SCHWARZ AND GRÜSS TYPE INEQUALITIES FOR C*-SEMINORMS AND POSITIVE LINEAR FUNCTIONALS ON BANACH $*$-MODULES

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Abstract. Let $\mathcal{A}$ be a unital Banach $*$-algebra, γ a C^*-seminorm or a positive linear functional on $\mathcal{A}$ and X be a semi-inner product $\mathcal{A}$-module. We define a real function Γ on X by $\Gamma(x) = (\gamma(\langle x, x\rangle))^{1/2}$ and show that the Schwarz inequality holds, therefore (X, Γ) is a semi-Hilbert $\mathcal{A}$-module. We also obtain some Grüss type inequalities for C^*-seminorms and positive linear functionals on $\mathcal{A}$.

1. Introduction

In 1934, G. Grüss [6] showed that for two Lebesgue integrable functions $f, g : [a, b] \to \mathbb{R}$,

$$\left|\frac{1}{b-a}\int_a^b f(t)g(t)dt - \frac{1}{b-a}\int_a^b f(t)dt\frac{1}{b-a}\int_a^b g(t)dt\right| \leq \frac{1}{4}(M-m)(N-n),$$

provided m, M, n, N are real numbers with the property $-\infty < m \leq f \leq M < \infty$ and $-\infty < n \leq f \leq N < \infty$ a.e. on $[a, b]$. The constant $\frac{1}{4}$ is best possible in the sense that it cannot be replaced by a smaller constant.

The following inequality of Grüss type in real or complex inner product spaces is known [3].

Theorem 1. *Let $(H; \langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{K}$ $(\mathbb{K} = \mathbb{C}, \mathbb{R})$ and $e \in H, \|e\| = 1$. If $\alpha, \beta, \lambda, \mu \in \mathbb{K}$ and $x, y \in H$ are such that conditions*

$$\mathrm{Re}\langle \alpha e - x, x - \beta e\rangle \geq 0, \quad \mathrm{Re}\langle \lambda e - y, y - \mu e\rangle \geq 0$$

or, equivalently,

$$\left\|x - \frac{\alpha+\beta}{2}e\right\| \leq \frac{1}{2}|\alpha - \beta|, \quad \left\|y - \frac{\lambda+\mu}{2}e\right\| \leq \frac{1}{2}|\lambda - \mu|$$

hold, then we have the inequality

$$|\langle x, y\rangle - \langle x, e\rangle\langle e, y\rangle| \leq \frac{1}{4}|\alpha - \beta||\lambda - \mu|. \tag{1.1}$$

The constant $\frac{1}{4}$ is best possible in (1.1).

Dragomir in [5] presented refinements of the Grüss type inequality (1.1) and some companions and applications. Iliśević and Varošaneć in [7] have proved a refinement of a Grüss type inequality in proper H^*-modules and C^*-modules.

2000 *Mathematics Subject Classification.* Primary 46L08, 46H25; Secondary 26D99, 46C99.

Key words and phrases. Inner product C^*-modules, inner product $*$-modules, Schwarz inequality, Grüss inequality, C^*-seminorms.

In this paper we obtain a version for the Schwarz inequality and provide inequalities of Grüss type in inner product Banach $*$-modules.

2. Preliminaries

Let $\mathcal{A}$ be a $*$-algebra. A seminorm γ on $\mathcal{A}$ is a real-valued function on $\mathcal{A}$ such that for $a, b \in \mathcal{A}$ and $\lambda \in \mathbb{C}$: $\gamma(a) \geq 0, \quad \gamma(\lambda a) = |\lambda|\gamma(a), \quad \gamma(a+b) \leq \gamma(a) + \gamma(b)$. A seminorm γ on $\mathcal{A}$ is called a C^*-seminorm if it satisfies the C^*-condition: $\gamma(a^*a) = (\gamma(a))^2 \quad (a \in \mathcal{A})$. By Sebestyen's theorem [2, Theorem 38.1] every C^*-seminorm γ on a $*$-algebra $\mathcal{A}$ is submultiplicative, i.e., $\gamma(ab) \leq \gamma(a)\gamma(b) \quad (a, b \in \mathcal{A})$, and by [1, Section 39, Lemma 2 (i)] $\gamma(a) = \gamma(a^*)$. For every $a \in \mathcal{A}$, the spectral radius of a is defined to be $r(a) = \sup\{|\lambda| : \lambda \in \sigma_{\mathcal{A}}(a)\}$.

The Pták function ρ on $*$-algebra $\mathcal{A}$ is defined to be $\rho : \mathcal{A} \to [0, \infty)$, where $\rho(a) = (r(a^*a))^{1/2}$. This function has important roles in Banach $*$-algebras, for example, on C^*-algebras, ρ is equal to the norm and on hermitian Banach $*$-algebras ρ is the greatest C^*-seminorm. By utilizing properties of the spectral radius and the Pták function, V. Pták [9] showed in 1970 that an elegant theory for Banach $*$-algebras arises from the inequality $r(a) \leq \rho(a)$.

This inequality characterizes hermitian (and symmetric) Banach $*$-algebras, and further characterizations of C^*-algebras follow as a result of Pták theory.

Let $\mathcal{A}$ be a $*$-algebra. We define $\mathcal{A}^+$ by

$$\mathcal{A}^+ = \left\{ \sum_{k=1}^{n} a_k^* a_k : n \in \mathbb{N}, a_k \in \mathcal{A} \text{ for } k = 1, 2, ..., n \right\},$$

and call the elements of $\mathcal{A}^*$ positive.

The set $\mathcal{A}^+$ of positive elements is obviously a convex cone (i.e., it is closed under convex combinations and multiplication by positive constants). Hence we call $\mathcal{A}^+$ the positive cone. By definition, zero belongs to $\mathcal{A}^+$. It is also clear that each positive element is hermitian.

Definition 1. *Let $\mathcal{A}$ be a $*$-algebra. A semi-inner product $\mathcal{A}$-module (or semi-inner product $*$-module) is a complex vector space which is also a right $\mathcal{A}$-module X with a sesquilinear semi-inner product $\langle \cdot, \cdot \rangle : X \times X \to \mathcal{A}$, fulfilling*

$$\langle x, ya \rangle = \langle x, y \rangle a \quad \text{for } x,\ y \in X,\ a \in \mathcal{A}, \qquad \text{(right linearity)}$$

$$\langle x, x \rangle \in \mathcal{A}^+ \quad \text{for } x \in X. \qquad \text{(positivity)}$$

Furthermore, if X satisfies the strict positivity condition

$$x = 0 \quad \text{if } \langle x, x \rangle = 0, \qquad \text{(strict positivity)}$$

then X is called an inner product $\mathcal{A}$-module (or inner product $*$-module).

Let γ be a seminorm or a positive linear functional on $\mathcal{A}$ and $\Gamma(x) = (\gamma(\langle x, x \rangle))^{1/2}$ $(x \in X)$. If Γ is a seminorm on a semi-inner product $\mathcal{A}$-module X, then (X, Γ) is said to be a semi-Hilbert $\mathcal{A}$-module.

If Γ is a norm on an inner product $\mathcal{A}$-module X, then (X, Γ) is said to be a pre-Hilbert $\mathcal{A}$-module.

A pre-Hilbert $\mathcal{A}$-module which is complete with respect to its norm is called a Hilbert $\mathcal{A}$-module.

3. Schwarz Inequality

Let φ be a positive linear functional on a $*$-algebra $\mathcal{A}$ and X be a semi-inner $\mathcal{A}$-module. We can define a sesquilinear form on $X \times X$ by $\sigma(x,y) = \varphi(\langle x,y\rangle)$; the Schwarz inequality for σ implies that

$$|\varphi(\langle x,y\rangle)|^2 \le \varphi(\langle x,x\rangle)\,\varphi(\langle y,y\rangle). \tag{3.1}$$

Therefore $\Phi(x) = (\phi(\langle x,x\rangle))^{1/2}$ is a seminorm on X and (X,Φ) is a semi-Hilbert $\mathcal{A}$-module.

Proposition 1. *Let $\mathcal{A}$ be a $*$-algebra and X be a semi-inner product $\mathcal{A}$-module. If γ is a C^*-seminorm on $\mathcal{A}$ and $\Gamma(x) = (\gamma(\langle x,x\rangle))^{1/2} \quad (x \in X)$ then the Schwarz inequality holds, that is*

$$(\gamma(\langle x,y\rangle))^2 \le \gamma(\langle x,x\rangle)\,\gamma(\langle y,y\rangle). \tag{3.2}$$

Therefore $\Gamma(x+y) \le \Gamma(x) + \Gamma(y)$ for every $x, y \in X$. Thus Γ is a seminorm on X and (X,Γ) is a semi-Hilbert $\mathcal{A}$-module.

Proof. First we show that γ has the following monotone property:

$$\gamma(a) \le \gamma(b) \quad \text{if} \quad 0 \le a \le b \quad (a, b \in \mathcal{A}).$$

Let J_γ be the ideal defined by

$$J_\gamma = \{a \in \mathcal{A} : \gamma(a) = 0\}$$

and $\tilde{\gamma}$ the algebra norm defined on the algebra $\mathcal{A}/J_\gamma$ by

$$\tilde{\gamma}(b) = \gamma(a) \qquad (a \in b \in \mathcal{A}/J_\gamma).$$

We denote by B_γ the completion of $\mathcal{A}/J_\gamma$ with respect to the norm $\tilde{\gamma}$ and we denote also by $\tilde{\gamma}$ the usual extension of this norm to B_γ. Suppose that $a, b \in \mathcal{A}$ and $0 \le a \le b$, then $[a] = a + J_\gamma \le b + J_\gamma = [b]$ in $\mathcal{A}/J_\gamma$. Since $\tilde{\gamma}$ is a C^*-norm on B_γ therefore $\tilde{\gamma}([a]) \le \tilde{\gamma}([b])$ [8, Theorem 2.2.5 (3)] consequently $\gamma(a) \le \gamma(b)$.

The rest of the proof is similar to the proof of Lemma 15.1.3 in [10]; we include it for the sake of completion:

If $\gamma(\langle x,y\rangle) = 0$, this is trivial. Suppose that $\gamma(\langle x,y\rangle) \ne 0$ put $a := \langle x,x\rangle$, $b := \langle y,y\rangle$, $c := \langle x,y\rangle$ and let λ be an arbitrary real number. Then

$$0 \le \langle x - y\lambda c^*, x - y\lambda c^*\rangle = a - 2\lambda cc^* + \lambda^2 cbc^*. \tag{3.3}$$

Adding the self-adjoint element $2\lambda cc^*$ at both sides and taking seminorms, we obtain

$$2|\lambda|\gamma(c)^2 \le \gamma(a + \lambda^2 cbc^*) \le \gamma(a) + \lambda^2\gamma(cbc^*) \le \gamma(a) + \lambda^2\gamma(c)^2\gamma(b).$$

Now, for all $\lambda \in \mathbf{R}$ we have the inequality

$$0 \le \gamma(c)^2\gamma(b)\cdot\lambda^2 - 2\gamma(c)^2|\lambda| + \gamma(a),$$

so that the discriminant $4\gamma(c)^4 - 4\gamma(c)^2\gamma(b)\gamma(a)$ must be non-positive. Since we suppose that $\gamma(c) \ne 0$, by dividing with $4\gamma(c)^2$ we deduce the desired inequality (3.2).

It follows that

$$\begin{aligned}\Gamma(x+y)^2 &= \gamma(\langle x+y, x+y\rangle)\\ &\le \gamma(\langle x,x\rangle)+\gamma(\langle x,y\rangle)+\gamma(\langle y,x\rangle)+\gamma(\langle y,y\rangle)\\ &\le \gamma(\langle x,x\rangle)+2(\gamma(\langle x,x\rangle))^{\frac{1}{2}}(\gamma(\langle y,y\rangle))^{\frac{1}{2}}+\gamma(\langle y,y\rangle)\\ &= \left((\gamma(\langle x,x\rangle))^{\frac{1}{2}}+(\gamma(\langle y,y\rangle))^{\frac{1}{2}}\right)^2\\ &= (\Gamma(x)+\Gamma(y))^2.\end{aligned}$$

Therefore Γ is a seminorm on X, and (X,Γ) is a semi-Hilbert $\mathcal{A}$-module. □

Example 1.

(a) *Let $\mathcal{A}$ be a $*$-algebra and γ a positive linear functional or a C^*-seminorm on $\mathcal{A}$. It is known that $(\mathcal{A},\gamma)$ is a semi-Hilbert $\mathcal{A}$-module over itself with the inner product defined by $\langle a,b\rangle := a^*b$, in this case $\Gamma=\gamma$.*

(b) *Let $\mathcal{A}$ be a hermitian Banach $*$-algebra and ρ be the Pták function on $\mathcal{A}$. If X is a semi-inner product $\mathcal{A}$-module and $P(x)=(\rho(\langle x,x\rangle))^{1/2} \quad (x\in X)$, then (X,P) is a semi-Hilbert $\mathcal{A}$-module.*

(c) *Let $\mathcal{A}$ be a A^*-algebra and $|\cdot|$ be the auxiliary norm on $\mathcal{A}$. If X is an inner product $\mathcal{A}$-module and $|x| = |\langle x,x\rangle|^{1/2} \quad (x\in X)$, then $(X,|\cdot|)$ is a pre-Hilbert $\mathcal{A}$-module.*

Remark 1. *Let φ be a positive linear functional on a unital Banach $*$-algebra $\mathcal{A}$, X a semi-inner $\mathcal{A}$-module and $x,y\in X$. Put $a:=\langle x,x\rangle, b:=\langle y,y\rangle$ and $c:=\langle x,y\rangle$. From (3.3) and* [1, Section 37 Lemma 6 (iii)] *we have*

$$2\lambda cc^* \le a+\lambda^2 cbc^*, \quad \textit{therefore} \quad \begin{aligned}2\lambda\varphi(cc^*) &\le \varphi(a+\lambda^2 cbc^*) = \varphi(a)+\lambda^2\varphi(cbc^*)\\ &\le \varphi(a)+\lambda^2\varphi(cc^*)r(b).\end{aligned}$$

Thus for all $\lambda\in\mathbf{R}$ inequality $0\le\lambda^2\varphi(cc^*)r(b)-2\lambda\varphi(cc^*)+\varphi(a)$ holds. So the discriminant $\varphi(cc^*)^2-\varphi(cc^*)\varphi(a)r(b)=\varphi(cc^*)(\varphi(cc^*)-\varphi(a)r(b))\le 0$. This implies that

$$\varphi(cc^*)\le\varphi(a)r(b) \quad \textit{or} \quad \varphi(\langle x,y\rangle\langle y,x\rangle)\le\varphi(\langle x,x\rangle)r(\langle y,y\rangle). \tag{3.4}$$

Now suppose that X is a C^*-module on C^*-algebra $\mathcal{A}$ and a,b,c are as above. By [8, Theorem 3.3.6] *there is a state φ on $\mathcal{A}$ such that $\varphi(cc^*)=\|cc^*\|=\|c\|^2$. Using inequality (3.4) we have*

$$\|c\|^2=\varphi(cc^*)\le\varphi(a)r(b)\le\|a\|\|b\|.$$

Therefore (3.4) is a refinement of Schwarz's inequality for C^*-modules [10, Lemma 15.1.3].

4. Grüss Type Inequalities

We assume, unless stated otherwise, throughout this section that $\mathcal{A}$ is a unital Banach $*$-algebra. The following Lemma 1 is a version of [4, Lemma 2.1] for a semi-inner product $\mathcal{A}$-module and the following Lemma 3 is a version of ([4, Lemma 2.4]) for an inner product $\mathcal{A}$-module.

Lemma 1. *Let X be an semi-inner product $\mathcal{A}$-module, and $x, y \in X, \alpha, \beta \in \mathbb{C}$. Then*

$$\operatorname{Re}\langle \alpha y - x, x - \beta y\rangle \geq 0$$

if and only if

$$\left\langle x - \frac{\alpha+\beta}{2}y, x - \frac{\alpha+\beta}{2}y\right\rangle \leq \frac{1}{4}|\alpha-\beta|^2 \langle y, y\rangle .$$

Proof. Follows from the equalities:

$$\begin{aligned}\operatorname{Re}\langle \alpha y - x, x - \beta y\rangle &= \frac{1}{2}(\langle \alpha y - x, x - \beta y\rangle + \langle x - \beta y, \alpha e - x\rangle)\\ &= \frac{\bar\alpha + \bar\beta}{2}\langle y, x\rangle - \frac{\bar\alpha\beta + \bar\beta\alpha}{2}\langle y, y\rangle - \langle x, x\rangle + \frac{\alpha+\beta}{2}\langle x, y\rangle\\ &= \frac{1}{4}|\alpha-\beta|^2\langle y, y\rangle - \left\langle x - \frac{\alpha+\beta}{2}y, x - \frac{\alpha+\beta}{2}y\right\rangle .\end{aligned}$$

□

Lemma 2. *Let X be an inner product $\mathcal{A}$-module and $x, y, e \in X$. If $\langle e, e\rangle$ is idempotent, then $e\langle e, e\rangle = e$, and therefore*

$$\langle e, e\rangle\langle e, x\rangle = \langle e, x\rangle , \quad \langle x, e\rangle = \langle x, e\rangle\langle e, e\rangle .$$

Proof. Observe that the equality

$$\begin{aligned}\langle e\langle e, e\rangle - e, e\langle e, e\rangle - e\rangle &= \langle e\langle e, e\rangle , e\langle e, e\rangle\rangle - \langle e\langle e, e\rangle , e\rangle - \langle e, e\langle e, e\rangle\rangle + \langle e, e\rangle\\ &= \langle e, e\rangle\langle e, e\rangle\langle e, e\rangle - \langle e, e\rangle\langle e, e\rangle - \langle e, e\rangle\langle e, e\rangle + \langle e, e\rangle\\ &= 0,\end{aligned}$$

implies that $e\langle e, e\rangle - e = 0$.

The rest follows from this fact and we omit the details. □

Lemma 3. *Let X be an inner product $\mathcal{A}$-module and γ be a C^*-seminorm or a positive linear functional on $\mathcal{A}$ and $\Gamma(x) = (\gamma(\langle x, x\rangle))^{1/2} \quad (x \in X)$. If $x, e \in X$ and $\langle e, e\rangle$ is an idempotent then*

$$0 \leq \langle x, x\rangle - \langle x, e\rangle\langle e, x\rangle$$

and

$$\gamma(\langle x, x\rangle - \langle x, e\rangle\langle e, x\rangle) \leq \inf_{\lambda\in\mathbb{C}} \Gamma(x - \lambda e)^2.$$

Proof. Observe, for any $a \in \mathcal{A}$, that

$$\begin{aligned}\langle x - ea, x - e\langle e, x\rangle\rangle &= \langle x, x\rangle - \langle x, e\langle e, x\rangle\rangle - \langle ea, x\rangle + \langle ea, e\langle e, x\rangle\rangle\\ &= \langle x, x\rangle - \langle x, e\rangle\langle e, x\rangle - a^*\langle e, x\rangle + a^*\langle e, e\rangle\langle e, x\rangle\\ &= \langle x, x\rangle - \langle x, e\rangle\langle e, x\rangle .\end{aligned}$$

This implies that

$$\langle x, x\rangle - \langle x, e\rangle\langle e, x\rangle = \langle x - e\langle e, x\rangle , x - e\langle e, x\rangle\rangle \geq 0.$$

Also observe, for any $\lambda \in \mathbb{C}$, that

$$\begin{aligned}\langle x - \lambda e, x - e\langle e, x\rangle\rangle &= \langle x, x\rangle - \langle x, e\langle e, x\rangle\rangle - \langle \lambda e, x\rangle + \langle \lambda e, e\langle e, x\rangle\rangle\\ &= \langle x, x\rangle - \langle x, e\rangle\langle e, x\rangle - \bar\lambda\langle e, x\rangle + \bar\lambda\langle e, e\rangle\langle e, x\rangle\\ &= \langle x, x\rangle - \langle x, e\rangle\langle e, x\rangle .\end{aligned}$$

Using Schwarz's inequality, we have

$$\begin{aligned}\gamma(\langle x,x\rangle - \langle x,e\rangle\langle e,x\rangle)^2 &= \gamma(\langle x-\lambda e, x - e\langle e,x\rangle\rangle)^2 \\ &\le \gamma(\langle x-\lambda e, x-\lambda e\rangle)\gamma(\langle x - e\langle e,x\rangle, x - e\langle e,x\rangle\rangle) \\ &= \gamma(\langle x-\lambda e, x-\lambda e\rangle)\gamma(\langle x,x\rangle - \langle x,e\rangle\langle e,x\rangle),\end{aligned}$$

therefore giving the bound

$$\gamma(\langle x,x\rangle - \langle x,e\rangle\langle e,x\rangle) \le \gamma(\langle x-\lambda e, x-\lambda e\rangle) = \Gamma(x-\lambda e)^2 \quad \lambda \in \mathbb{C}.$$

Taking the infimum in the above relation over $\lambda \in \mathbb{C}$, we deduce

$$\gamma(\langle x,x\rangle - \langle x,e\rangle\langle e,x\rangle) \le \inf_{\lambda\in\mathbb{C}} \Gamma(x-\lambda e)^2.$$

□

Let X be a semi-inner product $\mathcal{A}$-module, $x, y \in X, \alpha, \beta \in \mathbb{C}$ and γ be a C^*-seminorm or a positive linear functional on $\mathcal{A}$. Put $\Gamma(x) = (\gamma(\langle x,x\rangle))^{1/2}(x \in X)$. By Lemma 1, $\operatorname{Re}\langle \alpha y - x, x - \beta y\rangle \ge 0$ implies that

$$\begin{aligned}\Gamma\left(x - \frac{\alpha+\beta}{2}y\right)^2 &= \gamma\left(\left\langle x - \frac{\alpha+\beta}{2}y, x - \frac{\alpha+\beta}{2}y\right\rangle\right) \\ &\le \frac{1}{4}|\alpha-\beta|^2\gamma(\langle y,y\rangle) \\ &= \frac{1}{4}|\alpha-\beta|^2\Gamma(y)^2.\end{aligned}$$

Also, let $0 \ne e \in X$ and $\langle e,e\rangle$ be idempotent. If γ is a C^*-seminorm, then it is trivial that $\gamma(\langle e,e\rangle) = 0$ or $\gamma(\langle e,e\rangle) = 1$, i.e., $\Gamma(e) \le 1$.

Lemma 4. *Let X be an inner product $\mathcal{A}$-module, γ be a C^*-seminorm on $\mathcal{A}$ and $\Gamma(x) = (\gamma(\langle x,x\rangle))^{1/2} \quad (x \in X)$. If $x, y, e \in X$, $\langle e,e\rangle$ is idempotent and $\alpha, \beta, \lambda, \mu$ are real or complex numbers such that*

$$\Gamma\left(x - \frac{\alpha+\beta}{2}e\right) \le \frac{1}{2}|\alpha-\beta|, \quad \Gamma\left(y - \frac{\lambda+\mu}{2}e\right) \le \frac{1}{2}|\lambda-\mu|$$

hold, then one has the inequality

$$\gamma(\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle) \le \frac{1}{4}|\alpha-\beta||\lambda-\mu|.$$

Furthermore, if there is a non zero element f in X such that $\langle e,f\rangle = 0$ and $\Gamma(f) \ne 0$, then the constant $\frac{1}{4}$ is best possible.

Proof. By (3.2), Γ is a seminorm on X. It can be easily shown that,

$$\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle = \langle x - e\langle e,x\rangle, y - e\langle e,y\rangle\rangle.$$

From the Schwarz inequality, we obtain

$$\begin{aligned}&\gamma(\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle) \\ &= \gamma(\langle x - e\langle e,x\rangle, y - e\langle e,y\rangle\rangle) \\ &\le \gamma(\langle x - e\langle e,x\rangle, x - e\langle e,x\rangle\rangle)^{\frac{1}{2}}\gamma(\langle y - e\langle e,y\rangle, y - e\langle e,y\rangle\rangle)^{\frac{1}{2}} \\ &= \gamma(\langle x,x\rangle - \langle x,e\rangle\langle e,x\rangle)^{\frac{1}{2}}\gamma(\langle y,y\rangle - \langle y,e\rangle\langle e,y\rangle)^{\frac{1}{2}}.\end{aligned} \tag{4.1}$$

Using Lemma 3 and the above assumptions, we have that

$$\begin{aligned}\gamma(\langle x,x\rangle-\langle x,e\rangle\langle e,x\rangle)^{\frac{1}{2}} &\le \inf_{\lambda\in\mathbb{C}}\Gamma(x-\lambda e)\\ &\le \Gamma\left(x-\frac{\alpha+\beta}{2}e\right)\le\frac{1}{2}|\alpha-\beta|\end{aligned}$$

and

$$\begin{aligned}\gamma(\langle y,y\rangle-\langle y,e\rangle\langle e,y\rangle)^{\frac{1}{2}} &\le \inf_{\lambda\in\mathbb{C}}\Gamma(y-\lambda e)\\ &\le \Gamma\left(y-\frac{\lambda+\mu}{2}e\right)\le\frac{1}{2}|\lambda-\mu|.\end{aligned}$$

Therefore the desired inequality is obtained.

Now we show that the constant $\frac{1}{4}$ is best possible. If f is a non zero element of X with $\Gamma(f)\neq 0$ such that $\langle e,f\rangle=0$ and given $\epsilon>0$, then for

$$x_\epsilon=\frac{|\alpha-\beta|}{2(\Gamma(f)+\epsilon)}f+\frac{\alpha+\beta}{2}e,\quad y_\epsilon=\frac{|\lambda-\mu|}{2(\Gamma(f)+\epsilon)}f+\frac{\lambda+\mu}{2}e$$

the assumptions of the previous lemma hold and in this case

$$\gamma(\langle x_\epsilon,y_\epsilon\rangle-\langle x_\epsilon,e\rangle\langle e,y_\epsilon\rangle)=\frac{|\alpha-\beta||\lambda-\mu|}{4}\cdot\frac{\Gamma(f)^2}{(\Gamma(f)+\epsilon)^2}.$$

Now if c is a constant such that $0<c<\frac{1}{4}$ then there is a $\epsilon>0$ such that $\frac{\Gamma(f)^2}{4(\Gamma(f)+\epsilon)^2}>c$. Therefore

$$\gamma(\langle x_\epsilon,y_\epsilon\rangle-\langle x_\epsilon,e\rangle\langle e,y_\epsilon\rangle)>c|\alpha-\beta||\lambda-\mu|.$$

□

In the following lemma φ is a positive linear functional on $\mathcal{A}$. Putting $\Phi(x)=\varphi(\langle x,x\rangle)^{\frac{1}{2}}\quad(x\in X)$, by (3.1), Φ is a seminorm on X. Therefore we have:

Lemma 5. *Let X be an inner product $\mathcal{A}$-module and φ a positive linear functional on $\mathcal{A}$. If $x,y,e\in X$, $\langle e,e\rangle$ is idempotent and α,β,λ,μ are real or complex numbers such that*

$$\Phi\left(x-\frac{\alpha+\beta}{2}e\right)\le\frac{1}{2}|\alpha-\beta|\Phi(e),\quad \Phi\left(y-\frac{\lambda+\mu}{2}e\right)\le\frac{1}{2}|\lambda-\mu|\Phi(e)$$

hold, then one has the inequality

$$\varphi(\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle)\le\frac{1}{4}|\alpha-\beta||\lambda-\mu|\Phi(e)^2.$$

Furthermore, if there is a non zero element f in X such that $\langle e,f\rangle=0$ and $\Phi(f)\neq 0$ then the constant $\frac{1}{4}$ is best possible.

We are able now to state our first main result:

Theorem 2. *Let X be an inner product $\mathcal{A}$-module, γ a C^*-seminorm on $\mathcal{A}$ and $\Gamma(x)=(\gamma(\langle x,x\rangle))^{\frac{1}{2}}\quad(x\in X)$. If $x,y,e\in X$, $\langle e,e\rangle$ is idempotent and α,β,λ,μ are real or complex numbers such that*

$$\Gamma\left(x-\frac{\alpha+\beta}{2}e\right)\le\frac{1}{2}|\alpha-\beta|,\quad \Gamma\left(y-\frac{\lambda+\mu}{2}e\right)\le\frac{1}{2}|\lambda-\mu|$$

hold, then one has the inequality

$$\gamma(\langle x,y\rangle - \langle x,e\rangle \langle e,y\rangle) \le \frac{1}{4}|\alpha-\beta||\lambda-\mu| - \left(\frac{1}{4}|\alpha-\beta|^2 - \Gamma\left(x-\frac{\alpha+\beta}{2}e\right)^2\right)^{\frac{1}{2}} \times \left(\frac{1}{4}|\lambda-\mu|^2 - \Gamma\left(y-\frac{\lambda+\mu}{2}e\right)^2\right)^{\frac{1}{2}}. \tag{4.2}$$

Furthermore, if there is a non zero element f in X such that $\langle e,f\rangle = 0$ and $\Gamma(f)\neq 0$ then the constant $\frac{1}{4}$ is best possible.

Proof. A simple calculation shows that

$$\langle \alpha e - e\langle e,x\rangle, e\langle e,x\rangle - \beta e\rangle - \langle \alpha e - x, x-\beta e\rangle = \langle x,x\rangle - \langle x,e\rangle\langle e,x\rangle,$$

therefore

$$\mathrm{Re}\,\langle \alpha e - e\langle e,x\rangle, e\langle e,x\rangle - \beta e\rangle - \mathrm{Re}\,\langle \alpha e - x, x-\beta e\rangle = \langle x,x\rangle - \langle x,e\rangle\langle e,x\rangle.$$

Since $\langle a,b\rangle + \langle b,a\rangle \le \frac{1}{2}\langle a+b,a+b\rangle$, so

$$\mathrm{Re}\,\langle \alpha e - e\langle e,x\rangle, e\langle e,x\rangle - \beta e\rangle \le \frac{1}{4}|\alpha-\beta|^2\langle e,e\rangle.$$

As in the proof of Lemma 1

$$\mathrm{Re}\,\langle \alpha e - x, x-\beta e\rangle = \frac{1}{4}|\alpha-\beta|^2\langle e,e\rangle - \left\langle x-\frac{\alpha+\beta}{2}e, x-\frac{\alpha+\beta}{2}e\right\rangle,$$

therefore

$$\langle x,x\rangle - \langle x,e\rangle\langle e,x\rangle \le \left\langle x-\frac{\alpha+\beta}{2}e, x-\frac{\alpha+\beta}{2}e\right\rangle. \tag{4.3}$$

Analogously

$$\langle y,y\rangle - \langle y,e\rangle\langle e,y\rangle \le \left\langle y-\frac{\lambda+\mu}{2}e, y-\frac{\lambda+\mu}{2}e\right\rangle. \tag{4.4}$$

We obtain

$$\gamma(\langle x,x\rangle - \langle x,e\rangle\langle e,x\rangle)\gamma(\langle y,y\rangle - \langle y,e\rangle\langle e,y\rangle) \le \Gamma\left(x-\frac{\alpha+\beta}{2}e\right)^2\Gamma\left(y-\frac{\lambda+\mu}{2}e\right)^2.$$

Finally, using the elementary inequality for real numbers

$$(m^2-n^2)(p^2-q^2) \le (mp-nq)^2$$

on

$$m=\frac{1}{2}|\alpha-\beta|, \qquad n=\left[\frac{1}{4}|\alpha-\beta|^2 - \Gamma(x-\frac{\alpha+\beta}{2}e)^2\right]^{\frac{1}{2}},$$

$$p=\frac{1}{2}|\lambda-\mu|, \qquad q=\left[\frac{1}{4}|\lambda-\mu|^2 - \Gamma(y-\frac{\lambda+\mu}{2}e)^2\right]^{\frac{1}{2}},$$

we get

$$\Gamma\left(x-\frac{\alpha+\beta}{2}e\right)\Gamma\left(y-\frac{\lambda+\mu}{2}e\right) \leq \frac{1}{4}|\alpha-\beta||\lambda-\mu| - \left(\frac{1}{4}|\alpha-\beta|^2 - \Gamma\left(x-\frac{\alpha+\beta}{2}e\right)^2\right)^{\frac{1}{2}} \times \left(\frac{1}{4}|\lambda-\mu|^2 - \Gamma\left(y-\frac{\lambda+\mu}{2}e\right)^2\right)^{\frac{1}{2}}.$$

The fact that $\frac{1}{4}$ is the best constant can be proven in a similar manner to the one in the previous lemma. The details are omitted. □

Similarly for a positive linear functional φ, the following theorem holds.

Theorem 3. *Let X be an inner product $\mathcal{A}$-module, φ a positive linear functional $\mathcal{A}$ and $\Phi(x) = (\varphi(\langle x, x\rangle))^{\frac{1}{2}}$ $(x \in X)$. If $x, y, e \in X$, $\langle e, e\rangle$ is idempotent and $\alpha, \beta, \lambda, \mu$ are real or complex numbers such that*

$$\Phi\left(x-\frac{\alpha+\beta}{2}e\right) \leq \frac{1}{2}|\alpha-\beta|\Phi(e), \quad \Phi\left(y-\frac{\lambda+\mu}{2}e\right) \leq \frac{1}{2}|\lambda-\mu|\Phi(e)$$

hold, then one has the inequality

$$\varphi(\langle x, y\rangle - \langle x, e\rangle\langle e, y\rangle) \leq \frac{1}{4}|\alpha-\beta||\lambda-\mu|\Phi(e)^2 - \left(\frac{1}{4}|\alpha-\beta|^2\Phi(e)^2 - \Phi\left(x-\frac{\alpha+\beta}{2}e\right)^2\right)^{\frac{1}{2}} \times \left(\frac{1}{4}|\lambda-\mu|^2\Phi(e)^2 - \Phi\left(y-\frac{\lambda+\mu}{2}e\right)^2\right)^{\frac{1}{2}}. \tag{4.5}$$

Furthermore, if there is a non zero element f in X such that $\langle e, f\rangle = 0$ and $\Phi(f) \neq 0$, then the constant $\frac{1}{4}$ is best possible.

Remark 2.

(i) *If in the above theorem φ is a state on $\mathcal{A}$ then, obviously, inequality (4.5) becomes the following:*

$$\varphi(\langle x, y\rangle - \langle x, e\rangle\langle e, y\rangle) \leq \frac{1}{4}|\alpha-\beta||\lambda-\mu| - \left(\frac{1}{4}|\alpha-\beta|^2 - \Phi\left(x-\frac{\alpha+\beta}{2}e\right)^2\right)^{\frac{1}{2}} \times \left(\frac{1}{4}|\lambda-\mu|^2 - \Phi\left(y-\frac{\lambda+\mu}{2}e\right)^2\right)^{\frac{1}{2}}. \tag{4.6}$$

(ii) *Let X be a C^*-module and $x, y, e \in X$. If $\langle e, e\rangle$ is idempotent and $\alpha, \beta, \lambda, \mu$ are real or complex numbers such that*

$$\left\|x-\frac{\alpha+\beta}{2}e\right\| \leq \frac{1}{2}|\alpha-\beta|, \qquad \left\|y-\frac{\lambda+\mu}{2}e\right\| \leq \frac{1}{2}|\lambda-\mu| \tag{4.7}$$

hold, then by [8, Theorem 3.3.6] *there are states* φ *and* ψ *on a* C^**-algebra* $\mathcal{A}$ *such that*

$$\varphi(\langle x,x\rangle - \langle x,e\rangle\langle e,x\rangle) = \|\langle x,x\rangle - \langle x,e\rangle\langle e,x\rangle\|$$

and

$$\psi(\langle y,y\rangle - \langle y,e\rangle\langle e,y\rangle) = \|\langle y,y\rangle - \langle y,e\rangle\langle e,y\rangle\|.$$

Inequalities (4.3), (4.4) imply that

$$\varphi(\langle x,x\rangle - \langle x,e\rangle\langle e,x\rangle) \le \varphi\left(\left\langle x-\frac{\alpha+\beta}{2}e, x-\frac{\alpha+\beta}{2}e\right\rangle\right) \tag{4.8}$$

and

$$\psi(\langle y,y\rangle - \langle y,e\rangle\langle e,y\rangle) \le \psi\left(\left\langle y-\frac{\lambda+\mu}{2}e, y-\frac{\lambda+\mu}{2}e\right\rangle\right). \tag{4.9}$$

From Schwarz's inequality (4.1) and inequalities (4.8), (4.9) we get

$$\begin{aligned}\|\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle\| &\le \|\langle x,x\rangle - \langle x,e\rangle\langle e,x\rangle\|^{\frac{1}{2}}\|\langle y,y\rangle - \langle y,e\rangle\langle e,y\rangle\|^{\frac{1}{2}}\\ &= \varphi(\langle x,x\rangle - \langle x,e\rangle\langle e,x\rangle)^{\frac{1}{2}}\psi(\langle y,y\rangle - \langle y,e\rangle\langle e,y\rangle)^{\frac{1}{2}}\\ &\le \Phi\left(x-\frac{\alpha+\beta}{2}e\right)\Psi\left(y-\frac{\lambda+\mu}{2}e\right).\end{aligned}$$

By (4.7) we have

$$\begin{aligned}\Phi\left(x-\frac{\alpha+\beta}{2}e\right) &\le \left\|x-\frac{\alpha+\beta}{2}e\right\| \le \frac{|\alpha-\beta|}{2},\\ \Psi\left(y-\frac{\lambda+\mu}{2}e\right) &\le \left\|y-\frac{\lambda+\mu}{2}e\right\| \le \frac{|\lambda-\mu|}{2}.\end{aligned}$$

Using the elementary inequality for real numbers

$$(m^2-n^2)(p^2-q^2) \le (mp-nq)^2$$

for

$$m=\frac{1}{2}|\alpha-\beta|, \qquad n=\left[\frac{1}{4}|\alpha-\beta|^2 - \Phi\left(x-\frac{\alpha+\beta}{2}e\right)^2\right]^{\frac{1}{2}}$$

and

$$p=\frac{1}{2}|\lambda-\mu|, \qquad q=\left[\frac{1}{4}|\lambda-\mu|^2 - \Psi\left(y-\frac{\lambda+\mu}{2}e\right)^2\right]^{\frac{1}{2}},$$

we obtain

$$
\begin{aligned}
&\| \langle x,y\rangle - \langle x,e\rangle \langle e,y\rangle \| \\
&\leq \frac{1}{4}|\alpha-\beta||\lambda-\mu| - \left(\frac{1}{4}|\alpha-\beta|^2 - \Phi\left(x-\frac{\alpha+\beta}{2}e\right)^2\right)^{\frac{1}{2}} \\
&\qquad \times \left(\frac{1}{4}|\lambda-\mu|^2 - \Psi\left(y-\frac{\lambda+\mu}{2}e\right)^2\right)^{\frac{1}{2}} \\
&\leq \frac{1}{4}|\alpha-\beta||\lambda-\mu| - \left(\frac{1}{4}|\alpha-\beta|^2 - \left\|x-\frac{\alpha+\beta}{2}e\right\|^2\right)^{\frac{1}{2}} \\
&\qquad \times \left(\frac{1}{4}|\lambda-\mu|^2 - \left\|y-\frac{\lambda+\mu}{2}e\right\|^2\right)^{\frac{1}{2}},
\end{aligned} \tag{4.10}
$$

which is a refinement of the Grüss inequality for C^**-modules* [7, Theorem 5.1].

(iii) *By inequality (3.4) we may obtain another refinement of* [7, Theorem 5.1]*: Put* $G = \langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle$ *and* $R(x) = (r\langle x,x\rangle)^{\frac{1}{2}}$*. For every positive linear functional* φ *on* $\mathcal{A}$ *we have*

$$
\begin{aligned}
\varphi(GG^*) \leq \frac{1}{4}|\alpha-\beta||\lambda-\mu| &- \left(\frac{1}{4}|\alpha-\beta|^2 - \Phi\left(x-\frac{\alpha+\beta}{2}e\right)^2\right)^{\frac{1}{2}} \\
&\times \left(\frac{1}{4}|\lambda-\mu|^2 - R\left(y-\frac{\lambda+\mu}{2}e\right)^2\right)^{\frac{1}{2}},
\end{aligned}
$$

and we know that there is a state φ *on the* C^**-algebra* $\mathcal{A}$ *such that* $\varphi(GG^*) = \|GG^*\| = \|G\|^2$.

5. A Companion of the Grüss Inequality

The following companion of the Grüss inequality for positive linear functionals holds:

Theorem 4. *Let* X *be an inner product* $\mathcal{A}$*-module,* φ *a positive linear functional on* $\mathcal{A}$ *and* $x,y,e \in X$*. If* $\langle e,e\rangle$ *is idempotent and* α,β,λ,μ *are real or complex numbers such that*

$$\operatorname{Re}\langle \alpha e - x, x-\beta e\rangle \geq 0, \quad \operatorname{Re}\langle \lambda e - y, y-\mu e\rangle \geq 0$$

hold, then one has the inequality

$$
\begin{aligned}
\varphi(\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle) \leq{}& \frac{1}{4}|\alpha-\beta||\lambda-\mu|\Phi(e)^2 \\
&- \Phi\left(\frac{\alpha+\beta}{2}e - e\langle e,x\rangle\right)\Phi\left(\frac{\lambda+\mu}{2}e - e\langle e,x\rangle\right).
\end{aligned}
$$

Furthermore, if there is a non zero element f *in* X *such that* $\langle e,f\rangle = 0$ *and* $\Phi(f) \neq 0$ *then the constant* $\frac{1}{4}$ *is best possible.*

Proof. For every $k \in \mathbb{K}$ we have

$$\langle x,x\rangle - \langle x,e\rangle \langle e,x\rangle = \langle x-ke, x-ke\rangle - \langle ke - e\langle e,x\rangle, ke - e\langle e,x\rangle\rangle .$$

For $k = \frac{\alpha+\beta}{2}$ Lemma 1 implies that

$$\langle x,x\rangle - \langle x,e\rangle \langle e,x\rangle \leq \frac{1}{4}|\alpha-\beta|^2 \langle e,e\rangle - \left\langle \frac{\alpha+\beta}{2}e - e\langle e,x\rangle , \frac{\alpha+\beta}{2}e - e\langle e,x\rangle \right\rangle .$$

Therefore

$$\varphi(\langle x,x\rangle - \langle x,e\rangle \langle e,x\rangle) \leq \frac{1}{4}|\alpha-\beta|^2 \Phi(e)^2 - \Phi\left(\frac{\alpha+\beta}{2}e - e\langle e,x\rangle\right)^2 .$$

Analogously

$$\varphi(\langle y,y\rangle - \langle y,e\rangle \langle e,y\rangle) \leq \frac{1}{4}|\lambda-\mu|^2 \Phi(e)^2 - \Phi\left(\frac{\lambda+\mu}{2}e - e\langle e,x\rangle\right)^2 .$$

Using Schwarz's inequality and the elementary inequality for real numbers

$$(m^2-n^2)(p^2-q^2) \leq (mp-nq)^2$$

we obtain

$$\begin{aligned} &(\varphi(\langle x,y\rangle - \langle x,e\rangle \langle e,y\rangle))^2 \\ &\leq \varphi(\langle x,x\rangle - \langle x,e\rangle \langle e,x\rangle)\varphi(\langle y,y\rangle - \langle y,e\rangle \langle e,y\rangle) \\ &\leq \left[\frac{1}{4}|\alpha-\beta||\lambda-\mu|\Phi(e)^2 - \Phi\left(\frac{\alpha+\beta}{2}e - e\langle e,x\rangle\right)\Phi\left(\frac{\lambda+\mu}{2}e - e\langle e,x\rangle\right)\right]^2 . \end{aligned}$$

Therefore we get

$$\begin{aligned} &\varphi(\langle x,y\rangle - \langle x,e\rangle \langle e,y\rangle) \\ &\qquad \leq \frac{1}{4}|\alpha-\beta||\lambda-\mu|\Phi(e)^2 - \Phi\left(\frac{\alpha+\beta}{2}e - e\langle e,x\rangle\right)\Phi\left(\frac{\lambda+\mu}{2}e - e\langle e,x\rangle\right) . \end{aligned}$$

□

Other inequalities related to the Grüss inequality such as Theorem 18, Proposition 18, Theorem 20, Corollary 20 and Remark 23 in [5], have versions that are valid for positive linear functionals and C^*-seminorms on unital Banach $*$-algebras. However, the details are omitted.

Acknowledgement 1. *This work was done when the first author was at the Research Group in Mathematical Inequalities and Applications (RGMIA), Victoria University on sabbatical leave from Lorestan University. He thanks both institutions for their support.*

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