GRÜSS' TYPE INEQUALITIES FOR SOME CLASSES OF CONTINUOUS FUNCTIONS OF SELFADJOINT OPERATORS IN HILBERT SPACES

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Abstract. Some inequalities of Grüss' type for continuous functions of selfadjoint operators in Hilbert spaces, under suitable assumptions for the involved operators, are given. Applications for power and logarithmic functions are provided as well.

1. Introduction

In [11], in order to generalize Grüss' inequality in abstract structures the author has proved the following result:

Theorem 1 (Dragomir, 1999, [11])**.** *Let $(H, \langle .,. \rangle)$ be an inner product space over $\mathbb{K}\,(\mathbb{K} = \mathbb{R},\mathbb{C})$ and $e \in H, \|e\| = 1$. If $\varphi, \gamma, \Phi, \Gamma$ are real or complex numbers and x, y are vectors in H such that the conditions*

$$\operatorname{Re} \langle \Phi e - x, x - \varphi e\rangle \geq 0 \text{ and } \operatorname{Re} \langle \Gamma e - y, y - \gamma e\rangle \geq 0 \tag{1.1}$$

hold, then we have the inequality

$$|\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle| \leq \frac{1}{4} |\Phi - \varphi| \cdot |\Gamma - \gamma| . \tag{1.2}$$

The constant $\frac{1}{4}$ is best possible in the sense that it can not be replaced by a smaller quantity.

For other results of this type, see the recent monograph [14] and the references therein.

For discrete and integral inequalities of Grüss type, see Chapter X of the book [26]. For other related results see the papers [1]-[3], [4]-[6], [7]-[9], [10]-[17], [21], [29], [31] and the references therein.

In order to extend this result for functions of selfadjoint operators we need the following preparation.

Let A be a selfadjoint linear operator on a complex Hilbert space $(H; \langle .,. \rangle)$. The *Gelfand map* establishes a $*$-isometrically isomorphism Φ between the set $C(Sp(A))$ of all *continuous functions* defined on the *spectrum* of A, denoted $Sp(A)$, and the C^*-algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows (see for instance [23, p. 3]):

For any $f, g \in C(Sp(A))$ and any $\alpha, \beta \in \mathbb{C}$ we have

(i) $\Phi(\alpha f + \beta g) = \alpha \Phi(f) + \beta \Phi(g)$;

(ii) $\Phi(fg) = \Phi(f)\Phi(g)$ and $\Phi(\bar{f}) = \Phi(f)^*$;

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(iii) $\|\Phi(f)\| = \|f\| := \sup_{t\in Sp(A)} |f(t)|$;
(iv) $\Phi(f_0) = 1_H$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$, for $t \in Sp(A)$.
With this notation we define

$$f(A) := \Phi(f) \text{ for all } f \in C(Sp(A))$$

and we call it the *continuous functional calculus* for a selfadjoint operator A.

If A is a selfadjoint operator and f is a real valued continuous function on $Sp(A)$, then $f(t) \geq 0$ for any $t \in Sp(A)$ implies that $f(A) \geq 0$, *i.e.* $f(A)$ is a positive operator on H. Moreover, if both f and g are real valued functions on $Sp(A)$ then the following important property holds:

$$f(t) \geq g(t) \text{ for any } t \in Sp(A) \text{ implies that } f(A) \geq g(A) \tag{P}$$

in the operator order of $B(H)$.

For a recent monograph devoted to various inequalities for functions of selfadjoint operators, see [23] and the references therein. For other results, see [28], [25] and [30].

In the recent paper [19] we obtained amongst other the following refinement of the Grüss inequality for functions of selfadjoint operators:

Theorem 2 (Dragomir, 2009, [19])**.** *Let A be a selfadjoint operator on the Hilbert space $(H;\langle .,.\rangle)$ and assume that $Sp(A) \subseteq [m,M]$ for some scalars $m < M$. If f and g are continuous on $[m,M]$ and $\gamma := \min_{t\in[m,M]} f(t)$ and $\Gamma := \max_{t\in[m,M]} f(t)$ then*

$$\begin{aligned} &|\langle f(A)g(A)x,x\rangle - \langle f(A)x,x\rangle \cdot \langle g(A)x,x\rangle| \\ &\leq \frac{1}{2}\cdot(\Gamma-\gamma)\left[\|g(A)x\|^2 - \langle g(A)x,x\rangle^2\right]^{1/2} \leq \frac{1}{4}(\Gamma-\gamma)(\Delta-\delta) \end{aligned} \tag{1.3}$$

for each $x \in H$ with $\|x\| = 1$, where $\delta := \min_{t\in[m,M]} g(t)$ and $\Delta := \max_{t\in[m,M]} g(t)$.

Remark 1. *The inequality between the first and the last term of (1.3) is the operator version of the Grüss inequality and was firstly obtained by Mond and Pečarić in* [27].

The following result for one Lipschitzian function and the second continuous, can be stated as well:

Theorem 3 (Dragomir, 2009, [20])**.** *Let A be a selfadjoint operator with $Sp(A) \subseteq [m,M]$ for some real numbers $m < M$. If $f : [m,M] \longrightarrow \mathbb{R}$ is Lipschitzian with the constant $L > 0$ and $g : [m,M] \longrightarrow \mathbb{R}$ is continuous with $\delta := \min_{t\in[m,M]} g(t)$ and $\Delta := \max_{t\in[m,M]} g(t)$, then*

$$\begin{aligned} &|\langle f(A)g(A)x,x\rangle - \langle f(A)x,x\rangle \cdot \langle g(A)x,x\rangle| \\ &\leq \frac{1}{2}(\Delta-\delta)L\langle \ell_{A,x}(A)x,x\rangle \leq \frac{\sqrt{2}}{2}(\Delta-\delta)L\left[\|Ax\|^2 - \langle Ax,x\rangle^2\right]^{1/2} \end{aligned} \tag{1.4}$$

for any $x \in H$ with $\|x\| = 1$, where

$$\ell_{A,x}(t) := \langle |t\cdot 1_H - A| x,x\rangle$$

is a continuous function on $[m,M]$.

When both functions are Lipschitzian, then we have the result:

Theorem 4 (Dragomir, 2009, [20])**.** *Let A be a selfadjoint operator with $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$. If $f, g : [m, M] \longrightarrow \mathbb{R}$ are Lipschitzian with the constants $L, K > 0$, then*

$$|\langle f(A) g(A) x, x\rangle - \langle f(A) x, x\rangle \cdot \langle g(A) x, x\rangle| \leq LK \left[\|Ax\|^2 - \langle Ax, x\rangle^2\right], \tag{1.5}$$

for any $x \in H$ with $\|x\| = 1$.

In order to provide some new vector Grüss' type inequalities for continuous functions of selfadjoint operators in Hilbert spaces, we need the following facts concerning the spectral representation of such functions.

Let U be a selfadjoint operator on the complex Hilbert space $(H, \langle ., .\rangle)$ with the spectrum $Sp(U)$ included in the interval $[m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its *spectral family.* Then for any continuous function $f : [m, M] \to \mathbb{C}$, it is well known that we have the following *spectral representation in terms of the Riemann-Stieltjes integral*:

$$f(U) = \int_{m-0}^{M} f(\lambda)\, dE_\lambda, \tag{1.6}$$

which, in terms of vectors, can be written as

$$\langle f(U) x, y\rangle = \int_{m-0}^{M} f(\lambda)\, d\langle E_\lambda x, y\rangle, \tag{1.7}$$

for any $x, y \in H$. The function $g_{x,y}(\lambda) := \langle E_\lambda x, y\rangle$ is of *bounded variation* on the interval $[m, M]$ and

$$g_{x,y}(m - 0) = 0 \text{ and } g_{x,y}(M) = \langle x, y\rangle$$

for any $x, y \in H$. It is also well known that $g_x(\lambda) := \langle E_\lambda x, x\rangle$ is *monotonic nondecreasing* and *right continuous* on $[m, M]$.

Motivated by the above results we consider in the present paper the more general problem of finding bounds for the quantity

$$|\langle f(A) x, g(B) x\rangle - \langle f(A) x, x\rangle \langle g(B) x, x\rangle|$$

where A, B are two selfadjoint operators in the Hilbert space H with the spectra $Sp(A), Sp(B) \subseteq [m, M]$ for some real numbers $m < M$ and $f, g : [m, M] \to \mathbb{C}$ belong to different subclasses of continuous functions on the compact interval $[m, M]$. Applications for some elementary functions of selfadjoint operators are provided as well.

2. Some Representation Results

We start with the following representation result that will play a key role in obtaining various bounds for different choices of functions including continuous functions of bounded variation, Lipschitzian functions or monotonic and continuous functions.

Theorem 5. *Let A, B be two selfadjoint operators in the Hilbert space H with the spectra $Sp(A), Sp(B) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be* *the spectral family of A and $\{F_\mu\}_\mu$ the spectral family of B. If $f, g : [m, M] \to \mathbb{C}$ are continuous, then we have the representation*

$$\begin{aligned} &\langle f(A)x, g(B)x\rangle - \langle f(A)x, x\rangle \langle x, g(B)x\rangle \\ &= \int_{m-0}^{M} \left(\int_{m-0}^{M} \left[\langle E_\lambda x, x\rangle \langle x, F_\mu x\rangle - \langle E_\lambda x, F_\mu x\rangle \right] d(g(\mu)) \right) d(f(\lambda)) \end{aligned} \tag{2.1}$$

for any $x \in H$ with $\|x\| = 1$.

Proof. Integrating by parts in the Riemann-Stieltjes integral and making use of the spectral representation (1.7) we have

$$\begin{aligned} &\int_{m-0}^{M} \left[\langle E_\lambda x, y\rangle - \langle E_\lambda x, x\rangle \langle x, y\rangle \right] df(\lambda) \\ &= \left[\langle E_\lambda x, y\rangle - \langle E_\lambda x, x\rangle \langle x, y\rangle \right] f(\lambda)\Big|_{m-0}^{M} \\ &- \int_{m-0}^{M} f(\lambda)\, d\left[\langle E_\lambda x, y\rangle - \langle E_\lambda x, x\rangle \langle x, y\rangle \right] \\ &= \langle x, y\rangle \int_{m-0}^{M} f(\lambda)\, d\langle E_\lambda x, x\rangle - \int_{m-0}^{M} f(\lambda)\, d\langle E_\lambda x, y\rangle \\ &= \langle x, y\rangle \langle f(A)x, x\rangle - \langle f(A)x, y\rangle \end{aligned} \tag{2.2}$$

for any $x, y \in H$ with $\|x\| = 1$.

Now, if we chose $y = g(B)x$ in (2.2) then we get that

$$\begin{aligned} &\int_{m-0}^{M} \left[\langle E_\lambda x, g(B)x\rangle - \langle E_\lambda x, x\rangle \langle x, g(B)x\rangle \right] df(\lambda) \\ &= \langle x, g(B)x\rangle \langle f(A)x, x\rangle - \langle f(A)x, g(B)x\rangle \end{aligned} \tag{2.3}$$

for any $x \in H$ with $\|x\| = 1$.

Utilising the spectral representation for B we also have for each fixed $\lambda \in [m, M]$ that

$$\begin{aligned} &\langle E_\lambda x, g(B)x\rangle - \langle E_\lambda x, x\rangle \langle x, g(B)x\rangle \\ &= \left\langle E_\lambda x, \int_{m-0}^{M} g(\mu)\, dF_\mu x \right\rangle - \langle E_\lambda x, x\rangle \left\langle x, \int_{m-0}^{M} g(\mu)\, dF_\mu x \right\rangle \\ &= \int_{m-0}^{M} g(\mu)\, d(\langle E_\lambda x, F_\mu x\rangle) - \langle E_\lambda x, x\rangle \int_{m-0}^{M} g(\mu)\, d(\langle x, F_\mu x\rangle) \end{aligned} \tag{2.4}$$

for any $x \in H$ with $\|x\| = 1$.

Integrating by parts in the Riemann-Stieltjes integral we have

$$\begin{aligned} \int_{m-0}^{M} g(\mu)\, d(\langle E_\lambda x, F_\mu x\rangle) &= g(\mu) \langle E_\lambda x, F_\mu x\rangle\big]_{m-0}^{M} - \int_{m-0}^{M} \langle E_\lambda x, F_\mu x\rangle\, dg(\mu) \\ &= g(M) \langle E_\lambda x, x\rangle - \int_{m-0}^{M} \langle E_\lambda x, F_\mu x\rangle\, d(g(\mu)) \end{aligned}$$

and

$$\int_{m-0}^{M} g(\mu)\, d(\langle x, F_\mu x\rangle) = g(\mu)\langle x, F_\mu x\rangle]_{m-0}^{M} - \int_{m-0}^{M} \langle x, F_\mu x\rangle\, d(g(\mu))$$
$$= g(M) - \int_{m-0}^{M} \langle x, F_\mu x\rangle\, d(g(\mu)),$$

therefore

$$\begin{aligned}(2.5)\quad &\int_{m-0}^{M} g(\mu)\, d(\langle E_\lambda x, F_\mu x\rangle) - \langle E_\lambda x, x\rangle \int_{m-0}^{M} g(\mu)\, d(\langle x, F_\mu x\rangle)\\ &= g(M)\langle E_\lambda x, x\rangle - \int_{m-0}^{M} \langle E_\lambda x, F_\mu x\rangle\, d(g(\mu))\\ &- \langle E_\lambda x, x\rangle \left(g(M) - \int_{m-0}^{M} \langle x, F_\mu x\rangle\, d(g(\mu))\right)\\ &= \langle E_\lambda x, x\rangle \int_{m-0}^{M} \langle x, F_\mu x\rangle\, d(g(\mu)) - \int_{m-0}^{M} \langle E_\lambda x, F_\mu x\rangle\, d(g(\mu))\\ &= \int_{m-0}^{M} \left[\langle E_\lambda x, x\rangle \langle x, F_\mu x\rangle - \langle E_\lambda x, F_\mu x\rangle\right] d(g(\mu))\end{aligned}$$

for any $x \in H$ with $\|x\| = 1$ and $\lambda \in [m, M]$.

Utilising (2.3)-(2.5) we deduce the desired result (2.1). □

Remark 2. *In particular, if we take $B = A$, then we get from (2.1) the equality*

$$\begin{aligned}(2.6)\quad &\langle f(A)x, g(A)x\rangle - \langle f(A)x, x\rangle\langle x, g(A)x\rangle\\ &= \int_{m-0}^{M}\left(\int_{m-0}^{M} \left[\langle E_\lambda x, x\rangle\langle x, E_\mu x\rangle - \langle E_\lambda x, E_\mu x\rangle\right] d(g(\mu))\right) d(f(\lambda))\end{aligned}$$

for any $x \in H$ with $\|x\| = 1$, which for $g = f$ produces the representation result for the variance of the selfadjoint operator $f(A)$,

$$\begin{aligned}(2.7)\quad &\|f(A)x\|^2 - \langle f(A)x, x\rangle^2\\ &= \int_{m-0}^{M}\left(\int_{m-0}^{M} \left[\langle E_\lambda x, x\rangle\langle x, E_\mu x\rangle - \langle E_\lambda x, E_\mu x\rangle\right] d(f(\mu))\right) d(f(\lambda))\end{aligned}$$

for any $x \in H$ with $\|x\| = 1$.

3. Bounds for f of Bounded Variation

The first vectorial Grüss' type inequality when one of the functions is of bounded variation is as follows:

Theorem 6. *Let A, B be two selfadjoint operators in the Hilbert space H with the spectra $Sp(A), Sp(B) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be the spectral family of A and $\{F_\mu\}_\mu$ the spectral family of B. Also, assume that $f : [m, M] \to \mathbb{C}$ is continuous and of bounded variation on $[m, M]$.*

1. If $g:[m,M]\to\mathbb{C}$ is continuous and of bounded variation on $[m,M]$, then we have the inequality

$$
\begin{aligned}
&\left|\langle f(A)x, g(B)x\rangle - \langle f(A)x,x\rangle\langle x,g(B)x\rangle\right| \\
&\le \max_{(\lambda,\mu)\in[m,M]^2}\left|\langle E_\lambda x,x\rangle\langle x,F_\mu x\rangle - \langle E_\lambda x,F_\mu x\rangle\right|\bigvee_m^M(g)\bigvee_m^M(f) \\
&\le \max_{\lambda\in[m,M]}\left[\langle E_\lambda x,x\rangle\langle(1_H-E_\lambda)x,x\rangle\right]^{1/2} \\
&\times \max_{\mu\in[m,M]}\left[\langle F_\mu x,x\rangle\langle(1_H-F_\mu)x,x\rangle\right]^{1/2}\bigvee_m^M(g)\bigvee_m^M(f) \le \frac{1}{4}\bigvee_m^M(g)\bigvee_m^M(f)
\end{aligned}
\tag{3.1}
$$

for any $x\in H$ with $\|x\|=1$.

2. If $g:[m,M]\to\mathbb{C}$ is Lipschitzian with the constant $K>0$ on $[m,M]$, then we have the inequality

$$
\begin{aligned}
&\left|\langle f(A)x, g(B)x\rangle - \langle f(A)x,x\rangle\langle x,g(B)x\rangle\right| \\
&\le K\max_{\lambda\in[m,M]}\left[\int_{m-0}^{M}\left|\langle E_\lambda x,x\rangle\langle x,F_\mu x\rangle - \langle E_\lambda x,F_\mu x\rangle\right|d\mu\right]\bigvee_m^M(f) \\
&\le K\bigvee_m^M(f)\max_{\lambda\in[m,M]}\left[\langle E_\lambda x,x\rangle\langle(1_H-E_\lambda)x,x\rangle\right]^{1/2} \\
&\times\int_{m-0}^{M}\left[\langle F_\mu x,x\rangle\langle(1_H-F_\mu)x,x\rangle\right]^{1/2}d\mu \\
&\le \frac{1}{2}K\bigvee_m^M(f)\langle(M1_H-B)x,x\rangle^{1/2}\langle(B-m1_H)x,x\rangle^{1/2} \\
&\le \frac{1}{4}K(M-m)\bigvee_m^M(f)
\end{aligned}
\tag{3.2}
$$

for any $x\in H$ with $\|x\|=1$.

3. If $g:[m,M]\to\mathbb{R}$ is continuous and monotonic nondecreasing on $[m,M]$, then we have the inequality

$$
\begin{aligned}
(3.3)\qquad & \left|\langle f(A)x, g(B)x\rangle - \langle f(A)x,x\rangle\langle x, g(B)x\rangle\right| \\
&\leq \max_{\lambda\in[m,M]}\left[\int_{m-0}^{M}\left|\langle E_\lambda x,x\rangle\langle x,F_\mu x\rangle - \langle E_\lambda x, F_\mu x\rangle\right| dg(\mu)\right]\bigvee_m^M(f) \\
&\leq \bigvee_m^M(f)\max_{\lambda\in[m,M]}\left[\langle E_\lambda x,x\rangle\langle(1_H-E_\lambda)x,x\rangle\right]^{1/2} \\
&\times\int_{m-0}^{M}\left[\langle F_\mu x,x\rangle\langle(1_H-F_\mu)x,x\rangle\right]^{1/2}dg(\mu) \\
&\leq \frac{1}{2}\bigvee_m^M(f)\langle(g(M)1_H-g(B))x,x\rangle^{1/2}\langle(g(B)-g(m)1_H)x,x\rangle^{1/2} \\
&\leq \frac{1}{4}\left[g(M)-g(m)\right]\bigvee_m^M(f)
\end{aligned}
$$

for any $x\in H$ with $\|x\|=1$.

Proof. 1. It is well known that if $p:[a,b]\to\mathbb{C}$ is a continuous function, $v:[a,b]\to\mathbb{C}$ is of bounded variation then the Riemann-Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and the following inequality holds

$$
(3.4)\qquad \left|\int_a^b p(t)\,dv(t)\right| \leq \max_{t\in[a,b]}|p(t)|\bigvee_a^b(v),
$$

where $\bigvee_a^b(v)$ denotes the total variation of v on $[a,b]$.

Now, on utilizing the property (3.4) and the identity (2.1) we have

$$
\begin{aligned}
(3.5)\qquad & \left|\langle f(A)x, g(B)x\rangle - \langle f(A)x,x\rangle\langle x, g(B)x\rangle\right| \\
&\leq \max_{\lambda\in[m,M]}\left|\int_{m-0}^{M}\left[\langle E_\lambda x,x\rangle\langle x,F_\mu x\rangle - \langle E_\lambda x, F_\mu x\rangle\right] d(g(\mu))\right|\bigvee_m^M(f)
\end{aligned}
$$

for any $x\in[m,M]$.

The same inequality (3.4) produces the bound

$$
\begin{aligned}
(3.6)\qquad & \max_{\lambda\in[m,M]}\left|\int_{m-0}^{M}\left[\langle E_\lambda x,x\rangle\langle x,F_\mu x\rangle - \langle E_\lambda x, F_\mu x\rangle\right] d(g(\mu))\right| \\
&\leq \max_{\lambda\in[m,M]}\left[\max_{\mu\in[m,M]}\left|\langle E_\lambda x,x\rangle\langle x,F_\mu x\rangle - \langle E_\lambda x, F_\mu x\rangle\right|\right]\bigvee_m^M(f) \\
&= \max_{(\lambda,\mu)\in[m,M]^2}\left|\langle E_\lambda x,x\rangle\langle x,F_\mu x\rangle - \langle E_\lambda x, F_\mu x\rangle\right|\bigvee_m^M(f)
\end{aligned}
$$

for any $x\in[m,M]$.

By making use of (3.5) and (3.6) we deduce the first part of (3.1).

Further, we notice that by the Schwarz inequality in H we have for any $u, v, e \in H$ with $\|e\| = 1$ that

$$|\langle u, v\rangle - \langle u, e\rangle \langle e, v\rangle| \leq \left(\|u\|^2 - |\langle u, e\rangle|^2\right)^{1/2} \left(\|v\|^2 - |\langle v, e\rangle|^2\right)^{1/2}. \tag{3.7}$$

Indeed, if we write Schwarz's inequality for the vectors $u - \langle u, e\rangle e$ and $v - \langle v, e\rangle e$ we have

$$|\langle u - \langle u, e\rangle e, v - \langle v, e\rangle e\rangle| \leq \|u - \langle u, e\rangle e\| \|v - \langle v, e\rangle e\|$$

which, by performing the calculations, is equivalent with (3.7).

Now, on utilizing (3.7), we can state that

$$\begin{aligned} &|\langle E_\lambda x, x\rangle \langle x, F_\mu x\rangle - \langle E_\lambda x, F_\mu x\rangle| \\ &\leq \left(\|E_\lambda x\|^2 - |\langle E_\lambda x, x\rangle|^2\right)^{1/2} \left(\|F_\mu x\|^2 - |\langle F_\mu x, x\rangle|^2\right)^{1/2} \end{aligned} \tag{3.8}$$

for any $\lambda, \mu \in [m, M]$.

Since E_λ and F_μ are projections and $E_\lambda, F_\mu \geq 0$ then

$$\begin{aligned} \|E_\lambda x\|^2 - |\langle E_\lambda x, x\rangle|^2 &= \langle E_\lambda x, x\rangle - \langle E_\lambda x, x\rangle^2 \\ &= \langle E_\lambda x, x\rangle \langle (1_H - E_\lambda) x, x\rangle \leq \frac{1}{4} \end{aligned} \tag{3.9}$$

and

$$\|F_\mu x\|^2 - |\langle F_\mu x, x\rangle|^2 = \langle F_\mu x, x\rangle \langle (1_H - F_\mu) x, x\rangle \leq \frac{1}{4} \tag{3.10}$$

for any $\lambda, \mu \in [m, M]$ and $x \in H$ with $\|x\| = 1$.

Now, if we use (3.8)-(3.10) then we get the second part of (3.1).

2. Further, recall that if $p : [a, b] \to \mathbb{C}$ is a Riemann integrable function and $v : [a, b] \to \mathbb{C}$ is Lipschitzian with the constant $L > 0$, i.e.,

$$|f(s) - f(t)| \leq L |s - t| \text{ for any } t, s \in [a, b],$$

then the Riemann-Stieltjes integral $\int_a^b p(t)\, dv(t)$ exists and the following inequality holds

$$\left|\int_a^b p(t)\, dv(t)\right| \leq L \int_a^b |p(t)|\, dt. \tag{3.11}$$

If we use the inequality (3.11), then we have in the case when g is Lipschitzian with the constant $K > 0$ that

$$\begin{aligned} &\max_{\lambda \in [m,M]} \left|\int_{m-0}^{M} [\langle E_\lambda x, x\rangle \langle x, F_\mu x\rangle - \langle E_\lambda x, F_\mu x\rangle]\, d(g(\mu))\right| \\ &\leq K \max_{\lambda \in [m,M]} \left[\int_{m-0}^{M} |\langle E_\lambda x, x\rangle \langle x, F_\mu x\rangle - \langle E_\lambda x, F_\mu x\rangle|\, d\mu\right] \end{aligned} \tag{3.12}$$

for any $x \in H$ with $\|x\| = 1$ and the first part of (3.2) is proved.

Further, by employing (3.8)-(3.10) we also get that

$$\begin{aligned}\max_{\lambda\in[m,M]}\int_{m-0}^{M}&\left|\langle E_{\lambda}x,x\rangle\langle x,F_{\mu}x\rangle-\langle E_{\lambda}x,F_{\mu}x\rangle\right|d\mu\\&\le\max_{\lambda\in[m,M]}\left[\langle E_{\lambda}x,x\rangle\langle(1_{H}-E_{\lambda})x,x\rangle\right]^{1/2}\\&\times\int_{m-0}^{M}\left[\langle F_{\mu}x,x\rangle\langle(1_{H}-F_{\mu})x,x\rangle\right]^{1/2}d\mu\end{aligned}\tag{3.13}$$

for any $x\in H$ with $\|x\|=1$.

If we use the Cauchy-Bunyakovsky-Schwarz integral inequality and the spectral representation (1.7), then we have successively

$$\begin{aligned}\int_{m-0}^{M}&\left(\langle F_{\mu}x,x\rangle\langle(1_{H}-F_{\mu})x,x\rangle\right)^{1/2}d\mu\\&\le\left[\int_{m-0}^{M}\langle F_{\mu}x,x\rangle d\mu\right]^{1/2}\left[\int_{m-0}^{M}\langle(1_{H}-F_{\mu})x,x\rangle d\mu\right]^{1/2}\\&=\left[\langle F_{\mu}x,x\rangle\mu|_{m-0}^{M}-\int_{m-0}^{M}\mu d\langle F_{\mu}x,x\rangle\right]^{1/2}\\&\times\left[\langle(1_{H}-F_{\mu})x,x\rangle\mu|_{m-0}^{M}-\int_{m-0}^{M}\mu d\langle(1_{H}-F_{\mu})x,x\rangle\right]\\&=\langle(M1_{H}-B)x,x\rangle^{1/2}\langle(B-m1_{H})x,x\rangle^{1/2},\end{aligned}\tag{3.14}$$

for any $x\in H$ with $\|x\|=1$.

On employing now (3.12)-(3.14) we deduce the second part of (3.2).

The last part of (3.2) follows by the elementary inequality

$$\alpha\beta\le\frac{1}{4}(\alpha+\beta)^{2},\alpha\beta\ge0\tag{3.15}$$

for the choice $\alpha=\langle(M1_{H}-B)x,x\rangle$ and $\beta=\langle(B-m1_{H})x,x\rangle$ and the details are omitted.

3. Further, from the theory of Riemann-Stieltjes integral it is also well known that if $p:[a,b]\to\mathbb{C}$ is of bounded variation and $v:[a,b]\to\mathbb{R}$ is continuous and monotonic nondecreasing, then the Riemann-Stieltjes integrals $\int_a^b p(t)\,dv(t)$ and $\int_a^b |p(t)|\,dv(t)$ exist and

$$\left|\int_{a}^{b}p(t)\,dv(t)\right|\le\int_{a}^{b}|p(t)|\,dv(t).\tag{3.16}$$

Now, if we assume that g is monotonic nondecreasing on $[m,M]$, then by (3.16) we have that

$$\begin{aligned}\max_{\lambda\in[m,M]}&\left|\int_{m-0}^{M}\left[\langle E_{\lambda}x,x\rangle\langle x,F_{\mu}x\rangle-\langle E_{\lambda}x,F_{\mu}x\rangle\right]d(g(\mu))\right|\\&\le\max_{\lambda\in[m,M]}\left[\int_{m-0}^{M}\left|\langle E_{\lambda}x,x\rangle\langle x,F_{\mu}x\rangle-\langle E_{\lambda}x,F_{\mu}x\rangle\right|dg(\mu)\right]\end{aligned}\tag{3.17}$$

for any $x\in H$ with $\|x\|=1$.

Further, by employing (3.8)-(3.10) we also get that

$$\max_{\lambda\in[m,M]}\int_{m-0}^{M}\left|\left\langle E_{\lambda}x,x\right\rangle\left\langle x,F_{\mu}x\right\rangle-\left\langle E_{\lambda}x,F_{\mu}x\right\rangle\right|dg\left(\mu\right) \tag{3.18}$$
$$\leq\max_{\lambda\in[m,M]}\left[\left\langle E_{\lambda}x,x\right\rangle\left\langle\left(1_{H}-E_{\lambda}\right)x,x\right\rangle\right]^{1/2}$$
$$\times\int_{m-0}^{M}\left[\left\langle F_{\mu}x,x\right\rangle\left\langle\left(1_{H}-F_{\mu}\right)x,x\right\rangle\right]^{1/2}dg\left(\mu\right)$$

for any $x\in H$ with $\|x\|=1$. These prove the first part of (3.3).

If we use the Cauchy-Bunyakovsky-Schwarz integral inequality for the Riemann-Stieltjes integral with monotonic nondecreasing integrators and the spectral representation (1.7), then we have successively

$$\int_{m-0}^{M}\left(\left\langle F_{\mu}x,x\right\rangle\left\langle\left(1_{H}-F_{\mu}\right)x,x\right\rangle\right)^{1/2}dg\left(\mu\right) \tag{3.19}$$
$$\leq\left[\int_{m-0}^{M}\left\langle F_{\mu}x,x\right\rangle dg\left(\mu\right)\right]^{1/2}\left[\int_{m-0}^{M}\left\langle\left(1_{H}-F_{\mu}\right)x,x\right\rangle dg\left(\mu\right)\right]^{1/2}$$
$$=\left[\left\langle F_{\mu}x,x\right\rangle g\left(\mu\right)\Big|_{m-0}^{M}-\int_{m-0}^{M}g\left(\mu\right)d\left\langle F_{\mu}x,x\right\rangle\right]^{1/2}$$
$$\times\left[\left\langle\left(1_{H}-F_{\mu}\right)x,x\right\rangle g\left(\mu\right)\Big|_{m-0}^{M}-\int_{m-0}^{M}g\left(\mu\right)d\left\langle\left(1_{H}-F_{\mu}\right)x,x\right\rangle\right]^{1/2}$$
$$=\left\langle\left(g\left(M\right)1_{H}-g\left(B\right)\right)x,x\right\rangle^{1/2}\left\langle\left(g\left(B\right)-g\left(m\right)1_{H}\right)x,x\right\rangle^{1/2},$$

for any $x\in H$ with $\|x\|=1$.

Utilising (3.19) we then deduce the last part of (3.3). The details are omitted. □

Now, in order to provide other results that are similar to the Grüss' type inequalities stated in the introduction, we can state the following corollary:

Corollary 1. *Let A be a selfadjoint operators in the Hilbert space H with the spectrum $Sp\left(A\right)\subseteq\left[m,M\right]$ for some real numbers $m<M$ and let $\left\{E_{\lambda}\right\}_{\lambda}$ be the spectral family of A. Also, assume that $f:\left[m,M\right]\to\mathbb{C}$ is continuous and of bounded variation on $\left[m,M\right]$.*

1. If $g:\left[m,M\right]\to\mathbb{C}$ is continuous and of bounded variation on $\left[m,M\right]$, then we have the inequality

$$\left|\left\langle f\left(A\right)x,g\left(A\right)x\right\rangle-\left\langle f\left(A\right)x,x\right\rangle\left\langle x,g\left(A\right)x\right\rangle\right| \tag{3.20}$$
$$\leq\max_{(\lambda,\mu)\in[m,M]^{2}}\left|\left\langle E_{\lambda}x,x\right\rangle\left\langle x,E_{\mu}x\right\rangle-\left\langle E_{\lambda}x,E_{\mu}x\right\rangle\right|\bigvee_{m}^{M}\left(g\right)\bigvee_{m}^{M}\left(f\right)$$
$$\leq\max_{\lambda\in[m,M]}\left[\left\langle E_{\lambda}x,x\right\rangle\left\langle\left(1_{H}-E_{\lambda}\right)x,x\right\rangle\right]\bigvee_{m}^{M}\left(g\right)\bigvee_{m}^{M}\left(f\right)\leq\frac{1}{4}\bigvee_{m}^{M}\left(g\right)\bigvee_{m}^{M}\left(f\right)$$

for any $x\in H$ with $\|x\|=1$.

2. If $g:[m,M]\to\mathbb{C}$ is Lipschitzian with the constant $K>0$ on $[m,M]$, then we have the inequality

$$\begin{aligned}
(3.21)\qquad & |\langle f(A)x,g(A)x\rangle-\langle f(A)x,x\rangle\langle x,g(A)x\rangle| \\
&\le K\max_{\lambda\in[m,M]}\left[\int_{m-0}^{M}|\langle E_\lambda x,x\rangle\langle x,E_\mu x\rangle-\langle E_\lambda x,E_\mu x\rangle|\,d\mu\right]\bigvee_m^M(f) \\
&\le K\bigvee_m^M(f)\max_{\lambda\in[m,M]}\left[\langle E_\lambda x,x\rangle\langle(1_H-E_\lambda)x,x\rangle\right]^{1/2} \\
&\times\int_{m-0}^{M}\left[\langle E_\mu x,x\rangle\langle(1_H-E_\mu)x,x\rangle\right]^{1/2}d\mu \\
&\le\frac{1}{2}K\bigvee_m^M(f)\langle(M1_H-A)x,x\rangle^{1/2}\langle(A-m1_H)x,x\rangle^{1/2} \\
&\le\frac{1}{4}K(M-m)\bigvee_m^M(f)
\end{aligned}$$

for any $x\in H$ with $\|x\|=1$.

3. If $g:[m,M]\to\mathbb{R}$ is continuous and monotonic nondecreasing on $[m,M]$, then we have the inequality

$$\begin{aligned}
(3.22)\qquad & |\langle f(A)x,g(A)x\rangle-\langle f(A)x,x\rangle\langle x,g(A)x\rangle| \\
&\le\max_{\lambda\in[m,M]}\left[\int_{m-0}^{M}|\langle E_\lambda x,x\rangle\langle x,E_\mu x\rangle-\langle E_\lambda x,E_\mu x\rangle|\,dg(\mu)\right]\bigvee_m^M(f) \\
&\le\bigvee_m^M(f)\max_{\lambda\in[m,M]}\left[\langle E_\lambda x,x\rangle\langle(1_H-E_\lambda)x,x\rangle\right]^{1/2} \\
&\times\int_{m-0}^{M}\left[\langle E_\mu x,x\rangle\langle(1_H-E_\mu)x,x\rangle\right]^{1/2}dg(\mu) \\
&\le\frac{1}{2}\bigvee_m^M(f)\langle(g(M)1_H-g(A))x,x\rangle^{1/2}\langle(g(A)-g(m)1_H)x,x\rangle^{1/2} \\
&\le\frac{1}{4}[g(M)-g(m)]\bigvee_m^M(f)
\end{aligned}$$

for any $x\in H$ with $\|x\|=1$.

Remark 3. *The following inequality for the variance of $f(A)$ under the assumptions that A is a selfadjoint operators in the Hilbert space H with the spectrum $Sp(A)\subseteq[m,M]$ for some real numbers $m<M$, $\{E_\lambda\}_\lambda$ is the spectral family of A*

and $f:[m,M]\to\mathbb{C}$ is continuous and of bounded variation on $[m,M]$ can be stated

$$0\le\|f(A)x\|^2-\langle f(A)x,x\rangle^2 \tag{3.23}$$
$$\le\max_{(\lambda,\mu)\in[m,M]^2}|\langle E_\lambda x,x\rangle\langle x,E_\mu x\rangle-\langle E_\lambda x,E_\mu x\rangle|\left[\bigvee_m^M(f)\right]^2$$
$$\le\max_{\lambda\in[m,M]}[\langle E_\lambda x,x\rangle\langle(1_H-E_\lambda)x,x\rangle]\left[\bigvee_m^M(f)\right]^2\le\frac{1}{4}\left[\bigvee_m^M(f)\right]^2$$

for any $x\in H$ with $\|x\|=1$.

4. Bounds for f Lipschitzian

The case when the first function is Lipschitzian is as follows:

Theorem 7. *Let A,B be two selfadjoint operators in the Hilbert space H with the spectra $Sp(A),Sp(B)\subseteq[m,M]$ for some real numbers $m<M$ and let $\{E_\lambda\}_\lambda$ be the spectral family of A and $\{F_\mu\}_\mu$ the spectral family of B. Also, assume that $f:[m,M]\to\mathbb{C}$ is Lipschitzian with the constant $L>0$ on $[m,M]$.*

1. If $g:[m,M]\to\mathbb{C}$ is Lipschitzian with the constant $K>0$ on $[m,M]$, then we have the inequality

$$|\langle f(A)x,g(B)x\rangle-\langle f(A)x,x\rangle\langle x,g(B)x\rangle| \tag{4.1}$$
$$\le LK\int_{m-0}^{M}\int_{m-0}^{M}|\langle E_\lambda x,x\rangle\langle x,F_\mu x\rangle-\langle E_\lambda x,F_\mu x\rangle|\,d\mu d\lambda$$
$$\le LK\int_{m-0}^{M}[\langle E_\lambda x,x\rangle\langle(1_H-E_\lambda)x,x\rangle]^{1/2}\,d\lambda$$
$$\times\int_{m-0}^{M}[\langle F_\mu x,x\rangle\langle(1_H-F_\mu)x,x\rangle]^{1/2}\,d\mu$$
$$\le LK[\langle(M1_H-A)x,x\rangle\langle(A-m1_H)x,x\rangle]^{1/2}$$
$$\times[\langle(M1_H-B)x,x\rangle\langle(B-m1_H)x,x\rangle]^{1/2}\le\frac{1}{4}LK(M-m)^2$$

for any $x\in H$ with $\|x\|=1$.

2. If $g : [m, M] \to \mathbb{R}$ is continuous and monotonic nondecreasing on $[m, M]$, then we have the inequality

$$
\begin{aligned}
&|\langle f(A)x, g(B)x\rangle - \langle f(A)x, x\rangle \langle x, g(B)x\rangle| \\
&\le L \int_{m-0}^{M} \int_{m-0}^{M} |\langle E_\lambda x, x\rangle \langle x, F_\mu x\rangle - \langle E_\lambda x, F_\mu x\rangle| \, dg(\mu)\, d\lambda \\
&\le L \int_{m-0}^{M} [\langle E_\lambda x, x\rangle \langle (1_H - E_\lambda) x, x\rangle]^{1/2} d\lambda \\
&\times \int_{m-0}^{M} [\langle F_\mu x, x\rangle \langle (1_H - F_\mu) x, x\rangle]^{1/2} dg(\mu) \\
&\le L [\langle (M 1_H - A) x, x\rangle \langle (A - m 1_H) x, x\rangle]^{1/2} \\
&\times [\langle (g(M) 1_H - g(B)) x, x\rangle \langle (g(B) - g(m) 1_H) x, x\rangle]^{1/2} \\
&\le \frac{1}{4} L (M - m) [g(M) - g(m)]
\end{aligned} \tag{4.2}
$$

for any $x \in H$ with $\|x\| = 1$.

Proof. 1. We observe that, on utilizing the property (3.11) and the identity (2.1) we have

$$
\begin{aligned}
&|\langle f(A)x, g(B)x\rangle - \langle f(A)x, x\rangle \langle x, g(B)x\rangle| \\
&\le L \int_{m-0}^{M} \left| \int_{m-0}^{M} [\langle E_\lambda x, x\rangle \langle x, F_\mu x\rangle - \langle E_\lambda x, F_\mu x\rangle] \, d(g(\mu)) \right| d\lambda
\end{aligned} \tag{4.3}
$$

for any $x \in H, \|x\| = 1$.

By the same property (3.11) we also have

$$
\begin{aligned}
&\left| \int_{m-0}^{M} [\langle E_\lambda x, x\rangle \langle x, F_\mu x\rangle - \langle E_\lambda x, F_\mu x\rangle] \, d(g(\mu)) \right| \\
&\le K \int_{m-0}^{M} |\langle E_\lambda x, x\rangle \langle x, F_\mu x\rangle - \langle E_\lambda x, F_\mu x\rangle| \, d\mu
\end{aligned} \tag{4.4}
$$

for any $x \in H, \|x\| = 1$ and $\lambda \in [m, M]$.

Therefore, by (4.3) and (4.4) we get

$$
\begin{aligned}
&|\langle f(A)x, g(B)x\rangle - \langle f(A)x, x\rangle \langle x, g(B)x\rangle| \\
&\le LK \int_{m-0}^{M} \int_{m-0}^{M} |\langle E_\lambda x, x\rangle \langle x, F_\mu x\rangle - \langle E_\lambda x, F_\mu x\rangle| \, d\mu d\lambda
\end{aligned} \tag{4.5}
$$

for any $x \in H, \|x\| = 1$, which proves the first inequality in (4.1).

From (3.8)-(3.10) we have

$$
\begin{aligned}
&|\langle E_\lambda x, x\rangle \langle x, F_\mu x\rangle - \langle E_\lambda x, F_\mu x\rangle| \\
&\le [\langle E_\lambda x, x\rangle \langle (1_H - E_\lambda) x, x\rangle]^{1/2} [\langle F_\mu x, x\rangle \langle (1_H - F_\mu) x, x\rangle]^{1/2}
\end{aligned} \tag{4.6}
$$

for any $x \in H, \|x\| = 1$ and $\lambda, \mu \in [m, M]$.

Integrating on $[m,M]^2$ the inequality (4.6) and utilizing the Cauchy-Bunyakowsky-Schwarz integral inequality for the Riemann integral we have

$$\begin{aligned}
(4.7)\qquad & \int_{m-0}^{M}\int_{m-0}^{M}\left|\langle E_\lambda x,x\rangle\langle x,F_\mu x\rangle-\langle E_\lambda x,F_\mu x\rangle\right|d\mu d\lambda \\
& \le \int_{m-0}^{M}\left[\langle E_\lambda x,x\rangle\left\langle\left(1_H-E_\lambda\right)x,x\right\rangle\right]^{1/2}d\lambda \\
& \times \int_{m-0}^{M}\left[\langle F_\mu x,x\rangle\left\langle\left(1_H-F_\mu\right)x,x\right\rangle\right]^{1/2}d\mu \\
& \le \left[\int_{m-0}^{M}\langle E_\lambda x,x\rangle d\lambda\right]^{1/2}\left[\int_{m-0}^{M}\left\langle\left(1_H-E_\lambda\right)x,x\right\rangle d\lambda\right]^{1/2} \\
& \times \left[\int_{m-0}^{M}\langle F_\mu x,x\rangle d\mu\right]^{1/2}\left[\int_{m-0}^{M}\left\langle\left(1_H-F_\mu\right)x,x\right\rangle d\mu\right]^{1/2}.
\end{aligned}$$

Integrating by parts and utilizing the integral representation (1.7) we have

$$\begin{aligned}
\int_{m-0}^{M}\langle E_\lambda x,x\rangle d\lambda &= \langle E_\lambda x,x\rangle\lambda\Big|_{m-0}^{M}-\int_{m-0}^{M}\lambda d\langle E_\lambda x,x\rangle \\
&= M-\langle Ax,x\rangle=\langle(M1_H-A)x,x\rangle,
\end{aligned}$$

$$\int_{m-0}^{M}\left\langle\left(1_H-E_\lambda\right)x,x\right\rangle d\lambda=\langle(A-m1_H)x,x\rangle$$

and the similar equalities for B, providing the second part of (4.1).

The last part follows from (3.15) and we omit the details.

2. Utilising the inequality (3.16) we have

$$\begin{aligned}
(4.8)\qquad & \left|\int_{m-0}^{M}\left[\langle E_\lambda x,x\rangle\langle x,F_\mu x\rangle-\langle E_\lambda x,F_\mu x\rangle\right]d\left(g\left(\mu\right)\right)\right| \\
& \le \int_{m-0}^{M}\left|\langle E_\lambda x,x\rangle\langle x,F_\mu x\rangle-\langle E_\lambda x,F_\mu x\rangle\right|dg\left(\mu\right)
\end{aligned}$$

which, together with (4.3), produces the inequality

$$\begin{aligned}
(4.9)\qquad & \left|\langle f(A)x,g(B)x\rangle-\langle f(A)x,x\rangle\langle x,g(B)x\rangle\right| \\
& \le L\int_{m-0}^{M}\int_{m-0}^{M}\left|\langle E_\lambda x,x\rangle\langle x,F_\mu x\rangle-\langle E_\lambda x,F_\mu x\rangle\right|dg\left(\mu\right)d\lambda
\end{aligned}$$

for any $x\in H$, $\|x\|=1$.

Now, by utilizing (4.6) and a similar argument to the one outlined above, we deduce the desired result (4.2) and the details are omitted. □

The case of one operator is incorporated in

Corollary 2. *Let A be a selfadjoint operators in the Hilbert space H with the spectrum $Sp(A)\subseteq[m,M]$ for some real numbers $m<M$ and let $\{E_\lambda\}_\lambda$ be the spectral family of A. Also, assume that $f:[m,M]\to\mathbb{C}$ is Lipschitzian with the constant $L>0$ on $[m,M]$.*

1. If $g:[m,M]\to\mathbb{C}$ is Lipschitzian with the constant $K>0$ on $[m,M]$, then we have the inequality

$$\begin{aligned}
&\left|\left\langle f(A)x,g(A)x\right\rangle-\left\langle f(A)x,x\right\rangle\left\langle x,g(A)x\right\rangle\right| \\
&\leq LK\int_{m-0}^{M}\int_{m-0}^{M}\left|\left\langle E_{\lambda}x,x\right\rangle\left\langle x,E_{\mu}x\right\rangle-\left\langle E_{\lambda}x,E_{\mu}x\right\rangle\right|d\mu d\lambda \\
&\leq LK\left(\int_{m-0}^{M}\left[\left\langle E_{\lambda}x,x\right\rangle\left\langle\left(1_H-E_{\lambda}\right)x,x\right\rangle\right]^{1/2}d\lambda\right)^{2} \\
&\leq LK\left[\left\langle\left(M1_H-A\right)x,x\right\rangle\left\langle\left(A-m1_H\right)x,x\right\rangle\right]\leq\frac{1}{4}LK\left(M-m\right)^{2}
\end{aligned}\tag{4.10}$$

for any $x\in H$ with $\|x\|=1$.

2. If $g:[m,M]\to\mathbb{R}$ is continuous and monotonic nondecreasing on $[m,M]$, then we have the inequality

$$\begin{aligned}
&\left|\left\langle f(A)x,g(A)x\right\rangle-\left\langle f(A)x,x\right\rangle\left\langle x,g(A)x\right\rangle\right| \\
&\leq L\int_{m-0}^{M}\int_{m-0}^{M}\left|\left\langle E_{\lambda}x,x\right\rangle\left\langle x,F_{\mu}x\right\rangle-\left\langle E_{\lambda}x,F_{\mu}x\right\rangle\right|dg(\mu)\,d\lambda \\
&\leq L\int_{m-0}^{M}\left[\left\langle E_{\lambda}x,x\right\rangle\left\langle\left(1_H-E_{\lambda}\right)x,x\right\rangle\right]^{1/2}d\lambda \\
&\times\int_{m-0}^{M}\left[\left\langle E_{\mu}x,x\right\rangle\left\langle\left(1_H-E_{\mu}\right)x,x\right\rangle\right]^{1/2}dg(\mu) \\
&\leq L\left[\left\langle\left(M1_H-A\right)x,x\right\rangle\left\langle\left(A-m1_H\right)x,x\right\rangle\right]^{1/2} \\
&\times\left[\left\langle\left(g(M)1_H-g(A)\right)x,x\right\rangle\left\langle\left(g(A)-g(m)1_H\right)x,x\right\rangle\right]^{1/2} \\
&\leq\frac{1}{4}L\left(M-m\right)\left[g(M)-g(m)\right]
\end{aligned}\tag{4.11}$$

for any $x\in H$ with $\|x\|=1$.

Remark 4. *The following inequality for the variance of $f(A)$ under the assumptions that A is a selfadjoint operators in the Hilbert space H with the spectrum $Sp(A)\subseteq[m,M]$ for some real numbers $m<M$, $\{E_\lambda\}_\lambda$ is the spectral family of A and $f:[m,M]\to\mathbb{C}$ is Lipschitzian with the constant $L>0$ on $[m,M]$ can be stated*

$$\begin{aligned}
&0\leq\left\|f(A)x\right\|^{2}-\left\langle f(A)x,x\right\rangle^{2} \\
&\leq L^{2}\int_{m-0}^{M}\int_{m-0}^{M}\left|\left\langle E_{\lambda}x,x\right\rangle\left\langle x,E_{\mu}x\right\rangle-\left\langle E_{\lambda}x,E_{\mu}x\right\rangle\right|d\mu d\lambda \\
&\leq L^{2}\left(\int_{m-0}^{M}\left[\left\langle E_{\lambda}x,x\right\rangle\left\langle\left(1_H-E_{\lambda}\right)x,x\right\rangle\right]^{1/2}d\lambda\right)^{2} \\
&\leq L^{2}\left[\left\langle\left(M1_H-A\right)x,x\right\rangle\left\langle\left(A-m1_H\right)x,x\right\rangle\right]\leq\frac{1}{4}L^{2}\left(M-m\right)^{2}
\end{aligned}\tag{4.12}$$

for any $x\in H$ with $\|x\|=1$.

5. Bounds for f Monotonic Nondecreasing

Finally, for the case of two monotonic functions we have the following result as well:

Theorem 8. *Let A, B be two selfadjoint operators in the Hilbert space H with the spectra $Sp(A), Sp(B) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be the spectral family of A and $\{F_\mu\}_\mu$ the spectral family of B. If $f, g : [m, M] \to \mathbb{C}$ are continuous and monotonic nondecreasing on $[m, M]$, then*

$$\begin{aligned}
&|\langle f(A)x, g(B)x\rangle - \langle f(A)x, x\rangle \langle x, g(B)x\rangle| \\
&\leq \int_{m-0}^{M} \int_{m-0}^{M} |\langle E_\lambda x, x\rangle \langle x, F_\mu x\rangle - \langle E_\lambda x, F_\mu x\rangle| \, dg(\mu) \, df(\lambda) \\
&\leq \int_{m-0}^{M} [\langle E_\lambda x, x\rangle \langle (1_H - E_\lambda) x, x\rangle]^{1/2} df(\lambda) \\
&\times \int_{m-0}^{M} [\langle F_\mu x, x\rangle \langle (1_H - F_\mu) x, x\rangle]^{1/2} dg(\mu) \\
&\leq [\langle (f(M) 1_H - f(A)) x, x\rangle \langle (f(A) - f(m) 1_H) x, x\rangle]^{1/2} \\
&\times [\langle (g(M) 1_H - g(B)) x, x\rangle \langle (g(B) - g(m) 1_H) x, x\rangle]^{1/2} \\
&\leq \frac{1}{4} [f(M) - f(m)] [g(M) - g(m)]
\end{aligned} \tag{5.1}$$

for any $x \in H, \|x\| = 1$.

The details of the proof are omitted.

In particular we have:

Corollary 3. *Let A be a selfadjoint operators in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be the spectral family of A. If $f, g : [m, M] \to \mathbb{C}$ are continuous and monotonic nondecreasing on $[m, M]$, then*

$$\begin{aligned}
&|\langle f(A)x, g(A)x\rangle - \langle f(A)x, x\rangle \langle x, g(A)x\rangle| \\
&\leq \int_{m-0}^{M} \int_{m-0}^{M} |\langle E_\lambda x, x\rangle \langle x, E_\mu x\rangle - \langle E_\lambda x, E_\mu x\rangle| \, dg(\mu) \, df(\lambda) \\
&\leq \int_{m-0}^{M} [\langle E_\lambda x, x\rangle \langle (1_H - E_\lambda) x, x\rangle]^{1/2} df(\lambda) \\
&\times \int_{m-0}^{M} [\langle E_\mu x, x\rangle \langle (1_H - E_\mu) x, x\rangle]^{1/2} dg(\mu) \\
&\leq [\langle (f(M) 1_H - f(A)) x, x\rangle \langle (f(A) - f(m) 1_H) x, x\rangle]^{1/2} \\
&\times [\langle (g(M) 1_H - g(A)) x, x\rangle \langle (g(A) - g(m) 1_H) x, x\rangle]^{1/2} \\
&\leq \frac{1}{4} [f(M) - f(m)] [g(M) - g(m)]
\end{aligned} \tag{5.2}$$

for any $x \in H, \|x\| = 1$.

In particular, the following inequality for the variance of $f(A)$ in the case of monotonic nondecreasing functions f holds:

$$
\begin{aligned}
(5.3)\qquad 0 &\leq \|f(A)x\|^2 - \langle f(A)x, x\rangle^2 \\
&\leq \int_{m-0}^{M}\int_{m-0}^{M} |\langle E_\lambda x, x\rangle \langle x, E_\mu x\rangle - \langle E_\lambda x, E_\mu x\rangle|\, df(\mu)\, df(\lambda) \\
&\leq \left(\int_{m-0}^{M} [\langle E_\lambda x, x\rangle \langle (1_H - E_\lambda)x, x\rangle]^{1/2}\, df(\lambda)\right)^2 \\
&\leq [\langle (f(M)1_H - f(A))x, x\rangle \langle (f(A) - f(m)1_H)x, x\rangle] \\
&\leq \frac{1}{4}[f(M) - f(m)]^2
\end{aligned}
$$

for any $x \in H, \|x\| = 1$.

6. Applications

By choosing different examples of elementary functions into the above inequalities, one can obtain various Grüss' type inequalities of interest.

For instance, if we choose $f, g : (0, \infty) \to (0, \infty)$ with $f(t) = t^p, g(t) = t^q$ and $p, q > 0$, then for any selfadjoint operators A, B with $Sp(A), Sp(B) \subseteq [m, M] \subset (0, \infty)$ we get from (5.1) the inequalities:

$$
\begin{aligned}
(6.1)\qquad & |\langle A^p x, B^q x\rangle - \langle A^p x, x\rangle \langle B^q x, x\rangle| \\
&\leq pq \int_{m-0}^{M}\int_{m-0}^{M} |\langle E_\lambda x, x\rangle \langle x, F_\mu x\rangle - \langle E_\lambda x, F_\mu x\rangle|\, \mu^{q-1}\lambda^{p-1} d\mu d\lambda \\
&\leq pq \int_{m-0}^{M} [\langle E_\lambda x, x\rangle \langle (1_H - E_\lambda)x, x\rangle]^{1/2}\, \lambda^{p-1} d\lambda \\
&\times \int_{m-0}^{M} [\langle F_\mu x, x\rangle \langle (1_H - F_\mu)x, x\rangle]^{1/2}\, \mu^{q-1} d\mu \\
&\leq [\langle (M^p 1_H - A^p)x, x\rangle \langle (A^p - m^p 1_H)x, x\rangle]^{1/2} \\
&\times [\langle (M^q 1_H - B^q)x, x\rangle \langle (B^q - m^q 1_H)x, x\rangle]^{1/2} \\
&\leq \frac{1}{4}(M^p - m^p)(M^q - m^q)
\end{aligned}
$$

for any $x \in H$ with $\|x\| = 1$, where $\{E_\lambda\}_\lambda$ is the spectral family of A and $\{F_\mu\}_\mu$ is the spectral family of B.

When $B=A$ then by the Čebyšev's inequality for functions of same monotonicity the inequality (6.1) becomes

$$
\begin{aligned}
(6.2)\qquad & 0 \leq \langle A^{p}x, A^{q}x\rangle - \langle A^{p}x, x\rangle \langle A^{q}x, x\rangle \\
& \leq pq \int_{m-0}^{M}\int_{m-0}^{M} \left|\langle E_{\lambda}x, x\rangle \langle x, E_{\mu}x\rangle - \langle E_{\lambda}x, E_{\mu}x\rangle\right| \mu^{q-1}\lambda^{p-1} d\mu d\lambda \\
& \leq pq \int_{m-0}^{M} \left[\langle E_{\lambda}x, x\rangle \langle (1_{H}-E_{\lambda})x, x\rangle\right]^{1/2} \lambda^{p-1} d\lambda \\
& \times \int_{m-0}^{M} \left[\langle E_{\mu}x, x\rangle \langle (1_{H}-E_{\mu})x, x\rangle\right]^{1/2} \mu^{q-1} d\mu \\
& \leq \left[\langle (M^{p}1_{H}-A^{p})x, x\rangle \langle (A^{p}-m^{p}1_{H})x, x\rangle\right]^{1/2} \\
& \times \left[\langle (M^{q}1_{H}-B^{q})x, x\rangle \langle (B^{q}-m^{q}1_{H})x, x\rangle\right]^{1/2} \\
& \leq \frac{1}{4}\left(M^{p}-m^{p}\right)\left(M^{q}-m^{q}\right)
\end{aligned}
$$

for any $x \in H$ with $\|x\|=1$ and $p, q > 0$.

Now. define the coefficients

$$
(6.3)\qquad \Delta_{p} := p \times \begin{cases} M^{p-1}-m^{p-1} \text{ if } p \geq 1 \\ \\ \frac{M^{1-p}-m^{1-p}}{M^{1-p}m^{1-p}} \text{ if } 0<p<1. \end{cases}
$$

On utilizing the inequality (4.1) for the same power functions considered above, we can state the inequality

$$
\begin{aligned}
(6.4)\qquad & \left|\langle A^{p}x, B^{q}x\rangle - \langle A^{p}x, x\rangle \langle B^{q}x, x\rangle\right| \\
& \leq \Delta_{p}\Delta_{q} \int_{m-0}^{M}\int_{m-0}^{M} \left|\langle E_{\lambda}x, x\rangle \langle x, F_{\mu}x\rangle - \langle E_{\lambda}x, F_{\mu}x\rangle\right| d\mu d\lambda \\
& \leq \Delta_{p}\Delta_{q} \int_{m-0}^{M} \left[\langle E_{\lambda}x, x\rangle \langle (1_{H}-E_{\lambda})x, x\rangle\right]^{1/2} d\lambda \\
& \times \int_{m-0}^{M} \left[\langle F_{\mu}x, x\rangle \langle (1_{H}-F_{\mu})x, x\rangle\right]^{1/2} d\mu \\
& \leq \Delta_{p}\Delta_{q} \left[\langle (M1_{H}-A)x, x\rangle \langle (A-m1_{H})x, x\rangle\right]^{1/2} \\
& \times \left[\langle (M1_{H}-B)x, x\rangle \langle (B-m1_{H})x, x\rangle\right]^{1/2} \leq \frac{1}{4}\Delta_{p}\Delta_{q}\left(M-m\right)^{2}
\end{aligned}
$$

for any $x \in H$ with $\|x\|=1$ and $p, q > 0$.

In particular, for $B = A$ we have

$$0 \le \langle A^p x, A^q x\rangle - \langle A^p x, x\rangle \langle A^q x, x\rangle \tag{6.5}$$
$$\le \Delta_p \Delta_q \int_{m-0}^{M} \int_{m-0}^{M} |\langle E_\lambda x, x\rangle \langle x, E_\mu x\rangle - \langle E_\lambda x, E_\mu x\rangle| \, d\mu d\lambda$$
$$\le \Delta_p \Delta_q \left(\int_{m-0}^{M} [\langle E_\lambda x, x\rangle \langle (1_H - E_\lambda) x, x\rangle]^{1/2} d\lambda \right)^2$$
$$\le \Delta_p \Delta_q [\langle (M1_H - A) x, x\rangle \langle (A - m1_H) x, x\rangle] \le \frac{1}{4} \Delta_p \Delta_q (M - m)^2$$

for any $x \in H$ with $\|x\| = 1$ and $p, q > 0$.

Similar results can be stated if $p < 0$ or $q < 0$. However the details are left to the interest reader.

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