (2.20).

(3) Using the substitution rules for integration, we have the following identities:

$$\begin{aligned} F_1(t) = \frac{1}{4(y-x)^2} \Bigg\{ &\int_x^y \int_x^y \left[f(ts + (1-t)u) + f(ts + (1-t)(y+y'-u))\right] dsdu \\ &+ \int_x^y \int_x^y \left[f(t(y+y'-s) + (1-t)u)\right. \\ &\qquad \left. + f(t(y+y'-s) + (1-t)(y+y'-u))\right] dsdu \Bigg\} \end{aligned}$$

and

$$\frac{H_1(t)+H_2(t)}{2}=\frac{1}{4(y-x)^2}\left\{\int_x^y\int_x^y\left[f(ts+(1-t)y)+f(ts+(1-t)y')\right]dsdu\right.$$
$$+\int_x^y\int_x^y\left[f(t(y+y'-s)+(1-t)y)\right.$$
$$\left.\left.+f(t(y+y'-s)+(1-t)y')\right]dsdu\right\}$$

for all $t\in[0,1]$. By Lemma 2, the following inequalities hold for all $t\in[0,1]$, $s\in[x,y]$ and $u\in[x,y]$

$$f(ts+(1-t)y)+f(ts+(1-t)y')\le f(ts+(1-t)u)+f(ts+(1-t)(y+y'-u)),$$

$$f(t(y+y'-s)+(1-t)y)+f(t(y+y'-s)+(1-t)y')$$
$$\le f(t(y+y'-s)+(1-t)u)+f(t(y+y'-s)+(1-t)(y+y'-u)).$$

Dividing the above inequalities by $4(y-x)^2$, integrating them over s on $[x,y]$, over u on $[x,y]$ and using the above identities, we derive the inequality (2.21).

From the inequalities (2.3), (2.21) and the monotonicity of H_1, we have

$$\frac{f(y)+2f\left(\frac{y+y'}{2}\right)+f(y')}{4}\le\frac{H_1(0)+H_2\left(\frac{1}{2}\right)}{2}$$
$$\le\frac{H_1\left(\frac{1}{2}\right)+H_2\left(\frac{1}{2}\right)}{2}$$
$$\le F_1\left(\frac{1}{2}\right),$$

from which we derive the inequality (2.22).

This completes the proof. □

Remark 3. *Let $x=a, y=y'=\frac{a+b}{2}$ and $x'=b$ in Theorem 3. Then $F_1(t)=F(t)$, $H_1(t)=H_2(t)=H(t)$ $(t\in[0,1])$ and Theorem 3 reduces to Theorem B.*

Theorem 4. *Let $x,y,y',x',f,H_1,P_1,G_1,G_2$ be defined as above. Then we have the following results:*

(1) *G_1 and G_2 are convex on $[0,1]$.*

(2) *G_1 is increasing on $[0,1]$, G_2 is decreasing on $\left[0,\frac{y'-y}{2(y'-x)}\right]$ and increasing on $\left[\frac{y'-y}{2(y'-x)},1\right]$, and the inequalities*

$$\frac{f(y)+f(y')}{2}=G_1(0)\le G_1(t)\le G_1(1)=\frac{f(x)+f(x')}{2}\tag{2.26}$$

and

$$f\left(\frac{x+x'}{2}\right)=G_2\left(\frac{y'-y}{2(y'-x)}\right)\le G_2(t)\le G_2(1)=\frac{f(x)+f(x')}{2}\tag{2.27}$$

hold for all $t\in[0,1]$.

(3) *The inequalities*

$$H_1(t) \le G_1(t) \le P_1(t) \tag{2.28}$$

and

$$G_2(t) \le G_1(t) \tag{2.29}$$

hold for all $t \in [0,1]$.

(4) *The inequalities*

$$\begin{aligned} \frac{1}{y-x}\int_{\left[\frac{x+y}{2},y\right]\cup\left[y',\frac{x'+y'}{2}\right]} f(s)\,ds &\le \frac{1}{2}\left[f\left(\frac{x+y}{2}\right)+f\left(\frac{x'+y'}{2}\right)\right] \\ &\le \int_0^1 G_1(t)\,dt \\ &\le \frac{f(x)+f(x')+f(y)+f(y')}{4} \end{aligned} \tag{2.30}$$

and

$$\begin{aligned} \frac{1}{y-x}\int_{\frac{x+y'}{2}}^{\frac{x'+y}{2}} f(s)\,ds &\le \frac{1}{2}\left[f\left(\frac{x+y'}{2}\right)+f\left(\frac{x'+y}{2}\right)\right] \\ &\le \int_0^1 G_2(t)\,dt \\ &\le \frac{f(x)+f(x')+f(y)+f(y')}{4} \end{aligned} \tag{2.31}$$

hold.

(5) *The inequalities*

$$0 \le H_1(t) - \frac{f(y)+f(y')}{2} \le G_1(t) - H_1(t) \tag{2.32}$$

and

$$0 \le P_1(t) - G_1(t) \le \frac{f(x)+f(x')}{2} - P_1(t) \tag{2.33}$$

hold for all $t \in [0,1]$.

Proof. (1) It is easily observed from the convexity of f that G_1 and G_2 are convex on $[0,1]$.

(2) Let $0 \le t_1 < t_2 \le 1$ and $0 \le u_1 < u_2 \le \frac{y'-y}{2(y'-x)} \le s_1 < s_2 \le 1$. By Lemma 2, the following inequalities hold:

$$\begin{aligned} G_1(t_1) &= \frac{1}{2}\left[f(t_1x+(1-t_1)y)+f(t_1x'+(1-t_1)y')\right] \\ &\le \frac{1}{2}\left[f(t_2x+(1-t_2)y)+f(t_2x'+(1-t_2)y')\right] \\ &= G_1(t_2), \end{aligned}$$

$$\begin{aligned} G_2(u_2) &= \frac{1}{2}\left[f(u_2x'+(1-u_2)y)+f(u_2x+(1-u_2)y')\right] \\ &\le \frac{1}{2}\left[f(u_1x'+(1-u_1)y)+f(u_1x+(1-u_1)y')\right] \\ &= G_2(u_1) \end{aligned}$$

and

$$\begin{aligned}G_2(s_1) &= \frac{1}{2}\left[f\left(s_1x+(1-s_1)y'\right)+f\left(s_1x'+(1-s_1)y\right)\right]\\ &\le \frac{1}{2}\left[f\left(s_2x+(1-s_2)y'\right)+f\left(s_2x'+(1-s_2)y\right)\right]\\ &= G_2(s_2).\end{aligned}$$

Thus, G_1 is increasing on $[0,1]$ and G_2 is decreasing on $\left[0,\frac{y'-y}{2(y'-x)}\right]$ and increasing on $\left[\frac{y'-y}{2(y'-x)},1\right]$. From the monotonicity of G_1 and G_2, we get the inequalities

$$\frac{f(y)+f(y')}{2}=G_1(0)\le G_1(t)\le G_1(1)=\frac{f(x)+f(x')}{2}$$

and

$$\begin{aligned}f\left(\frac{x+x'}{2}\right) &= G_2\left(\frac{y'-y}{2(y'-x)}\right)\le G_2(t)\\ &\le \max\left\{G_2(0),G_2(1)\right\}=\frac{f(x)+f(x')}{2}\end{aligned}$$

for all $t\in[0,1]$.

(3) By Lemma 2, the following inequalities hold for all $t\in[0,1]$ and $s\in[x,y]$:

$$\begin{aligned}&f\left(ts+(1-t)y\right)+f\left(t\left(y+y'-s\right)+(1-t)y'\right)\\ &\qquad\le f\left(tx+(1-t)y\right)+f\left(tx'+(1-t)y'\right)\end{aligned}$$

and

$$f\left(tx+(1-t)y\right)+f\left(tx'+(1-t)y'\right)\le f\left(tx+(1-t)s\right)+f\left(tx'+(1-t)\left(x+x'-s\right)\right).$$

Integrating the above inequalities over s on $[x,y]$, and dividing both sides by $2(y-x)$, we have

$$H_1(t)\le G_1(t)\le P_1(t)$$

for all $t\in[0,1]$.

Again, using Lemma 2, the inequality

$$f\left(tx+(1-t)y'\right)+f\left(tx'+(1-t)y\right)\le f\left(tx+(1-t)y\right)+f\left(tx'+(1-t)y'\right)$$

holds for all $t\in[0,1]$. Using the above result, we get $G_2(t)\le G_1(t)$ on $[0,1]$.

Using simple techniques of integration, we have the following identities

$$\begin{aligned}&\frac{1}{y-x}\int_{\left[\frac{x+y}{2},y\right]\cup\left[y',\frac{x'+y'}{2}\right]}f(s)\,ds\\ &\qquad=\frac{1}{y-x}\int_x^y\int_0^{\frac{1}{2}}\left[f\left(\frac{s+y}{2}\right)+f\left(\frac{x+x'+y'-s}{2}\right)\right]dtds,\end{aligned}$$

$$\frac{1}{2}\left[f\left(\frac{x+y}{2}\right)+f\left(\frac{x'+y'}{2}\right)\right]=\frac{1}{y-x}\int_x^y\int_0^{\frac{1}{2}}f\left[\left(\frac{x+y}{2}\right)+f\left(\frac{x'+y'}{2}\right)\right]dtds,$$

$$\begin{aligned}\int_0^1 G_1(t)\,dt &= \frac{1}{(y-x)}\int_x^y\int_0^{\frac{1}{2}}\frac{1}{2}\left[f\left(ty+(1-t)x\right)+f\left(tx+(1-t)y\right)\right]dtds\\ &\quad+\frac{1}{(y-x)}\int_x^y\int_0^{\frac{1}{2}}\frac{1}{2}\left[f\left(tx'+(1-t)y'\right)+f\left(ty'+(1-t)x'\right)\right]dtds\end{aligned}$$

and

$$\frac{1}{2}\left[\frac{f(x)+f(x')}{2}+\frac{f(y)+f(y')}{2}\right] = \frac{1}{(y-x)}\int_x^y\int_0^{\frac{1}{2}}\frac{1}{2}\left[f(x)+f(x')+f(y)+f(y')\right]dtds.$$

By Lemma 2, the following inequalities hold for all $t\in\left[0,\frac{1}{2}\right]$ and $s\in[x,y]$:

$$f\left(\frac{s+y}{2}\right)+f\left(\frac{x+x'+y'-s}{2}\right)\le f\left(\frac{x+y}{2}\right)+f\left(\frac{x'+y'}{2}\right),$$

$$f\left(\frac{x+y}{2}\right)\le\frac{1}{2}\left[f(ty+(1-t)x)+f(tx+(1-t)y)\right],$$

$$f\left(\frac{x'+y'}{2}\right)\le\frac{1}{2}\left[f(tx'+(1-t)y')+f(ty'+(1-t)x')\right],$$

$$\frac{1}{2}\left[f(ty+(1-t)x)+f(tx+(1-t)y)\right]\le\frac{f(x)+f(y)}{2}$$

and

$$\frac{1}{2}\left[f(tx'+(1-t)y')+f(ty'+(1-t)x')\right]\le\frac{f(x')+f(y')}{2}.$$

Integrating the above inequalities over t on $\left[0,\frac{1}{2}\right]$, over s on $[x,y]$, dividing both sides by $(y-x)$ and using the above identities, we derive (2.30).

Again, using simple techniques of integration, we have the following identities

$$\frac{1}{y-x}\int_{\frac{x+y'}{2}}^{\frac{x'+y}{2}}f(s)\,ds=\frac{1}{y-x}\int_{\frac{x+y'}{2}}^{\frac{x'+y}{2}}\int_0^{\frac{1}{2}}\left[f(s)+f(x+x'-s)\right]dtds,$$

$$\frac{1}{2}\left[f\left(\frac{x+y'}{2}\right)+f\left(\frac{x'+y}{2}\right)\right] = \frac{1}{y-x}\int_{\frac{x+y'}{2}}^{\frac{x'+y}{2}}\int_0^{\frac{1}{2}}\left[f\left(\frac{x+y'}{2}\right)+f\left(\frac{x'+y}{2}\right)\right]dtds,$$

$$\int_0^1 G_2(t)\,dt=\frac{1}{y-x}\int_{\frac{x+y'}{2}}^{\frac{x'+y}{2}}\int_0^{\frac{1}{2}}\frac{1}{2}\left[f(tx+(1-t)y')+f(ty'+(1-t)x)\right]dtds + \frac{1}{y-x}\int_{\frac{x+y'}{2}}^{\frac{x'+y}{2}}\int_0^{\frac{1}{2}}\frac{1}{2}\left[f(tx'+(1-t)y)+f(ty+(1-t)x')\right]dtds$$

and

$$\frac{f(x)+f(x')+f(y)+f(y')}{4}=\frac{1}{y-x}\int_{\frac{x+y'}{2}}^{\frac{y+x'}{2}}\int_0^{\frac{1}{2}}\frac{1}{2}\left[f(x)+f(x')+f(y)+f(y')\right]dtds.$$

By Lemma 2, the following inequalities hold for all $t\in\left[0,\frac{1}{2}\right]$ and $s\in\left[\frac{x+y'}{2},\frac{x'+y}{2}\right]$:

$$f(s)+f(x+x'-s)\le f\left(\frac{x+y'}{2}\right)+f\left(\frac{x'+y}{2}\right),$$

$$f\left(\frac{x+y'}{2}\right)\le\frac{1}{2}\left[f(ty'+(1-t)x)+f(tx+(1-t)y')\right],$$

$$f\left(\frac{x'+y}{2}\right) \le \frac{1}{2}\left[f\left(tx'+(1-t)\,y\right)+f\left(ty+(1-t)\,x'\right)\right],$$

$$f\left(ty'+(1-t)\,x\right)+f\left(tx+(1-t)\,y'\right) \le f(x)+f(y')$$

and

$$f\left(tx'+(1-t)\,y\right)+f\left(ty+(1-t)\,x'\right) \le f(y)+f(x').$$

Integrating the above inequalities over t on $\left[0,\frac{1}{2}\right]$, over s on $\left[\frac{x+y'}{2},\frac{x'+y}{2}\right]$, dividing both sides by $(y-x)$ and using the above identities, we derive (2.31).

(5) Using simple techniques of integration, we have the following identities

$$2H_1(t) = \frac{1}{(y-x)}\int_x^y \frac{1}{2}\left[f\left(ts+(1-t)\,y\right)+f\left(t\,(x+y-s)+(1-t)\,y\right)\right.$$
$$\left.+\ f\left(t\,(y+y'-s)+(1-t)\,y'\right)+f\left(t\,(y'-x+s)+(1-t)\,y'\right)\right]ds$$

and

$$2P_1(t) = \frac{1}{(y-x)}\int_x^y \frac{1}{2}\left[f\left(tx+(1-t)\,s\right)+f\left(tx+(1-t)\,(x+y-s)\right)\right.$$
$$\left.+\ f\left(tx'+(1-t)\,(x+x'-s)\right)+f\left(tx'+(1-t)\,(y'-x+s)\right)\right]ds$$

for all $t\in[0,1]$. By Lemma 2, the following inequalities hold for all $t\in[0,1]$ and $s\in[x,y]$:

$$\frac{1}{2}\left[f\left(ts+(1-t)\,y\right)+f\left(t\,(x+y-s)+(1-t)\,y\right)\right] \le \frac{1}{2}\left[f\left(tx+(1-t)\,y\right)+f(y)\right],$$

$$\frac{1}{2}\left[f\left(t\,(y+y'-s)+(1-t)\,y'\right)+f\left(t\,(y'-x+s)+(1-t)\,y'\right)\right]$$
$$\le \frac{1}{2}\left[f(y')+f\left(tx'+(1-t)\,y'\right)\right],$$

$$\frac{1}{2}\left[f\left(tx+(1-t)\,s\right)+f\left(tx+(1-t)\,(x+y-s)\right)\right] \le \frac{1}{2}\left[f(x)+f\left(tx+(1-t)\,y\right)\right]$$

and

$$\frac{1}{2}\left[f\left(tx'+(1-t)\,(x+x'-s)\right)+f\left(tx'+(1-t)\,(y'-x+s)\right)\right]$$
$$\le \frac{1}{2}\left[f\left(tx'+(1-t)\,y'\right)+f(x')\right].$$

Integrating the above inequalities over s on $[x,y]$, dividing both sides by $(y-x)$ and using the above identities, we obtain

$$2H_1(t) \le G_1(t)+\frac{f(y)+f(y')}{2} \tag{2.34}$$

and

$$2P_1(t) \le G_1(t)+\frac{f(x)+f(x')}{2}. \tag{2.35}$$

Using (2.1), (2.28), (2.34) and (2.35), we derive (2.32) and (2.33).

This completes the proof. □

Remark 4. *Let $x=a, y=y'=\frac{a+b}{2}$ and $x'=b$ in Theorem 4. Then $H_1(t)=H(t)$, $G_1(t)=G_2(t)=G(t)$ $(t\in[0,1])$ and Theorem 4 reduces to Theorem E.*

Theorem 5. *Let $x, y, y', x', \Omega, f, L_1, P_1, G_1, G_2, H_1, H_2, F_1$ be defined as above. Then we have the following results:*

(1) *L_1 is convex on $[0,1]$.*

(2) *The following inequalities*

$$\frac{G_1(t)+G_2(t)}{2} \leq L_1(t) \leq P_1(t) \qquad (t\in[0,1]) \tag{2.36}$$

and

$$\sup_{t\in[0,1]} L_1(t) = L_1(1) = \frac{f(x)+f(x')}{2} \tag{2.37}$$

hold.

(3) *The following inequalities*

$$\frac{H_1(1-t)+H_2(1-t)}{2} \leq F_1(t) \leq L_1(t), \tag{2.38}$$

$$\frac{H_1(t)+H_2(t)}{2} \leq F_1(t) \leq L_1(t) \tag{2.39}$$

and

$$\frac{H_1(t)+H_2(t)+H_1(1-t)+H_2(1-t)}{4} \leq F_1(t) \leq L_1(t) \tag{2.40}$$

hold for all $t\in[0,1]$.

(4) *The following inequality*

$$0 \leq F_1(t) - \frac{H_1(t)+H_2(t)}{2} \leq L_1(1-t) - F_1(t) \tag{2.41}$$

holds for all $t\in[0,1]$.

Proof. (1) It is easily observed from the convexity of f that L_1 is convex on $[0,1]$.

(2) Using simple techniques of integration, we have the following identities

$$\frac{G_1(t)+G_2(t)}{2} = \frac{1}{4(y-x)}\int_x^y [f(tx+(1-t)y)+f(tx+(1-t)y') + f(tx'+(1-t)y)+f(tx'+(1-t)y')]\,ds$$

and

$$\begin{aligned} L_1(t) &= \frac{1}{4(y-x)}\int_x^y [f(tx+(1-t)s)+f(tx+(1-t)(x+x'-s)) \\ &\quad + f(tx'+(1-t)s)+f(tx'+(1-t)(x+x'-s))]\,ds \\ &= \frac{1}{2}P_1(t) + \frac{1}{4(y-x)}\int_x^y [f(tx'+(1-t)s) \\ &\quad + f(tx+(1-t)(x+x'-s))]\,ds \end{aligned}$$

for all $t\in[0,1]$.

By Lemma 2, the following inequalities hold for all $t\in[0,1]$ and $s\in[x,y]$:

$$f(tx+(1-t)y)+f(tx+(1-t)y') \leq f(tx+(1-t)s)+f(tx+(1-t)(x+x'-s)),$$

$$f(tx'+(1-t)y)+f(tx'+(1-t)y') \leq f(tx'+(1-t)s)+f(tx'+(1-t)(x+x'-s))$$

and

$$f\left(tx'+(1-t)s\right)+f\left(tx+(1-t)\left(x+x'-s\right)\right)$$
$$\leq f\left(tx+(1-t)s\right)+f\left(tx'+(1-t)\left(x+x'-s\right)\right).$$

Integrating the above inequalities over s on $[x,y]$, dividing both sides by $4(y-x)$ and using the above identities and (2.12), we derive (2.36) and (2.37).

(3) Using simple techniques of integration, we have the following identity

$$F_1(t)=\frac{1}{4(y-x)^2}\int_\Omega\int_x^y\left[f\left(tu+(1-t)s\right)+f\left(t\left(x+x'-u\right)+(1-t)s\right)\right]duds$$

for all $t\in[0,1]$.

By Lemma 2, the following inequality holds for all $t\in[0,1]$, $u\in[x,y]$ and $s\in\Omega$.

$$f\left(tu+(1-t)s\right)+f\left(t\left(x+x'-u\right)+(1-t)s\right)\leq f\left(tx+(1-t)s\right)+f\left(tx'+(1-t)s\right).$$

Integrating the above inequality over u on $[x,y]$, over s on Ω, dividing both sides by $4(y-x)^2$ and the above identity and the definition of L_1, we obtain

$$F_1(t)\leq L_1(t) \tag{2.42}$$

for all $t\in[0,1]$. Using (2.21), (2.42) and the symmetry of F_1, we derive (2.38) – (2.40).

(4) Using simple techniques of integration, we have the following identity

$$F_1(t)=\frac{1}{4(y-x)^2}\int_\Omega\int_x^{\frac{x+y}{2}}\left[f\left(ts+(1-t)u\right)+f\left(ts+(1-t)(x+y-u)\right)\right.$$
$$\left.+f\left(ts+(1-t)\left(x'-y+u\right)\right)+f\left(ts+(1-t)\left(y+y'-u\right)\right)\right]duds$$

for all $t\in[0,1]$.

By Lemma 2, the following inequalities hold for all $t\in[0,1]$, $s\in\Omega$ and $u\in\left[x,\frac{x+y}{2}\right]$:

$$f\left(ts+(1-t)u\right)+f\left(ts+(1-t)(x+y-u)\right)\leq f\left(ts+(1-t)x\right)+f\left(ts+(1-t)y\right)$$

and

$$f\left(ts+(1-t)\left(x'-y+u\right)\right)+f\left(ts+(1-t)\left(y+y'-u\right)\right)$$
$$\leq f\left(ts+(1-t)y'\right)+f\left(ts+(1-t)x'\right).$$

Integrating the above inequalities over s on Ω, over u on $\left[x,\frac{x+y}{2}\right]$, dividing both sides by $4(y-x)^2$ and the above identity and the definitions of H_1,H_2 and L_1, we obtain

$$F_1(t)\leq\frac{1}{2}\left[L_1(1-t)+\frac{H_1(t)+H_2(t)}{2}\right] \tag{2.43}$$

for all $t\in[0,1]$. Using (2.39) and (2.43), we derive (2.41).

This completes the proof. □

Remark 5. *Let $x=a$, $y=y'=\frac{a+b}{2}$ and $x'=b$ in Theorem 5. Then $H(t)=H_1(t)=H_2(t)$, $F(t)=F_1(t)$, $P(t)=P_1(t)$, $G(t)=G_1(t)=G_2(t)$, $L(t)=L_1(t)$ $(t\in[0,1])$ and we have the following results:*

1. *Theorem 5 and the inequality* (2.12) *reduce to Theorem F and* (2.36), (2.39) – (2.40) *refine* (1.11) – (1.13)*, respectively.*
2. *Theorem 2 and the inequality* (2.36) *reduce to Theorem G.*

(3) *The inequality* (2.41) *reduces to Theorem H.*

The following corollary is a natural consequence of Theorems 1 – 5.

Corollary 1. *The following inequalities hold for all* $t \in [0,1]$

$$\frac{f(y)+f(y')}{2} \le H_1(t) \le \frac{1}{2(y-x)} \int_\Omega f(s)\,ds \le P_1(t) \le \frac{f(x)+f(x')}{2}; \tag{2.44}$$

$$\frac{1}{2}\left[f\left(\frac{y+y'}{2}\right) + \frac{f(y)+f(y')}{2}\right] \le \frac{H_1(t)+H_2(t)}{2} \le F_1(t); \tag{2.45}$$

$$F_1(t) \le \frac{1}{2(y-x)} \int_\Omega f(s)\,ds \ \le P_1(t) \le \frac{f(x)+f(x')}{2}; \tag{2.46}$$

$$\frac{1}{2}\left[f\left(\frac{y+y'}{2}\right) + \frac{f(y)+f(y')}{2}\right] \le \frac{H_1(1-t)+H_2(1-t)}{2} \le F_1(t); \tag{2.47}$$

$$F_1(t) \le \frac{1}{2}\left[\frac{H_1(1-t)+H_2(1-t)}{2} + L_1(t)\right] \le L_1(t) \le P_1(t) \le \frac{f(x)+f(x')}{2}; \tag{2.48}$$

$$\frac{1}{2}\left[f\left(\frac{y+y'}{2}\right) + \frac{f(y)+f(y')}{2}\right] \le \frac{G_1(t)+G_2(t)}{2} \le L_1(t); \tag{2.49}$$

$$\frac{f(y)+f(y')}{2} \le H_1(t) \le \frac{1}{2}\left[\frac{f(y)+f(y')}{2} + G_1(t)\right] \le G_1(t) \tag{2.50}$$

and

$$G_1(t) \le P_1(t) \le \frac{1}{2}\left[G_1(t) + \frac{f(x)+f(x')}{2}\right] \le \frac{f(x)+f(x')}{2}. \tag{2.51}$$

3. Applications

Using Corollary 1, we have the following propositions:

Proposition 1. *Let* $0 < a < b < \infty$, $p \in \mathbb{R}\setminus\{0,-1,-2\}$, $t \in (0,1)$ *and let* x, y, y', x' *be defined as above. Define*

$$H_1 = \frac{y^{p+1} - (tx+(1-t)y)^{p+1} + (tx'+(1-t)y')^{p+1} - y'^{p+1}}{2t(p+1)(y-x)},$$

$$H_1' = \frac{y^{p+1} - ((1-t)x+ty)^{p+1} + ((1-t)x'+ty')^{p+1} - y'^{p+1}}{2(1-t)(p+1)(y-x)},$$

$$H_2 = \frac{(ty+(1-t)y')^{p+1} - (tx+(1-t)y')^{p+1}}{2t(p+1)(y-x)} + \frac{(tx'+(1-t)y)^{p+1} - (ty'+(1-t)y)^{p+1}}{2t(p+1)(y-x)},$$

$$H_2' = \frac{((1-t)\,y + ty')^{p+1} - ((1-t)\,x + ty')^{p+1}}{2\,(1-t)\,(p+1)\,(y-x)} + \frac{((1-t)\,x' + ty)^{p+1} - ((1-t)\,y' + ty)^{p+1}}{2\,(1-t)\,(p+1)\,(y-x)},$$

$$P_1 = \frac{(tx + (1-t)\,y)^{p+1} - x^{p+1} + x'^{p+1} - (tx' + (1-t)\,y')^{p+1}}{2\,(1-t)\,(p+1)\,(y-x)},$$

$$\begin{aligned} F_1 = {} & \frac{y^{p+2} - (ty + (1-t)\,x)^{p+2} + (ty + (1-t)\,x')^{p+2} - (ty + (1-t)\,y')^{p+2}}{4t\,(1-t)\,(p+1)\,(p+2)\,(y-x)^2} \\ & - \frac{(tx + (1-t)\,y)^{p+2} - x^{p+2} + (tx + (1-t)\,x')^{p+2} - (tx + (1-t)\,y')^{p+2}}{4t\,(1-t)\,(p+1)\,(p+2)\,(y-x)^2} \\ & + \frac{(tx' + (1-t)\,y)^{p+2} - (tx' + (1-t)\,x)^{p+2} + x'^{p+2} - (tx' + (1-t)\,y')^{p+2}}{4t\,(1-t)\,(p+1)\,(p+2)\,(y-x)^2} \\ & - \frac{(ty' + (1-t)\,y)^{p+2} - (ty' + (1-t)\,x)^{p+2} + (ty' + (1-t)\,x')^{p+2} - y'^{p+2}}{4t\,(1-t)\,(p+1)\,(p+2)\,(y-x)^2}, \end{aligned}$$

$$G_1 = \frac{1}{2}\left[(tx + (1-t)\,y)^p + (tx' + (1-t)\,y')^p\right],$$

$$G_2 = \frac{1}{2}\left[(tx + (1-t)\,y')^p + (tx' + (1-t)\,y)^p\right]$$

and

$$\begin{aligned} L_1 = {} & \frac{(tx + (1-t)\,y)^{p+1} - x^{p+1} + (tx + (1-t)\,x')^{p+1} - (tx + (1-t)\,y')^{p+1}}{4\,(1-t)\,(p+1)\,(y-x)} \\ & + \frac{(tx' + (1-t)\,y)^{p+1} - (tx' + (1-t)\,x)^{p+1} + x'^{p+1} - (tx' + (1-t)\,y')^{p+1}}{4\,(1-t)\,(p+1)\,(y-x)}. \end{aligned}$$

Then we have the following results:

(1) *Let* $p \in (-\infty, 0) \cup [1, \infty)$ $(p \neq -1, -2)$. *Then the following inequalities hold:*

$$\frac{y^p + y'^p}{2} \leq H_1 \leq \frac{y^{p+1} - x^{p+1} + x'^{p+1} - y'^{p+1}}{2\,(p+1)\,(y-x)} \leq P_1 \leq \frac{x^p + x'^p}{2}, \tag{3.1}$$

$$\frac{1}{2}\left[\left(\frac{y+y'}{2}\right)^p + \frac{y^p + y'^p}{2}\right] \leq \frac{H_1 + H_2}{2} \leq F_1, \tag{3.2}$$

$$F_1 \leq \frac{y^{p+1} - x^{p+1} + x'^{p+1} - y'^{p+1}}{2\,(p+1)\,(y-x)} \leq P_1 \leq \frac{x^p + x'^p}{2}, \tag{3.3}$$

$$\frac{1}{2}\left[\left(\frac{y+y'}{2}\right)^p + \frac{y^p + y'^p}{2}\right] \leq \frac{H_1' + H_2'}{2} \leq F_1, \tag{3.4}$$

$$F_1 \leq \frac{1}{2}\left[\frac{H_1' + H_2'}{2} + L_1\right] \leq L_1 \leq P_1 \leq \frac{x^p + x'^p}{2}, \tag{3.5}$$

$$\frac{1}{2}\left[\left(\frac{y+y'}{2}\right)^p + \frac{y^p + y'^p}{2}\right] \leq \frac{G_1 + G_2}{2} \leq L_1, \tag{3.6}$$

$$\frac{y^p + y'^p}{2} \le H_1 \le \frac{1}{2}\left[\frac{y^p + y'^p}{2} + G_1\right] \le G_1 \tag{3.7}$$

and

$$G_1 \le P_1 \le \frac{1}{2}\left[G_1 + \frac{x^p + x'^p}{2}\right] \le \frac{x^p + x'^p}{2}. \tag{3.8}$$

(2) *Let* $p \in (0,1)$. *Then the inequalities* $(3.1) - (3.8)$ *are reversed.*

The proof of Proposition 1 follows from Corollary 1 applied to

$$f(s) = \begin{cases} s^p & \text{as } p \in (-\infty, 0) \cup [1, \infty)\ (p \ne -1, -2) \\ -s^p & \text{as } p \in (0,1) \end{cases}$$

for all $s \in (0, \infty)$.

Proposition 2. *Let* $0 < a < b < \infty$, $t \in (0,1)$ *and let* x, y, y', x' *be defined as above. Define*

$$H_3 = \frac{\ln y\,(tx' + (1-t)\,y') - \ln y'\,(tx + (1-t)\,y)}{2t\,(y-x)},$$

$$H_3' = \frac{\ln y\,((1-t)\,x' + ty') - \ln y'\,((1-t)\,x + ty)}{2\,(1-t)\,(y-x)},$$

$$H_4 = \frac{\ln (ty + (1-t)\,y')\,(tx' + (1-t)\,y)}{2t\,(y-x)} - \frac{\ln (tx + (1-t)\,y')\,(ty' + (1-t)\,y)}{2t\,(y-x)},$$

$$H_4' = \frac{\ln ((1-t)\,y + ty')\,((1-t)\,x' + ty)}{2\,(1-t)\,(y-x)} - \frac{\ln ((1-t)\,x + ty')\,((1-t)\,y' + ty)}{2\,(1-t)\,(y-x)},$$

$$P_2 = \frac{\ln x'\,(tx + (1-t)\,y) - \ln x\,(tx' + (1-t)\,y')}{2\,(1-t)\,(y-x)},$$

$$\begin{aligned}
F_2 = {} & \frac{y\,(\ln y - 1) - (ty + (1-t)\,x)\,[\ln (ty + (1-t)\,x) - 1]}{4t\,(1-t)\,(y-x)^2} \\
& + \frac{(ty + (1-t)\,x')\,[\ln (ty + (1-t)\,x') - 1] - (ty + (1-t)\,y')\,[\ln (ty + (1-t)\,y') - 1]}{4t\,(1-t)\,(y-x)^2} \\
& - \frac{(tx + (1-t)\,y)\,[\ln (tx + (1-t)\,y) - 1] - x\,(\ln x - 1)}{4t\,(1-t)\,(y-x)^2} \\
& - \frac{(tx + (1-t)\,x')\,[\ln (tx + (1-t)\,x') - 1] - (tx + (1-t)\,y')\,[\ln (tx + (1-t)\,y') - 1]}{4t\,(1-t)\,(y-x)^2} \\
& + \frac{(tx' + (1-t)\,y)\,[\ln (tx' + (1-t)\,y) - 1] - (tx' + (1-t)\,x)\,[\ln (tx' + (1-t)\,x) - 1]}{4t\,(1-t)\,(y-x)^2} \\
& + \frac{x'\,(\ln x' - 1) - (tx' + (1-t)\,y')\,[\ln (tx' + (1-t)\,y') - 1]}{4t\,(1-t)\,(y-x)^2} \\
& - \frac{(ty' + (1-t)\,y)\,[\ln (ty' + (1-t)\,y) - 1] - (ty' + (1-t)\,x)\,[\ln (ty' + (1-t)\,x) - 1]}{4t\,(1-t)\,(y-x)^2} \\
& - \frac{(ty' + (1-t)\,x')\,[\ln (ty' + (1-t)\,x') - 1] - y'\,[\ln y' - 1]}{4t\,(1-t)\,(y-x)^2},
\end{aligned}$$

$$G_3 = \frac{y + y'}{2\,(tx + (1-t)\,y)\,(tx' + (1-t)\,y')},$$

$$G_4 = \frac{y+y'}{2\left(tx+(1-t)y'\right)\left(tx'+(1-t)y\right)}$$

and

$$L_2 = \frac{\ln\left(tx+(1-t)y\right)\left(tx+(1-t)x'\right)-\ln x\left(tx+(1-t)y'\right)}{4(1-t)(y-x)} + \frac{\ln x'\left(tx'+(1-t)y\right)-\ln\left(tx'+(1-t)x\right)\left(tx'+(1-t)y'\right)}{4(1-t)(y-x)}.$$

Then we have the following inequalities

$$\frac{y+y'}{2yy'} \le H_3 \le \frac{\ln x'y-\ln xy'}{2(y-x)} \le P_2 \le \frac{x+x'}{2xx'},$$

$$\frac{1}{2}\left[\frac{2}{y+y'}+\frac{y+y'}{2yy'}\right] \le \frac{H_3+H_4}{2} \le F_2,$$

$$F_2 \le \frac{\ln x'y-\ln xy'}{2(y-x)} \le P_2 \le \frac{x+x'}{2xx'},$$

$$\frac{1}{2}\left[\frac{2}{y+y'}+\frac{y+y'}{2yy'}\right] \le \frac{H_3'+H_4'}{2} \le F_2,$$

$$F_2 \le \frac{1}{2}\left[\frac{H_3'+H_4'}{2}+L_2\right] \le L_2 \le P_2 \le \frac{x+x'}{2xx'},$$

$$\frac{1}{2}\left[\frac{2}{y+y'}+\frac{y+y'}{2yy'}\right] \le \frac{G_3+G_4}{2} \le L_2,$$

$$\frac{y+y'}{2yy'} \le H_3 \le \frac{1}{2}\left[\frac{y+y'}{2yy'}+G_3\right] \le G_3$$

and

$$G_3 \le P_2 \le \frac{1}{2}\left[G_3+\frac{x+x'}{2xx'}\right] \le \frac{x+x'}{2xx'}.$$

The proof of Proposition 2 follows from Corollary 1 applied to $f(s) = \frac{1}{s}\ (s \in (0,\infty))$.

The interested reader may obtain various particular inqualities of interest by utilising other convex functions such as $f(t) = -\ln t$ or $f(t) = t\ln t, t > 0$. The details are omitted.

References

[1] J. Hadamard, Étude sur les propriétés des fonctions entières en particulier d'une fonction considérée par Riemann, *J. Math. Pures Appl.* 58 (1893), 171-215.

[2] S. S. Dragomir, Two Mappings in Connection to Hadamard's Inequalities, *J. Math. Anal. Appl.* 167 (1992), 49-56.

[3] S. S. Dragomir, A Refinement of Hadamard's Inequality for Isotonic Linear Functionals, *Tamkang. J. Math.*, 24 (1993), 101-106.

[4] S. S. Dragomir, On the Hadamard's Inequality for Convex on the Co-ordinates in a Rectangle from the Plane, *Taiwanese J. Math.*, 5 (4) (2001), 775-788.

[5] S. S. Dragomir, Further Proprtities of Some Mapping Associoateed with Hermite-Hadamard Inequalities, *Tamkang. J. Math.* 34 (1) (2003), 45-57.

[6] S. S. Dragomir, Y.-J. Cho and S.-S. Kim, Inequalities of Hadamard's type for Lipschitzian Mappings and Their Applications, *J. Math. Anal. Appl.* 245 (2000),489-501.

[7] S. S. Dragomir, D. S. Milošević and József Sándor, On Some Refinements of Hadamard's Inequalities and Applications, *Univ. Belgrad. Publ. Elek. Fak. Sci. Math.* 4 (1993), 3-10.

[8] L. Fejér, Über die Fourierreihen, II, *Math. Naturwiss. Anz Ungar. Akad. Wiss.* 24 (1906),369-390.(In Hungarian).

[9] D.-Y. Hwang, K.-L. Tseng and G.-S. Yang, Some Hadamard's Inequalities for Co-ordinated Convex Functions in a Rectangle from the Plane, *Taiwanese J. Math.*, 11 (1) (2007), 63-73.

[10] K.-L. Tseng, S.-R. Hwang and S. S. Dragomir, On Some New Inequalities of Hermite-Hadamard-Fejér Type Involving Convex Functions, *Demonstratio Math.* XL (1) (2007), 51-64.

[11] K.-L. Tseng, G.-S. Yang and K.-C. Hsu, On Some Inequalities of Hadamard's Type and Applications, *Taiwanese J. Math.*, 13 (6B) (2009), 1929-1948.

[12] G.-S. Yang and M.-C. Hong, A Note on Hadamard's Inequality, *Tamkang. J. Math.* 28 (1) (1997), 33-37.

[13] G.-S. Yang and K.-L. Tseng, On Certain Integral Inequalities Related to Hermite-Hadamard Inequalities, *J. Math. Anal. Appl.* 239 (1999), 180-187.

[14] G.-S. Yang and K.-L. Tseng, Inequalities of Hadamard's Type for Lipschitzian Mappings, *J. Math. Anal. Appl.* 260 (2001),230-238.

[15] G.-S. Yang and K.-L. Tseng, On Certain Multiple Integral Inequalities Related to Hermite-Hadamard Inequalities, *Utilitas Math.,* 62 (2002), 131-142.

[16] G.-S. Yang and K.-L. Tseng, Inequalities of Hermite-Hadamard-Fejér Type for Convex Functions and Lipschitzian Functions, *Taiwanese J. Math.*, 7 (3) (2003), 433-440.

(Kuei-Lin Tseng) Department of Mathematics, Aletheia University, Tamsui, Taiwan 25103

E-mail address, Kuei-Lin Tseng: `kltseng@email.au.edu.tw`

(Shiow-Ru Hwang) China University of Science and Technology, Nankang, Taipei, Taiwan 11522

E-mail address: `hsru@cc.cust.edu.tw`

(Sever S. Dragomir) School of Computer Science & Mathematics, Victoria University, Melbourne, Victoria, Australia

E-mail address, Sever S. Dragomir: `sever.dragomir@vu.edu.au`

URL: `http://rgmia.vu.edu.au/dragomir`