COMPARISON BETWEEN FUNCTIONS OF SELFADJOINT OPERATORS IN HILBERT SPACES AND INTEGRAL MEANS

S.S. DRAGOMIR

Abstract. Comparison between functions of selfadjoint operators in Hilbert spaces and integral means under suitable conditions for the functions and operators involved are given. Applications for particular instances of interest are provided as well.

1. Scalar Ostrowski's Type Inequalities

In the scalar case, comparison between functions and integral means are incorporated in Ostrowski type inequalities as mentioned below.

The first result in this direction is known in the literature as Ostrowski's inequality [29].

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with the property that $|f'(t)|\le M$ for all $t\in(a,b)$. Then*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)M \tag{1.1}$$

for all $x\in[a,b]$.
The constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller quantity.

The following Ostrowski type result for absolutely continuous functions holds (see [17] – [19]).

Theorem 2. *Let $f:[a,b]\to\mathbb{R}$ be absolutely continuous on $[a,b]$. Then, for all $x\in[a,b]$, we have:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{1.2}$$
$$\le\begin{cases}\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\|f'\|_\infty & \text{if } f'\in L_\infty[a,b];\\ \frac{1}{(p+1)^{\frac{1}{p}}}\left[\left(\frac{x-a}{b-a}\right)^{p+1}+\left(\frac{b-x}{b-a}\right)^{p+1}\right]^{\frac{1}{p}}(b-a)^{\frac{1}{p}}\|f'\|_q & \text{if } f'\in L_q[a,b],\\ & \frac{1}{p}+\frac{1}{q}=1,\ p>1;\\ \left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\|f'\|_1;\end{cases}$$

1991 *Mathematics Subject Classification.* 26D15, 47A63; 47A99.
Key words and phrases. Riemann-Stieltjes integrals, Selfadjoint operators, Functions of Selfadjoint operators, Ostrowski's type inequalities, Vector inequalities.

where $\|\cdot\|_r$ $(r \in [1,\infty])$ *are the usual Lebesgue norms on* $L_r[a,b]$, *i.e.,*

$$\|g\|_\infty := ess \sup_{t\in[a,b]} |g(t)|$$

and

$$\|g\|_r := \left(\int_a^b |g(t)|^r dt\right)^{\frac{1}{r}}, \ r \in [1,\infty).$$

The constants $\frac{1}{4}$, $\frac{1}{(p+1)^{\frac{1}{p}}}$ *and* $\frac{1}{2}$ *respectively are sharp in the sense presented in Theorem 1.*

The above inequalities can also be obtained from the Fink result in [20] on choosing $n = 1$ and performing some appropriate computations.

If one drops the condition of absolute continuity and assumes that f is Hölder continuous, then one may state the result (see for instance [15] and the references therein for earlier contributions):

Theorem 3. *Let* $f : [a,b] \to \mathbb{R}$ *be of* $r - H-$*Hölder type, i.e.,*

$$|f(x) - f(y)| \le H|x-y|^r, \text{ for all } x,y \in [a,b], \tag{1.3}$$

where $r \in (0,1]$ *and* $H > 0$ *are fixed. Then, for all* $x \in [a,b]$, *we have the inequality:*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{H}{r+1}\left[\left(\frac{b-x}{b-a}\right)^{r+1} + \left(\frac{x-a}{b-a}\right)^{r+1}\right](b-a)^r. \tag{1.4}$$

The constant $\frac{1}{r+1}$ *is also sharp in the above sense.*

Note that if $r = 1$, i.e., f is Lipschitz continuous, then we get the following version of Ostrowski's inequality for Lipschitzian functions (with L instead of H) (see for instance [8])

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)L. \tag{1.5}$$

Here the constant $\frac{1}{4}$ is also best.

Moreover, if one drops the condition of the continuity of the function, and assumes that it is of bounded variation, then the following result may be stated (see [4]).

Theorem 4. *Assume that* $f : [a,b] \to \mathbb{R}$ *is of bounded variation and denote by* $\bigvee_a^b(f)$ *its total variation. Then*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \left[\frac{1}{2} + \left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\bigvee_a^b(f) \tag{1.6}$$

for all $x \in [a,b]$.
The constant $\frac{1}{2}$ *is the best possible.*

If we assume more about f, i.e., f is monotonically increasing, then the inequality (1.6) may be improved in the following manner [7] (see also the monograph [16]).

Theorem 5. *Let $f : [a,b] \to \mathbb{R}$ be monotonic nondecreasing. Then for all $x \in [a,b]$, we have the inequality:*

$$\begin{aligned} (1.7) \qquad & \left| f(x) - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \\ & \le \frac{1}{b-a} \left\{ [2x-(a+b)]\, f(x) + \int_a^b sgn\,(t-x)\, f(t)\,dt \right\} \\ & \le \frac{1}{b-a} \left\{ (x-a)\,[f(x)-f(a)] + (b-x)\,[f(b)-f(x)] \right\} \\ & \le \left[\frac{1}{2} + \left| \frac{x-\frac{a+b}{2}}{b-a} \right| \right] [f(b)-f(a)]\,. \end{aligned}$$

All the inequalities in (1.7) are sharp and the constant $\frac{1}{2}$ is the best possible.

For other scalar Ostrowski's type inequalities, see [1]-[2] and [9].

In order to extend the above results for functions of selfadjoint operators in Hilbert spaces we need some preparations as follows.

Let U be a selfadjoint operator on the complex Hilbert space $(H, \langle .,. \rangle)$ with the spectrum $Sp\,(U)$ included in the interval $[m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its *spectral family.* Then for any continuous function $f : [m, M] \to \mathbb{R}$, it is well known that we have the following *spectral representation* in terms of the Riemann-Stieltjes integral:

$$(1.8) \qquad \langle f(U)\,x, y \rangle = \int_{m-0}^{M} f(\lambda)\, d\left(\langle E_\lambda x, y \rangle \right),$$

for any $x, y \in H$. The function $g_{x,y}(\lambda) := \langle E_\lambda x, y \rangle$ is of *bounded variation* on the interval $[m, M]$ and

$$g_{x,y}(m-0) = 0 \text{ and } g_{x,y}(M) = \langle x, y \rangle$$

for any $x, y \in H$. It is also well known that $g_x(\lambda) := \langle E_\lambda x, x \rangle$ is *monotonic nondecreasing* and *right continuous* on $[m, M]$.

2. Ostrowski's Type Vector Inequalities

The following result holds:

Theorem 6. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp\,(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f : [m, M] \to \mathbb{C}$ is a continuous function of bounded variation on $[m, M]$,* *then we have the inequality*

$$\left|\langle x,y\rangle \frac{1}{M-m}\int_m^M f(s)\,ds-\langle f(A)x,y\rangle\right| \tag{2.1}$$
$$\le \frac{1}{M-m}\bigvee_m^M(f)\max_{t\in[m,M]}\Big[(M-t)\langle E_t x,x\rangle^{1/2}\langle E_t y,y\rangle^{1/2}$$
$$+(t-m)\langle (1_H-E_t)x,x\rangle^{1/2}\langle (1_H-E_t)y,y\rangle^{1/2}\Big]$$
$$\le \|x\|\,\|y\|\bigvee_m^M(f)$$

for any $x,y\in H$.

Proof. Assume that $f:[m,M]\to\mathbb{C}$ is a continuous function on $[m,M]$. Then under the assumptions of the theorem for A and $\{E_\lambda\}_\lambda$, we have the following representation

$$\langle x,y\rangle \frac{1}{M-m}\int_m^M f(s)\,ds-\langle f(A)x,y\rangle \tag{2.2}$$
$$=\frac{1}{M-m}\int_{m-0}^M \langle[(M-t)E_t+(t-m)(E_t-1_H)]x,y\rangle\,df(t)$$

for any $x,y\in H$.

Indeed, integrating by parts in the Riemann-Stieltjes integral and using the spectral representation (1.8) we have

$$\frac{1}{M-m}\int_{m-0}^M \langle[(M-t)E_t+(t-m)(E_t-1_H)]x,y\rangle\,df(t)$$
$$=\int_{m-0}^M\left(\langle E_t x,y\rangle-\frac{t-m}{M-m}\langle x,y\rangle\right)df(t)$$
$$=\left(\langle E_t x,y\rangle-\frac{t-m}{M-m}\langle x,y\rangle\right)f(t)\Big|_{m-0}^M$$
$$-\int_{m-0}^M f(t)\,d\left(\langle E_t x,y\rangle-\frac{t-m}{M-m}\langle x,y\rangle\right)$$
$$=-\int_{m-0}^M f(t)\,d\langle E_t x,y\rangle+\langle x,y\rangle\frac{1}{M-m}\int_m^M f(t)\,dt$$
$$=\langle x,y\rangle\frac{1}{M-m}\int_m^M f(t)\,dt-\langle f(A)x,y\rangle$$

for any $x,y\in H$ and the equality (2.2) is proved.

It is well known that if $p:[a,b]\to\mathbb{C}$ is a continuous function and $v:[a,b]\to\mathbb{C}$ is of bounded variation, then the Riemann-Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and the following inequality holds

$$\left|\int_a^b p(t)\,dv(t)\right|\le\max_{t\in[a,b]}|p(t)|\bigvee_a^b(v)$$

where $\bigvee_a^b (v)$ denotes the total variation of v on $[a,b]$.

Utilising this property we have from (2.2) that

$$\left|\langle x,y\rangle \frac{1}{M-m}\int_m^M f(s)\,ds - \langle f(A)x,y\rangle\right| \leq \frac{1}{M-m}\max_{t\in[m,M]}\left|\langle[(M-t)E_t+(t-m)(E_t-1_H)]x,y\rangle\right|\bigvee_m^M (f) \tag{2.3}$$

for any $x,y\in H$.

Now observe that

$$\begin{aligned}&\left|\langle[(M-t)E_t+(t-m)(E_t-1_H)]x,y\rangle\right| \\ &= \left|(M-t)\langle E_t x,y\rangle+(t-m)\langle(E_t-1_H)x,y\rangle\right| \\ &\leq (M-t)\left|\langle E_t x,y\rangle\right|+(t-m)\left|\langle(E_t-1_H)x,y\rangle\right|\end{aligned} \tag{2.4}$$

for any $x,y\in H$ and $t\in[m,M]$.

If P is a nonnegative operator on H, i.e., $\langle Px,x\rangle\geq 0$ for any $x\in H$, then the following inequality is a generalization of the Schwarz inequality in H

$$\left|\langle Px,y\rangle\right|^2 \leq \langle Px,x\rangle\langle Py,y\rangle \tag{2.5}$$

for any $x,y\in H$.

On applying the inequality (2.5) we have

$$\begin{aligned}&(M-t)\left|\langle E_t x,y\rangle\right|+(t-m)\left|\langle(E_t-1_H)x,y\rangle\right| \\ &\leq (M-t)\langle E_t x,x\rangle^{1/2}\langle E_t y,y\rangle^{1/2} \\ &+(t-m)\langle(1_H-E_t)x,x\rangle^{1/2}\langle(1_H-E_t)y,y\rangle^{1/2} \\ &\leq \max\{M-t,t-m\} \\ &\times\left[\langle E_t x,x\rangle^{1/2}\langle E_t y,y\rangle^{1/2}+\langle(1_H-E_t)x,x\rangle^{1/2}\langle(1_H-E_t)y,y\rangle^{1/2}\right] \\ &\leq \max\{M-t,t-m\} \\ &\times\left[\langle E_t x,x\rangle+\langle(1_H-E_t)x,x\rangle\right]^{1/2}\left[\langle E_t y,y\rangle+\langle(1_H-E_t)y,y\rangle\right]^{1/2} \\ &= \max\{M-t,t-m\}\|x\|\|y\|,\end{aligned} \tag{2.6}$$

where for the last inequality we used the elementary fact

$$a_1b_1+a_2b_2 \leq \left(a_1^2+a_2^2\right)^{1/2}\left(b_1^2+b_2^2\right)^{1/2} \tag{2.7}$$

that holds for a_1,b_1,a_2,b_2 positive real numbers.

Utilising the inequalities (2.3), (2.4) and (2.6) we deduce the desired result (2.1). □

The case of Lipschitzian functions is embodied in the following result:

Theorem 7. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A)\subseteq[m,M]$ for some real numbers $m<M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f:[m,M]\to\mathbb{C}$ is a Lipschitzian function with the constant $L>0$ on*

$[m,M]$, then we have the inequality

$$\left|\langle x,y\rangle \frac{1}{M-m}\int_m^M f(s)\,ds - \langle f(A)x,y\rangle\right| \tag{2.8}$$
$$\leq \frac{L}{M-m}\int_{m-0}^{M}\Big[(M-t)\langle E_t x,x\rangle^{1/2}\langle E_t y,y\rangle^{1/2}$$
$$+(t-m)\langle(1_H-E_t)x,x\rangle^{1/2}\langle(1_H-E_t)y,y\rangle^{1/2}\Big]\,dt$$
$$\leq \frac{3}{4}L(M-m)\|x\|\|y\|$$

for any $x,y\in H$.

Proof. It is well known that if $p:[a,b]\to\mathbb{C}$ is a Riemann integrable function and $v:[a,b]\to\mathbb{C}$ is Lipschitzian with the constant $L>0$, i.e.,

$$|f(s)-f(t)|\leq L|s-t| \text{ for any } t,s\in[a,b],$$

then the Riemann-Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and the following inequality holds

$$\left|\int_a^b p(t)\,dv(t)\right|\leq L\int_a^b |p(t)|\,dt.$$

Now, on applying this property of the Riemann-Stieltjes integral, we have from the representation (2.2) that

$$\left|\langle x,y\rangle \frac{1}{M-m}\int_m^M f(s)\,ds - \langle f(A)x,y\rangle\right| \tag{2.9}$$
$$\leq \frac{L}{M-m}\int_{m-0}^{M}|\langle[(M-t)E_t+(t-m)(E_t-1_H)]x,y\rangle|\,dt.$$

Since, from the proof of Theorem 6, we have

$$|\langle[(M-t)E_t+(t-m)(E_t-1_H)]x,y\rangle| \tag{2.10}$$
$$\leq (M-t)\langle E_t x,x\rangle^{1/2}\langle E_t y,y\rangle^{1/2}$$
$$+(t-m)\langle(1_H-E_t)x,x\rangle^{1/2}\langle(1_H-E_t)y,y\rangle^{1/2}$$
$$\leq \max\{M-t,t-m\}\|x\|\|y\|$$
$$=\left[\frac{1}{2}(M-m)+\left|t-\frac{m+M}{2}\right|\right]\|x\|\|y\|$$

for any $x,y\in H$ and $t\in[m,M]$, then integrating (2.10) and taking into account that

$$\int_m^M\left|t-\frac{m+M}{2}\right|dt=\frac{1}{4}(M-m)^2$$

we deduce the desired result (2.8). □

Finally for the section, we provide here the case of monotonic nondecreasing functions as well:

Theorem 8. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f : [m, M] \to \mathbb{R}$ is a continuous monotonic nondecreasing function on $[m, M]$, then we have the inequality*

$$\begin{aligned} &\left| \langle x, y\rangle \frac{1}{M-m} \int_m^M f(s)\, ds - \langle f(A) x, y\rangle \right| \\ &\le \frac{1}{M-m} \int_{m-0}^M \Big[(M-t) \langle E_t x, x\rangle^{1/2} \langle E_t y, y\rangle^{1/2} \\ &+ (t-m) \langle (1_H - E_t) x, x\rangle^{1/2} \langle (1_H - E_t) y, y\rangle^{1/2} \Big] df(t) \\ &\le \left[f(M) - f(m) - \frac{1}{M-m} \int_m^M sgn\left(t - \frac{m+M}{2}\right) f(t)\, dt \right] \|x\| \|y\| \\ &\le [f(M) - f(m)] \|x\| \|y\| \end{aligned} \tag{2.11}$$

for any $x, y \in H$.

Proof. From the theory of Riemann-Stieltjes integral is well known that if $p : [a, b] \to \mathbb{C}$ is of bounded variation and $v : [a, b] \to \mathbb{R}$ is continuous and monotonic nondecreasing, then the Riemann-Stieltjes integrals $\int_a^b p(t)\, dv(t)$ and $\int_a^b |p(t)|\, dv(t)$ exist and

$$\left| \int_a^b p(t)\, dv(t) \right| \le \int_a^b |p(t)|\, dv(t).$$

Now, on applying this property of the Riemann-Stieltjes integral, we have from the representation (2.2) that

$$\begin{aligned} &\left| \langle x, y\rangle \frac{1}{M-m} \int_m^M f(s)\, ds - \langle f(A) x, y\rangle \right| \\ &\le \frac{1}{M-m} \int_{m-0}^M |\langle [(M-t) E_t + (t-m)(E_t - 1_H)] x, y\rangle|\, df(t). \end{aligned} \tag{2.12}$$

Further on, by utilizing the inequality (2.10) we also have that

$$\begin{aligned} &\int_{m-0}^M |\langle [(M-t) E_t + (t-m)(E_t - 1_H)] x, y\rangle|\, df(t) \\ &\le \int_{m-0}^M \Big[(M-t) \langle E_t x, x\rangle^{1/2} \langle E_t y, y\rangle^{1/2} \\ &+ (t-m) \langle (1_H - E_t) x, x\rangle^{1/2} \langle (1_H - E_t) y, y\rangle^{1/2} \Big] df(t) \\ &\le \left[\frac{1}{2} (M-m) [f(M) - f(m)] + \int_m^M \left| t - \frac{m+M}{2} \right| df(t) \right] \|x\| \|y\|. \end{aligned} \tag{2.13}$$

Now, integrating by parts in the Riemann-Stieltjes integral we have

$$\begin{aligned}
&\int_m^M \left|t-\frac{m+M}{2}\right| df(t) \\
&= \int_m^{\frac{M+m}{2}} \left(\frac{m+M}{2}-t\right) df(t) + \int_{\frac{m+M}{2}}^M \left(t-\frac{m+M}{2}\right) df(t) \\
&= \left(\frac{m+M}{2}-t\right) f(t)\Big|_m^{\frac{M+m}{2}} + \int_m^{\frac{M+m}{2}} f(t)\,dt \\
&+ \left(t-\frac{m+M}{2}\right) f(t)\Big|_{\frac{m+M}{2}}^M - \int_{\frac{m+M}{2}}^M f(t)\,dt \\
&= \frac{1}{2}(M-m)\left[f(M)-f(m)\right] - \int_m^M sgn\left(t-\frac{m+M}{2}\right) f(t)\,dt,
\end{aligned}$$

which together with (2.13) produces the second inequality in (2.11).

Since the functions $sgn\left(\cdot-\frac{m+M}{2}\right)$ and $f(\cdot)$ have the same monotonicity, then by the Čebyšev inequality we have

$$\begin{aligned}
&\int_m^M sgn\left(t-\frac{m+M}{2}\right) f(t)\,dt \\
&\geq \frac{1}{M-m}\int_m^M sgn\left(t-\frac{m+M}{2}\right) dt \int_m^M f(t)\,dt = 0
\end{aligned}$$

and the last part of (2.11) is proved. □

3. Applications for Particular Functions

It is obvious that the above results can be applied for various particular functions. However, we will restrict here only to the power and logarithmic functions.

1. Consider now the power function $f:(0,\infty)\to\mathbb{R}$, $f(t)=t^p$ with $p>0$. This function is monotonic increasing on $(0,\infty)$ and applying Theorem 8 we can state the following proposition:

Proposition 1. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A)\subseteq[m,M]$ for some real numbers $0<m<M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. Then for any $x,y\in H$ we have the inequalities*

$$\begin{aligned}
&\left|\langle A^p x,y\rangle - \frac{M^{p+1}-m^{p+1}}{(p+1)(M-m)}\langle x,y\rangle\right| \\
&\leq \frac{p}{M-m}\int_{m-0}^M \Big[(M-t)\langle E_t x,x\rangle^{1/2}\langle E_t y,y\rangle^{1/2} \\
&+(t-m)\langle(1_H-E_t)x,x\rangle^{1/2}\langle(1_H-E_t)y,y\rangle^{1/2}\Big] t^{p-1}dt \\
&\leq \left[M^p-m^p-\frac{M^{p+1}+m^{p+1}-2^p(M+m)^{p+1}}{(p+1)(M-m)}\right]\|x\|\,\|y\|\,.
\end{aligned}\tag{3.1}$$

On applying now Theorem 7 to the same power function, then we can state the following result as well:

Proposition 2. *With the same assumptions from Proposition 1 we have*

$$\begin{aligned} (3.2) \qquad & \left| \langle A^p x, y\rangle - \frac{M^{p+1} - m^{p+1}}{(p+1)(M-m)} \langle x, y\rangle \right| \\ & \leq \frac{B_p}{M-m} \int_{m-0}^{M} \Big[(M-t) \langle E_t x, x\rangle^{1/2} \langle E_t y, y\rangle^{1/2} \\ & + (t-m) \langle (1_H - E_t) x, x\rangle^{1/2} \langle (1_H - E_t) y, y\rangle^{1/2} \Big] dt \\ & \leq \frac{3}{4} B_p (M-m) \|x\| \|y\| \end{aligned}$$

for any $x, y \in H$, where

$$B_p = p \times \begin{cases} M^{p-1} & \text{if } p \geq 1 \\ \\ m^{p-1} & \text{if } 0 < p < 1, m > 0. \end{cases}$$

The case of negative powers except $p = -1$ goes likewise and we omit the details.

Now, if we apply Theorem 8 and 7 for the increasing function $f(t) = -\frac{1}{t}$ with $t > 0$, then we can state the following proposition:

Proposition 3. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $0 < m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. Then for any $x, y \in H$ we have the inequalities*

$$\begin{aligned} (3.3) \qquad & \left| \langle A^{-1} x, y\rangle - \frac{\ln M - \ln m}{M-m} \langle x, y\rangle \right| \\ & \leq \frac{1}{M-m} \int_{m-0}^{M} \Big[(M-t) \langle E_t x, x\rangle^{1/2} \langle E_t y, y\rangle^{1/2} \\ & + (t-m) \langle (1_H - E_t) x, x\rangle^{1/2} \langle (1_H - E_t) y, y\rangle^{1/2} \Big] t^2 dt \\ & \leq \left[\frac{M-m}{mM} - \frac{\ln \left[\left(\frac{m+M}{2}\right)^2 \right] - \ln(mM)}{M-m} \right] \|x\| \|y\| \end{aligned}$$

and

$$\begin{aligned} (3.4) \qquad & \left| \langle A^{-1} x, y\rangle - \frac{\ln M - \ln m}{M-m} \langle x, y\rangle \right| \\ & \leq \frac{1}{m^2 (M-m)} \int_{m-0}^{M} \Big[(M-t) \langle E_t x, x\rangle^{1/2} \langle E_t y, y\rangle^{1/2} \\ & + (t-m) \langle (1_H - E_t) x, x\rangle^{1/2} \langle (1_H - E_t) y, y\rangle^{1/2} \Big] dt \\ & \leq \frac{3}{4} \frac{M-m}{m^2} \|x\| \|y\| . \end{aligned}$$

2. Now, if we apply Theorems 8 and 7 to the function $f : (0, \infty) \to \mathbb{R}$, $f(t) = \ln t$, then we can state

Proposition 4. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $0 < m < M$ and let $\{E_\lambda\}_\lambda$ be its* *spectral family. Then for any $x, y \in H$ we have the inequalities*

$$\begin{aligned} (3.5) \qquad & |\langle \ln Ax, y\rangle - \langle x, y\rangle \ln I(m, M)| \\ & \leq \frac{1}{M-m} \int_{m-0}^{M} \Big[(M-t) \langle E_t x, x\rangle^{1/2} \langle E_t y, y\rangle^{1/2} \\ & + (t-m) \langle (1_H - E_t) x, x\rangle^{1/2} \langle (1_H - E_t) y, y\rangle^{1/2} \Big] t dt \\ & \leq \left[\ln\left(\frac{M}{m}\right) - \ln\left(\sqrt{\frac{I\left(\frac{m+M}{2}, M\right)}{I\left(m, \frac{m+M}{2}\right)}}\right) \right] \|x\| \|y\| \end{aligned}$$

and

$$\begin{aligned} (3.6) \qquad & |\langle \ln Ax, y\rangle - \langle x, y\rangle \ln I(m, M)| \\ & \leq \frac{1}{m(M-m)} \int_{m-0}^{M} \Big[(M-t) \langle E_t x, x\rangle^{1/2} \langle E_t y, y\rangle^{1/2} \\ & + (t-m) \langle (1_H - E_t) x, x\rangle^{1/2} \langle (1_H - E_t) y, y\rangle^{1/2} \Big] dt \\ & \leq \frac{3}{4} \left(\frac{M}{m} - 1\right) \|x\| \|y\|, \end{aligned}$$

where $I(m, M)$ is the identric mean of m and M and is defined by

$$I(m, M) = \frac{1}{e} \left(\frac{M^M}{m^m}\right)^{1/(M-m)}.$$

References

[1] G.A. Anastassiou, Univariate Ostrowski inequalities, revisited. *Monatsh. Math.,* **135** (2002), no. 3, 175–189.

[2] P. Cerone and S.S. Dragomir, New bounds for the three-point rule involving the Riemann-Stieltjes integrals, in *Advances in Statistics Combinatorics and Related Areas*, C. Gulati, et al. (Eds.), World Science Publishing, 2002, 53-62.

[3] S.S. Dragomir, An Ostrowski like inequality for convex functions and applications, *Revista Math. Complutense*, **16**(2) (2003), 373-382.

[4] S.S. Dragomir, On the Ostrowski's inequality for mappings of bounded variation and applications, *Math. Ineq. & Appl.,* **4**(1) (2001), 33-40.

[5] S.S. Dragomir, On the Ostrowski's inequality for Riemann-Stieltjes integral, *Korean J. Appl. Math.,* **7** (2000), 477-485.

[6] S.S. Dragomir, On the Ostrowski inequality for Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ where f is of Hölder type and u is of bounded variation and applications, *J. KSIAM,* **5**(1) (2001), 35-45.

[7] S.S. Dragomir, Ostrowski's inequality for monotonous mappings and applications, *J. KSIAM,* **3**(1) (1999), 127-135.

[8] S.S. Dragomir, The Ostrowski's integral inequality for Lipschitzian mappings and applications, *Comp. and Math. with Appl.,* **38** (1999), 33-37.

[9] S.S. Dragomir, Ostrowski type inequalities for isotonic linear functionals, *J. Inequal. Pure & Appl. Math.,* **3**(5) (2002), Art. 68. [ONLINE `http://jipam.vu.edu.au/article.php?sid=220`].

[10] S.S. Dragomir, Čebyšev's type inequalities for functions of selfadjoint operators in Hilbert spaces, Preprint *RGMIA Res. Rep. Coll.*, **11**(e) (2008), Art. 9. [ONLINE: http://www.staff.vu.edu.au/RGMIA/v11(E).asp]

[11] S.S. Dragomir, Grüss' type inequalities for functions of selfadjoint operators in Hilbert spaces, Preprint *RGMIA Res. Rep. Coll.*, **11**(e) (2008), Art. 11. [ONLINE: http://www.staff.vu.edu.au/RGMIA/v11(E).asp]

[12] S.S. Dragomir, Inequalities for the Čebyšev functional of two functions of selfadjoint operators in Hilbert spaces, Preprint *RGMIA Res. Rep. Coll.*, **11**(e) (2008), Art. . [ONLINE: http://www.staff.vu.edu.au/RGMIA/v11(E).asp]
[13] S.S. Dragomir, Ostrowski's type inequalities for some classes of continuous functions of selfadjoint operators in Hilbert spaces, Preprint *RGMIA Res. Rep. Coll. (to appear).*
[14] S.S. Dragomir, Some Ostrowski's type vector inequalities for functions of selfadjoint operators in Hilbert spaces, Preprint *RGMIA Res. Rep. Coll. (to appear).*
[15] S.S. Dragomir, P. Cerone, J. Roumeliotis and S. Wang, A weighted version of Ostrowski inequality for mappings of Hölder type and applications in numerical analysis, *Bull. Math. Soc. Sci. Math. Romanic,* **42(90)** (4) (1999), 301-314.
[16] S.S. Dragomir and Th. M. Rassias (Eds), Ostrowski Type Inequalities and Applications in Numerical Integration, Kluwer Academic Publisher, 2002.
[17] S.S. Dragomir and S. Wang, A new inequality of Ostrowski's type in L_1-norm and applications to some special means and to some numerical quadrature rules, *Tamkang J. of Math.*, **28** (1997), 239-244.
[18] S.S. Dragomir and S. Wang, Applications of Ostrowski's inequality to the estimation of error bounds for some special means and some numerical quadrature rules, *Appl. Math. Lett.*, **11** (1998), 105-109.
[19] S.S. Dragomir and S. Wang, A new inequality of Ostrowski's type in L_p-norm and applications to some special means and to some numerical quadrature rules, *Indian J. of Math.,* **40**(3) (1998), 245-304.
[20] A.M. Fink, Bounds on the deviation of a function from its averages, *Czechoslovak Math. J.,* **42**(117) (1992), No. 2, 298-310.
[21] T. Furuta, J. Mićić Hot, J. Pečarić and Y. Seo, *Mond-Pečarić Method in Operator Inequalities. Inequalities for Bounded Selfadjoint Operators on a Hilbert Space*, Element, Zagreb, 2005.
[22] P. Kumar, The Ostrowski type moments integral inequalities and moment bounds for continuous random variables, *Comp. and Math. with Appl.,* **49**(11-12) (2005), 1929-1940.
[23] Z. Liu, Refinement of an inequality of Grüss type for Riemann-Stieltjes integral, *Soochow J. Math.,* **30**(4) (2004), 483-489.
[24] Z. Liu, Refinement of an inequality of Grüss type for Riemann-Stieltjes integral, *Soochow J. Math.,* **30**(4) (2004), 483-489.
[25] A. Matković, J. Pečarić and I. Perić, A variant of Jensen's inequality of Mercer's type for operators with applications. *Linear Algebra Appl.* **418** (2006), No. 2-3, 551–564.
[26] C.A. McCarthy, c_p, *Israel J. Math.*, **5**(1967), 249-271.
[27] B. Mond and J. Pečarić, Convex inequalities in Hilbert spaces, *Houston J. Math.*, **19**(1993), 405-420.
[28] B. Mond and J. Pečarić, Classical inequalities for matrix functions, *Utilitas Math.*, **46**(1994), 155-166.
[29] A. Ostrowski, Über die Absolutabweichung einer differentienbaren Funktionen von ihren Integralmittelwert, *Comment. Math. Hel,* **10** (1938), 226-227.
[30] J. Pečarić, J. Mićić and Y. Seo, Inequalities between operator means based on the Mond-Pečarić method. *Houston J. Math.* **30** (2004), no. 1, 191–207.

Mathematics, School of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.
E-mail address: sever.dragomir@vu.edu.au
URL: http://www.staff.vu.edu.au/rgmia/dragomir/