OSTROWSKI'S TYPE INEQUALITIES FOR SOME CLASSES OF CONTINUOUS FUNCTIONS OF SELFADJOINT OPERATORS IN HILBERT SPACES

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Abstract. New Ostrowski's type inequalities for some classes of continuous functions of selfadjoint operators in Hilbert spaces, under suitable assumptions for the involved operators, are given.

1. Introduction

The following result is known in the literature as Ostrowski's inequality [27]:

Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with the property that $|f'(t)|\le M$ for all $t\in(a,b)$. Then

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)M \tag{1.1}$$

for all $x\in(a,b)$. The constant $\frac{1}{4}$ is best possible in the sense that it cannot be replaced by a smaller constant.

The above result has been naturally extended for absolutely continuous functions and Lebesgue $p-$norms of the derivative f' in [15] – [17] and can be stated as:

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be absolutely continuous on $[a,b]$. Then for all $x\in[a,b]$ we have:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{1.2}$$

$$\le\begin{cases}\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\|f'\|_\infty & \text{if } f'\in L_\infty[a,b];\\[2ex] \frac{1}{(p+1)^{\frac{1}{p}}}\left[\left(\frac{x-a}{b-a}\right)^{p+1}+\left(\frac{b-x}{b-a}\right)^{p+1}\right](b-a)^{\frac{1}{q}}\|f'\|_q \\ \qquad\qquad \text{if } f'\in L_p[a,b],\ \frac{1}{p}+\frac{1}{q}=1,\ p>1;\\[2ex] \left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\|f'\|_1,\end{cases}$$

where $\|\cdot\|_r$ $(r\in[1,\infty])$ are the usual Lebesgue norms on $L_r[a,b]$, i.e.,

$$\|g\|_\infty:=ess\sup_{t\in[a,b]}|g(t)|\quad and\quad \|g\|_r:=\left(\int_a^b|g(t)|^r\,dt\right)^{\frac{1}{r}},\quad r\in[1,\infty).$$

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The constants $\frac{1}{4}$, $\frac{1}{(p+1)^{1/p}}$ *and* $\frac{1}{2}$ *respectively are sharp in the sense mentioned above.*

The inequalities (1.2) can also be obtained, in a slightly different form, as particular cases of some results established by A.M. Fink in [18] for $n-$time differentiable functions.

For other Ostrowski type inequalities concerning Lipschitzian and $r-H-$Hölder type functions, see [8] and [13].

The cases of bounded variation functions and monotonic functions were considered in [4] and [7] while the case of convex functions was studied in [3].

For various generalizations, extensions and related Ostrowski type inequalities for functions of one or several variables see the monograph [14] and the references therein. For related results see [1]-[2] and [9].

In order to extend the Ostrowski inequality for absolutely continuous functions of selfadjoint operators we need some definitions and results related to the *continuous functional calculus.*

Let A be a selfadjoint linear operator on a complex Hilbert space $(H;\langle .,.\rangle)$. The *Gelfand map* establishes a $*$-isometrically isomorphism Φ between the set $C(Sp(A))$ of all *continuous functions* defined on the *spectrum* of A, denoted $Sp(A)$, and the C^*-algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows (see for instance [19, p. 3]):

For any $f,g\in C(Sp(A))$ and any $\alpha,\beta\in\mathbb{C}$ we have

(i) $\Phi(\alpha f+\beta g)=\alpha\Phi(f)+\beta\Phi(g)$;

(ii) $\Phi(fg)=\Phi(f)\Phi(g)$ and $\Phi(\bar{f})=\Phi(f)^*$;

(iii) $\|\Phi(f)\|=\|f\|:=\sup_{t\in Sp(A)}|f(t)|$;

(iv) $\Phi(f_0)=1_H$ and $\Phi(f_1)=A$, where $f_0(t)=1$ and $f_1(t)=t$, for $t\in Sp(A)$.

With this notation we define

$$f(A):=\Phi(f) \text{ for all } f\in C(Sp(A))$$

and we call it the *continuous functional calculus* for a selfadjoint operator A.

If A is a selfadjoint operator and f is a real valued continuous function on $Sp(A)$, then $f(t)\geq 0$ for any $t\in Sp(A)$ implies that $f(A)\geq 0$, *i.e.* $f(A)$ is a *positive operator* on H. Moreover, if both f and g are real valued functions on $Sp(A)$ then the following important property holds:

$$\text{(P)} \qquad f(t)\geq g(t) \text{ for any } t\in Sp(A) \text{ implies that } f(A)\geq g(A)$$

in the operator order of $B(H)$.

For a recent monograph devoted to various inequalities for functions of selfadjoint operators, see [19] and the references therein.

For other results see [10], [11], [12], [23], [25], [26] and [28].

Motivated by the above results we investigate in this paper some Ostrowski type inequalities for absolutely continuous functions of selfadjoint operators in Hilbert spaces. Applications for some particular functions of interest are provided as well.

2. Inequalities for Absolutely Continuous Functions of Selfadjoint Operators

We start with the following scalar inequality that is of interest in itself since it provides a generalization of the Ostrowski inequality (1.1) when upper and lower bounds for the derivative are provided:

Lemma 1. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function whose derivative is bounded above and below on $[a,b]$, i.e., there exists the real constants γ and $\Gamma,\gamma<\Gamma$ with the property that $\gamma\leq f'(s)\leq\Gamma$ for almost every $s\in[a,b]$. Then we have the double inequality*

$$\begin{aligned}(2.1)\qquad & -\frac{1}{2}\cdot\frac{\Gamma-\gamma}{b-a}\left[\left(s-\frac{b\Gamma-a\gamma}{\Gamma-\gamma}\right)^2-\Gamma\gamma\left(\frac{b-a}{\Gamma-\gamma}\right)^2\right]\\ &\leq f(s)-\frac{1}{b-a}\int_a^b f(t)\,dt\\ &\leq\frac{1}{2}\cdot\frac{\Gamma-\gamma}{b-a}\left[\left(s-\frac{a\Gamma-b\gamma}{\Gamma-\gamma}\right)^2-\Gamma\gamma\left(\frac{b-a}{\Gamma-\gamma}\right)^2\right]\end{aligned}$$

for any $s\in[a,b]$. The inequalities are sharp.

Proof. We start with *Montgomery's identity*

$$\begin{aligned}(2.2)\qquad & f(s)-\frac{1}{b-a}\int_a^b f(t)\,dt\\ &=\frac{1}{b-a}\int_a^s(t-a)f'(t)\,dt+\frac{1}{b-a}\int_s^b(t-b)f'(t)\,dt\end{aligned}$$

that holds for any $s\in[a,b]$.

Since $\gamma\leq f'(t)\leq\Gamma$ for almost every $t\in[a,b]$, then

$$\frac{\gamma}{b-a}\int_a^s(t-a)\,dt\leq\frac{1}{b-a}\int_a^s(t-a)f'(t)\,dt\leq\frac{\Gamma}{b-a}\int_a^s(t-a)\,dt$$

and

$$\frac{\Gamma}{b-a}\int_s^b(b-t)\,dt\leq\frac{1}{b-a}\int_s^b(b-t)f'(t)\,dt\leq\frac{\Gamma}{b-a}\int_s^b(b-t)\,dt$$

for any $s\in[a,b]$.

Now, due to the fact that

$$\int_a^s(t-a)\,dt=\frac{1}{2}(s-a)^2\ \text{ and }\ \int_s^b(b-t)\,dt=\frac{1}{2}(b-s)^2$$

then by (2.2) we deduce the following inequality that is of interest in itself:

$$\begin{aligned}(2.3)\qquad & -\frac{1}{2(b-a)}\left[\Gamma(b-s)^2-\gamma(s-a)^2\right]\\ &\leq f(s)-\frac{1}{b-a}\int_a^b f(t)\,dt\\ &\leq\frac{1}{2(b-a)}\left[\Gamma(s-a)^2-\gamma(b-s)^2\right]\end{aligned}$$

for any $s\in[a,b]$.

Further on, if we denote by

$$A:=\gamma(s-a)^2-\Gamma(b-s)^2\ \text{ and }\ B:=\Gamma(s-a)^2-\gamma(b-s)^2$$

then, after some elementary calculations, we derive that

$$A=-(\Gamma-\gamma)\left(s-\frac{b\Gamma-a\gamma}{\Gamma-\gamma}\right)^2+\frac{\Gamma\gamma}{\Gamma-\gamma}(b-a)^2$$

and

$$B=(\Gamma-\gamma)\left(s-\frac{a\Gamma-b\gamma}{\Gamma-\gamma}\right)^{2}-\frac{\Gamma\gamma}{\Gamma-\gamma}(b-a)^{2}$$

which, together with (2.3), produces the desired result (2.1).

The sharpness of the inequalities follow from the sharpness of some particular cases outlined below. The details are omitted. □

Corollary 1. *With the assumptions of Lemma 1 we have the inequalities*

$$\frac{1}{2}\gamma(b-a)\leq\frac{1}{b-a}\int_{a}^{b}f(t)\,dt-f(a)\leq\frac{1}{2}\Gamma(b-a) \tag{2.4}$$

and

$$\frac{1}{2}\gamma(b-a)\leq f(b)-\frac{1}{b-a}\int_{a}^{b}f(t)\,dt\leq\frac{1}{2}\Gamma(b-a) \tag{2.5}$$

and

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_{a}^{b}f(t)\,dt\right|\leq\frac{1}{8}(\Gamma-\gamma)(b-a) \tag{2.6}$$

respectively. The constant $\frac{1}{8}$ is best possible in (2.6).

The proof is obvious from (2.1) on choosing $s=a,s=b$ and $s=\frac{a+b}{2}$, respectively.

Corollary 2. *With the assumptions of Lemma 1 and if, in addition $\gamma=-\alpha$ and $\Gamma=\beta$ with $\alpha,\beta>0$ then*

$$\frac{1}{b-a}\int_{a}^{b}f(t)\,dt-f\left(\frac{b\beta+a\alpha}{\beta+\alpha}\right)\leq\frac{1}{2}\cdot\alpha\beta\left(\frac{b-a}{\beta+\alpha}\right) \tag{2.7}$$

and

$$f\left(\frac{a\beta+b\alpha}{\beta+\alpha}\right)-\frac{1}{b-a}\int_{a}^{b}f(t)\,dt\leq\frac{1}{2}\cdot\alpha\beta\left(\frac{b-a}{\beta+\alpha}\right). \tag{2.8}$$

The proof follows from (2.1) on choosing $s=\frac{b\beta+a\alpha}{\beta+\alpha}\in[a,b]$ and $s=\frac{a\beta+b\alpha}{\beta+\alpha}\in[a,b]$, respectively.

Remark 1. *If $f:[a,b]\to\mathbb{R}$ is absolutely continuous and $\|f'\|_{\infty}:=ess\sup_{t\in[a,b]}|f'(t)|<\infty$, then by choosing $\gamma=-\|f'\|_{\infty}$ and $\Gamma=\|f'\|_{\infty}$ in (2.1) we deduce the classical Ostrowski's inequality for absolutely continuous functions incorporated in the first inequality from Theorem 1. The constant $\frac{1}{4}$ in Ostrowski's inequality is best possible.*

We are able now to state the following result providing upper and lower bounds for absolutely convex functions of selfadjoint operators in Hilbert spaces whose derivatives are bounded below and above:

Theorem 2. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A)\subseteq[m,M]$ for some real numbers $m<M$. If $f:[m,M]\to\mathbb{R}$ is an absolutely continuous function such that there exists the real constants γ and*

$\Gamma, \gamma < \Gamma$ *with the property that* $\gamma \leq f'(s) \leq \Gamma$ *for almost every* $s \in [m, M]$, *then we have the following double inequality in the operator order of* $B(H)$:

$$-\frac{1}{2} \cdot \frac{\Gamma - \gamma}{M - m} \left[\left(A - \frac{M\Gamma - m\gamma}{\Gamma - \gamma} \cdot 1_H \right)^2 - \Gamma\gamma \left(\frac{M - m}{\Gamma - \gamma} \right)^2 \cdot 1_H \right] \tag{2.9}$$
$$\leq f(A) - \left(\frac{1}{M - m} \int_m^M f(t)\, dt \right) \cdot 1_H$$
$$\leq \frac{1}{2} \cdot \frac{\Gamma - \gamma}{M - m} \left[\left(A - \frac{m\Gamma - M\gamma}{\Gamma - \gamma} \cdot 1_H \right)^2 - \Gamma\gamma \left(\frac{M - m}{\Gamma - \gamma} \right)^2 \cdot 1_H \right].$$

The proof follows by the property (P) applied for the inequality (2.1) in Lemma 1.

Theorem 3. *With the assumptions in Theorem 2 we have in the operator order the following inequalities*

$$\left| f(A) - \left(\frac{1}{M - m} \int_m^M f(t)\, dt \right) \cdot 1_H \right| \tag{2.10}$$
$$\leq \begin{cases} \left[\frac{1}{4} 1_H + \left(\frac{A - \frac{m+M}{2} 1_H}{M - m} \right)^2 \right] (M - m) \|f'\|_\infty & \text{if } f' \in L_\infty[m, M]; \\ \frac{1}{(p+1)^{\frac{1}{p}}} \left[\left(\frac{A - m 1_H}{M - m} \right)^{p+1} + \left(\frac{M 1_H - A}{M - m} \right)^{p+1} \right] (M - m)^{\frac{1}{q}} \|f'\|_q & \\ \qquad \text{if } f' \in L_p[m, M],\ \frac{1}{p} + \frac{1}{q} = 1,\ p > 1; & \\ \left[\frac{1}{2} 1_H + \left| \frac{A - \frac{m+M}{2} 1_H}{M - m} \right| \right] \|f'\|_1 . & \end{cases}$$

The proof is obvious by the scalar inequalities from Theorem 1 and the property (P).

The third inequality in (2.10) can be naturally generalized for functions of bounded variation as follows:

Theorem 4. *Let* A *be a selfadjoint operator in the Hilbert space* H *with the spectrum* $Sp(A) \subseteq [m, M]$ *for some real numbers* $m < M$. *If* $f : [m, M] \to \mathbb{R}$ *is a continuous function of bounded variation on* $[m, M]$, *then we have the inequality*

$$\left| f(A) - \left(\frac{1}{M - m} \int_m^M f(t)\, dt \right) \cdot 1_H \right| \leq \left[\frac{1}{2} 1_H + \left| \frac{A - \frac{m+M}{2} 1_H}{M - m} \right| \right] \bigvee_m^M (f) \tag{2.11}$$

where $\bigvee_m^M (f)$ *denotes the total variation of* f *on* $[m, M]$. *The constant* $\frac{1}{2}$ *is best possible in (2.11).*

Proof. Follows from the scalar inequality obtained by the author in [4], namely

$$\left| f(s) - \frac{1}{b - a} \int_a^b f(t)\, dt \right| \leq \left[\frac{1}{2} + \left| \frac{s - \frac{a+b}{2}}{b - a} \right| \right] \bigvee_a^b (f) \tag{2.12}$$

for any $s \in [a, b]$, where f is a function of bounded variation on $[a, b]$. The constant $\frac{1}{2}$ is best possible in (2.12). □

3. Inequalities for Convex Functions of Selfadjoint Operators

The case of convex functions is important for applications.

We need the following lemma.

Lemma 2. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable convex function such that the derivative f' is continuous on (a,b) and with the lateral derivative finite and $f'_-(b)\neq f'_+(a)$. Then we have the following double inequality*

$$-\frac{1}{2}\cdot\frac{f'_-(b)-f'_+(a)}{b-a} \times\left[\left(s-\frac{bf'_-(b)-af'_+(a)}{f'_-(b)-f'_+(a)}\right)^2-f'_-(b)f'_+(a)\left(\frac{b-a}{f'_-(b)-f'_+(a)}\right)^2\right] \leq f(s)-\frac{1}{b-a}\int_a^b f(t)\,dt\leq f'(s)\left(s-\frac{a+b}{2}\right) \tag{3.1}$$

for any $s\in[a,b]$.

Proof. Since f is convex, then by the fact that f' is monotonic nondecreasing, we have

$$\frac{f'_+(a)}{b-a}\int_a^s(t-a)\,dt\leq\frac{1}{b-a}\int_a^s(t-a)f'(t)\,dt\leq\frac{f'(s)}{b-a}\int_a^s(t-a)\,dt$$

and

$$\frac{f'(s)}{b-a}\int_s^b(b-t)\,dt\leq\frac{1}{b-a}\int_s^b(b-t)f'(t)\,dt\leq\frac{f'_-(b)}{b-a}\int_s^b(b-t)\,dt$$

for any $s\in[a,b]$, where $f'_+(a)$ and $f'_-(b)$ are the lateral derivatives in a and b respectively.

Utilising the Montgomery identity (2.2) we then have

$$\frac{f'_+(a)}{b-a}\int_a^s(t-a)\,dt-\frac{f'_-(b)}{b-a}\int_s^b(b-t)\,dt \leq f(s)-\frac{1}{b-a}\int_a^b f(t)\,dt \leq\frac{f'(s)}{b-a}\int_a^s(t-a)\,dt-\frac{f'(s)}{b-a}\int_s^b(b-t)\,dt$$

which is equivalent with the following inequality that is of interest in itself

$$\frac{1}{2(b-a)}\left[f'_+(a)(s-a)^2-f'_-(b)(b-s)^2\right] \leq f(s)-\frac{1}{b-a}\int_a^b f(t)\,dt\leq f'(s)\left(s-\frac{a+b}{2}\right) \tag{3.2}$$

for any $s\in[a,b]$.

A simple calculation reveals now that the left side of (3.2) coincides with the same side of the desired inequality (3.1). □

We are able now to sate our result for convex functions of selfadjoint operators:

Theorem 5. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$. If $f : [m, M] \to \mathbb{R}$ is a differentiable convex function such that the derivative f' is continuous on (m, M) and with the lateral derivative finite and $f'_-(M) \neq f'_+(m)$, then we have the double inequality in the operator order of $B(H)$*

$$-\frac{1}{2} \cdot \frac{f'_-(M) - f'_+(m)}{M - m} \times \left[\left(A - \frac{Mf'_-(M) - mf'_+(m)}{f'_-(M) - f'_+(m)} \cdot 1_H \right)^2 - f'_-(M) f'_+(m) \left(\frac{M - m}{f'_-(M) - f'_+(m)} \right)^2 \cdot 1_H \right] \leq f(A) - \left(\frac{1}{M - m} \int_m^M f(t)\, dt \right) \cdot 1_H \leq \left(A - \frac{m + M}{2} \cdot 1_H \right) f'(A). \tag{3.3}$$

The proof follows from the scalar case in Lemma 2.

Remark 2. *We observe that one can drop the assumption of differentiability of the convex function and will still have the first inequality in (3.3). This follows from the fact that the class of differentiable convex functions is dense in the class of all convex functions defined on a given interval.*

A different lower bound for the quantity

$$f(A) - \left(\frac{1}{M - m} \int_m^M f(t)\, dt \right) \cdot 1_H$$

expressed only in terms of the operator A and not its second power as above, also holds:

Theorem 6. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$. If $f : [m, M] \to \mathbb{R}$ is a convex function on $[m, M]$, then we have the following inequality in the operator order of $B(H)$*

$$f(A) - \left(\frac{1}{M - m} \int_m^M f(t)\, dt \right) \cdot 1_H \geq \left(\frac{1}{M - m} \int_m^M f(t)\, dt \right) \cdot 1_H - \frac{f(M)(M \cdot 1_H - A) + f(m)(A - m \cdot 1_H)}{M - m}. \tag{3.4}$$

Proof. It suffices to prove for the case of differentiable convex functions defined on (m, M).

So, by the gradient inequality we have that

$$f(t) - f(s) \geq (t - s) f'(s)$$

for any $t, s \in (m, M)$.

Now, if we integrate this inequality over $s \in [m, M]$ we get

$$(3.5) \quad (M-m) f(t) - \int_m^M f(s)\,ds \geq \int_m^M (t-s) f'(s)\,ds = \int_m^M f(s)\,ds - (M-t) f(M) - (t-m) f(m)$$

for each $s \in [m, M]$.

Finally, if we apply to the inequality (3.5) the property (P), we deduce the desired result (3.4). □

Corollary 3. *With the assumptions of Theorem 6 we have the following double inequality in the operator order*

$$(3.6) \quad \frac{f(m) + f(M)}{2} \cdot 1_H \geq \frac{1}{2}\left[f(A) + \frac{f(M)(M \cdot 1_H - A) + f(m)(A - m \cdot 1_H)}{M-m}\right] \geq \left(\frac{1}{M-m}\int_m^M f(t)\,dt\right) \cdot 1_H.$$

Proof. The second inequality is equivalent with (3.4).

For the first inequality, we observe, by the convexity of f we have that

$$\frac{f(M)(t-m) + f(m)(M-t)}{M-m} \geq f(t)$$

for any $t \in [m, M]$, which produces the operator inequality

$$(3.7) \quad \frac{f(M)(A - m \cdot 1_H) + f(m)(M \cdot 1_H - A)}{M-m} \geq f(A).$$

Now, if in both sides of (3.7) we add the same quantity

$$\frac{f(M)(M \cdot 1_H - A) + f(m)(A - m \cdot 1_H)}{M-m}$$

and perform the calculations, then we obtain the first part of (3.6) and the proof is complete. □

4. Some Vector Inequalities

Let U be a selfadjoint operator on the complex Hilbert space $(H, \langle .,. \rangle)$ with the spectrum $Sp(U)$ included in the interval $[m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its *spectral family.* Then for any continuous function $f : [m, M] \to \mathbb{R}$, it is well known that we have the following spectral representation in terms of the Riemann-Stieltjes integral:

$$(4.1) \quad \langle f(U) x, y \rangle = \int_{m-0}^M f(\lambda)\, d(\langle E_\lambda x, y \rangle),$$

for any $x, y \in H$. The function $g_{x,y}(\lambda) := \langle E_\lambda x, y \rangle$ is of *bounded variation* and *right continuous* on the interval $[m, M]$ and

$$g_{x,y}(m-0) = 0 \text{ and } g_{x,y}(M) = \langle x, y \rangle$$

for any $x, y \in H$. It is also well known that $g_x(\lambda) := \langle E_\lambda x, x\rangle$ is *monotonic nondecreasing* and right continuous on $[m, M]$.

The following result holds:

Theorem 7. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f : [m, M] \to \mathbb{R}$ is an absolutely continuous function on $[m, M]$, then we have the inequalities*

$$|f(s)\langle x, y\rangle - \langle f(A)x, y\rangle| \le \bigvee_m^M \left(\langle E_{(\cdot)}x, y\rangle\right) \times \begin{cases} \left[\frac{1}{2}(M-m) + \left|s - \frac{m+M}{2}\right|\right] \|f'\|_\infty & \text{if } f' \in L_\infty[m, M] \\ \left[\frac{1}{2}(M-m) + \left|s - \frac{m+M}{2}\right|\right]^{1/q} \|f'\|_p & \text{if } f' \in L_p[m, M], p > 1, \\ & \frac{1}{p} + \frac{1}{q} = 1, \end{cases}$$
$$\le \|x\|\,\|y\| \begin{cases} \left[\frac{1}{2}(M-m) + \left|s - \frac{m+M}{2}\right|\right] \|f'\|_\infty & \text{if } f' \in L_\infty[m, M] \\ \left[\frac{1}{2}(M-m) + \left|s - \frac{m+M}{2}\right|\right]^{1/q} \|f'\|_p & \text{if } f' \in L_p[m, M], p > 1, \\ & \frac{1}{p} + \frac{1}{q} = 1, \end{cases} \tag{4.2}$$

for any $x, y \in H$ and $s \in [m, M]$.

Proof. Since f is absolutely continuous, then we have

$$|f(s) - f(t)| = \left|\int_s^t f'(u)\,du\right| \le \left|\int_s^t |f'(u)|\,du\right| \le \begin{cases} |t-s|\,\|f'\|_\infty & \text{if } f' \in L_\infty[m, M] \\ |t-s|^{1/q}\,\|f'\|_p & \text{if } f' \in L_p[m, M], p > 1, \frac{1}{p} + \frac{1}{q} = 1, \end{cases} \tag{4.3}$$

for any $s, t \in [m, M]$.

It is well known that if $p : [a, b] \to \mathbb{C}$ is a continuous functions and $v : [a, b] \to \mathbb{C}$ is of bounded variation, then the Riemann-Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and the following inequality holds

$$\left|\int_a^b p(t)\,dv(t)\right| \le \max_{t\in[a,b]} |p(t)| \bigvee_a^b (v),$$

where $\bigvee_a^b (v)$ denotes the total variation of v on $[a, b]$.

Now, by the above property of the Riemann-Stieltjes integral we have from the representation (4.1) that

$$
\begin{aligned}
(4.4)\qquad & |f(s)\langle x,y\rangle - \langle f(A)x,y\rangle| \\
&= \left|\int_{m-0}^{M}[f(s)-f(t)]\,d(\langle E_t x,y\rangle)\right| \\
&\leq \max_{t\in[m,M]}|f(s)-f(t)|\bigvee_m^M(\langle E_{(\cdot)}x,y\rangle) \\
&\leq \bigvee_m^M(\langle E_{(\cdot)}x,y\rangle) \\
&\times\begin{cases}\max_{t\in[m,M]}|t-s|\,\|f'\|_\infty & \text{if } f'\in L_\infty[m,M] \\ \max_{t\in[m,M]}|t-s|^{1/q}\,\|f'\|_p & \text{if } f'\in L_p[m,M],\ p>1, \\ & \frac{1}{p}+\frac{1}{q}=1,\end{cases} := F
\end{aligned}
$$

where $\bigvee_m^M(\langle E_{(\cdot)}x,y\rangle)$ denotes the total variation of $\langle E_{(\cdot)}x,y\rangle$ and $x,y\in H$.

Since, obviously, we have $\max_{t\in[m,M]}|t-s| = \frac{1}{2}(M-m)+\left|s-\frac{m+M}{2}\right|$, then

$$
\begin{aligned}
(4.5)\qquad F &= \bigvee_m^M(\langle E_{(\cdot)}x,y\rangle) \\
&\times\begin{cases}\left[\frac{1}{2}(M-m)+\left|s-\frac{m+M}{2}\right|\right]\|f'\|_\infty & \text{if } f'\in L_\infty[m,M] \\ \left[\frac{1}{2}(M-m)+\left|s-\frac{m+M}{2}\right|\right]^{1/q}\|f'\|_p & \text{if } f'\in L_p[m,M],\ p>1, \\ & \frac{1}{p}+\frac{1}{q}=1,\end{cases}
\end{aligned}
$$

for any $x,y\in H$.

If P is a nonnegative operator on H, i.e., $\langle Px,x\rangle\geq 0$ for any $x\in H$, then the following inequality is a generalization of the Schwarz inequality in H

$$|\langle Px,y\rangle|^2\leq\langle Px,x\rangle\langle Py,y\rangle$$

for any $x,y\in H$.

Now, if $d: m=t_0<t_1<...<t_{n-1}<t_n=M$ is an arbitrary partition of the interval $[m,M]$, then we have by Schwarz's inequality for nonnegative operators that

$$
\begin{aligned}
(4.6)\qquad & \bigvee_m^M(\langle E_{(\cdot)}x,y\rangle) \\
&= \sup_d\left\{\sum_{i=0}^{n-1}\left|\left\langle\left(E_{t_{i+1}}-E_{t_i}\right)x,y\right\rangle\right|\right\} \\
&\leq \sup_d\left\{\sum_{i=0}^{n-1}\left[\left\langle\left(E_{t_{i+1}}-E_{t_i}\right)x,x\right\rangle^{1/2}\left\langle\left(E_{t_{i+1}}-E_{t_i}\right)y,y\right\rangle^{1/2}\right]\right\} := I.
\end{aligned}
$$

By the Cauchy-Buniakovski-Schwarz inequality for sequences of real numbers we also have that

$$I \le \sup_d \left\{ \left[\sum_{i=0}^{n-1} \left\langle \left(E_{t_{i+1}} - E_{t_i} \right) x, x \right\rangle \right]^{1/2} \left[\sum_{i=0}^{n-1} \left\langle \left(E_{t_{i+1}} - E_{t_i} \right) y, y \right\rangle \right]^{1/2} \right\} \tag{4.7}$$

$$\le \sup_d \left\{ \left[\sum_{i=0}^{n-1} \left\langle \left(E_{t_{i+1}} - E_{t_i} \right) x, x \right\rangle \right]^{1/2} \sup_d \left[\sum_{i=0}^{n-1} \left\langle \left(E_{t_{i+1}} - E_{t_i} \right) y, y \right\rangle \right]^{1/2} \right\}$$

$$= \left[\bigvee_m^M \left(\langle E_{(\cdot)} x, x \rangle \right) \right]^{1/2} \left[\bigvee_m^M \left(\langle E_{(\cdot)} y, y \rangle \right) \right]^{1/2} = \|x\| \, \|y\|$$

for any $x, y \in H$.

On making use of (4.4), (4.5), (4.6) and (4.7), we deduce the desired result (4.2). □

Corollary 4. *With the assumptions of Theorem 7 we have the following inequalities*

$$\left| f\left(\frac{\langle Ax, x \rangle}{\|x\|^2} \right) \langle x, y \rangle - \langle f(A) x, y \rangle \right| \le \|x\| \, \|y\| \tag{4.8}$$

$$\times \begin{cases} \left[\frac{1}{2}(M - m) + \left| \frac{\langle Ax, x \rangle}{\|x\|^2} - \frac{m+M}{2} \right| \right] \|f'\|_\infty & \text{if } f' \in L_\infty [m, M] \\ \left[\frac{1}{2}(M - m) + \left| \frac{\langle Ax, x \rangle}{\|x\|^2} - \frac{m+M}{2} \right| \right]^{1/q} \|f'\|_p & \text{if } f' \in L_p [m, M], p > 1, \\ & \frac{1}{p} + \frac{1}{q} = 1, \end{cases}$$

and

$$\left| f\left(\frac{m + M}{2} \right) \langle x, y \rangle - \langle f(A) x, y \rangle \right| \le \|x\| \, \|y\| \tag{4.9}$$

$$\times \begin{cases} \frac{1}{2}(M - m) \|f'\|_\infty & \text{if } f' \in L_\infty [m, M] \\ \frac{1}{2^{1/q}}(M - m)^{1/q} \|f'\|_p & \text{if } f' \in L_p [m, M], p > 1, \\ & \frac{1}{p} + \frac{1}{q} = 1, \end{cases}$$

for any $x, y \in H$.

Remark 3. *In particular, we obtain from (2.8) the following inequalities*

$$|f(\langle Ax, x \rangle) - \langle f(A) x, x \rangle| \tag{4.10}$$

$$\le \begin{cases} \left[\frac{1}{2}(M - m) + \left| \langle Ax, x \rangle - \frac{m+M}{2} \right| \right] \|f'\|_\infty & \text{if } f' \in L_\infty [m, M] \\ \left[\frac{1}{2}(M - m) + \left| \langle Ax, x \rangle - \frac{m+M}{2} \right| \right]^{1/q} \|f'\|_p & \text{if } f' \in L_p [m, M], p > 1, \\ & \frac{1}{p} + \frac{1}{q} = 1, \end{cases}$$

and

$$
\left| f\left(\frac{m+M}{2}\right) -\left\langle f\left(A\right) x,x\right\rangle \right| \tag{4.11}
$$

$$
\leq \left\{
\begin{array}{ll}
\frac{1}{2}\left(M-m\right) \left\Vert f^{\prime }\right\Vert _{\infty } & \text{if } f^{\prime }\in L_{\infty }\left[m,M\right] \\
\frac{1}{2^{1/q}}\left(M-m\right) ^{1/q}\left\Vert f^{\prime }\right\Vert _{p} & \begin{array}{c}\text{if } f^{\prime }\in L_{p}\left[m,M\right] ,p>1, \\ \frac{1}{p}+\frac{1}{q}=1,\end{array}
\end{array}
\right.
$$

for any $x\in H$ *with* $\left\Vert x\right\Vert =1.$

Theorem 8. *Let* A *be a selfadjoint operator in the Hilbert space* H *with the spectrum* $Sp\left(A\right) \subseteq \left[m,M\right]$ *for some real numbers* $m<M$ *and let* $\left\{ E_{\lambda }\right\} _{\lambda }$ *be its spectral family. If* $f:\left[m,M\right] \rightarrow \mathbb{R}$ *is* $r-H$*-Hölder continuous on* $\left[m,M\right]$*, then we have the inequality*

$$
\begin{aligned}
&\left\vert f\left(s\right) \left\langle x,y\right\rangle -\left\langle f\left(A\right) x,y\right\rangle \right\vert \\
&\leq H\bigvee_{m}^{M}\left(\left\langle E_{\left(\cdot \right) }x,y\right\rangle \right) \left[\frac{1}{2}\left(M-m\right) +\left\vert s-\frac{m+M}{2}\right\vert \right] ^{r} \\
&\leq H\left\Vert x\right\Vert \left\Vert y\right\Vert \left[\frac{1}{2}\left(M-m\right) +\left\vert s-\frac{m+M}{2}\right\vert \right] ^{r}
\end{aligned} \tag{4.12}
$$

for any $x,y\in H$ *and* $s\in \left[m,M\right] .$

In particular, we have the inequalities

$$
\begin{aligned}
&\left\vert f\left(\frac{\left\langle Ax,x\right\rangle }{\left\Vert x\right\Vert ^{2}}\right) \left\langle x,y\right\rangle -\left\langle f\left(A\right) x,y\right\rangle \right\vert \\
&\leq H\left\Vert x\right\Vert \left\Vert y\right\Vert \left[\frac{1}{2}\left(M-m\right) +\left\vert \frac{\left\langle Ax,x\right\rangle }{\left\Vert x\right\Vert ^{2}}-\frac{m+M}{2}\right\vert \right] ^{r}
\end{aligned} \tag{4.13}
$$

and

$$
\left\vert f\left(\frac{m+M}{2}\right) \left\langle x,y\right\rangle -\left\langle f\left(A\right) x,y\right\rangle \right\vert \leq \frac{1}{2^{r}}H\left\Vert x\right\Vert \left\Vert y\right\Vert \left(M-m\right) ^{r} \tag{4.14}
$$

for any $x,y\in H.$

Proof. Utilising the inequality (4.4) and the fact that f is $r-H$-Hölder continuous we have successively

$$\begin{aligned} &|f(s)\langle x,y\rangle-\langle f(A)x,y\rangle| \\ &=\left|\int_{m-0}^{M}[f(s)-f(t)]\,d(\langle E_t x,y\rangle)\right| \\ &\leq \max_{t\in[m,M]}|f(s)-f(t)|\bigvee_m^M(\langle E_{(\cdot)}x,y\rangle) \\ &\leq H\max_{t\in[m,M]}|s-t|^r\bigvee_m^M(\langle E_{(\cdot)}x,y\rangle) \\ &=H\left[\frac{1}{2}(M-m)+\left|s-\frac{m+M}{2}\right|\right]^r\bigvee_m^M(\langle E_{(\cdot)}x,y\rangle) \end{aligned} \tag{4.15}$$

for any $x,y\in H$ and $s\in[m,M]$.

The argument follows now as in the proof of Theorem 7 and the details are omitted. □

5. Logarithmic Inequalities

Consider *the identric mean*

$$I=I(a,b):=\begin{cases} a & \text{if} \quad a=b, \\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} & \text{if} \quad a\neq b, \end{cases} \qquad a,b>0;$$

and observe that

$$\frac{1}{b-a}\int_a^b \ln t\,dt=\ln[I(a,b)].$$

If we apply Theorem 5 for the convex function $f(t)=-\ln t, t>0$, then we can state:

Proposition 1. *Let A be a positive selfadjoint operator in the Hilbert space H with the spectrum $Sp(A)\subseteq[m,M]$ for some positive numbers $0<m<M$. Then we have the double inequality in the operator order of $B(H)$*

$$-\frac{1}{2mM}\left(A^2-mM\cdot 1_H\right)\leq \ln I(m,M)\cdot 1_H-\ln A\leq\frac{m+M}{2}\cdot A^{-1}-1_H. \tag{5.1}$$

If we denote by $G(a,b):=\sqrt{ab}$ the geometric mean of the positive numbers a,b, then we can state the following result as well:

Proposition 2. *With the assumptions of Proposition 1, we have the inequalities in the operator order of $B(H)$*

$$\begin{aligned} &\ln G(m,M)\cdot 1_H \\ &\leq\frac{1}{2}\left[\ln A+\frac{\ln M\cdot(M\cdot 1_H-A)+\ln m\cdot(A-m\cdot 1_H)}{M-m}\right] \\ &\leq \ln I(m,M)\cdot 1_H. \end{aligned} \tag{5.2}$$

The inequality follows by Corollary 3 applied for the convex function $f(t) = -\ln t, t > 0.$

Finally, the following vector inequality may be stated

Proposition 3. *With the assumptions of Proposition 1, for any $x, y \in H$ we have the inequalities*

$$|\langle x, y\rangle \ln s - \langle \ln Ax, y\rangle| \le \|x\| \|y\| \begin{cases} \left[\frac{1}{2}(M-m) + \left|s - \frac{m+M}{2}\right|\right] \frac{1}{m}, \\ \\ \left[\frac{1}{2}(M-m) + \left|s - \frac{m+M}{2}\right|\right]^{1/q} \frac{M^{p-1}-m^{p-1}}{(p-1)M^{p-1}m^{p-1}}, \end{cases} \tag{5.3}$$

for any $s \in [m, M]$, where $p > 1, \frac{1}{p} + \frac{1}{q} = 1.$

References

[1] G.A. Anastassiou, Univariate Ostrowski inequalities, revisited. *Monatsh. Math.,* **135** (2002), no. 3, 175–189.

[2] P. Cerone and S.S. Dragomir, New bounds for the three-point rule involving the Riemann-Stieltjes integrals, in *Advances in Statistics Combinatorics and Related Areas*, C. Gulati, et al. (Eds.), World Science Publishing, 2002, 53-62.

[3] S.S. Dragomir, An Ostrowski like inequality for convex functions and applications, *Revista Math. Complutense*, **16**(2) (2003), 373-382.

[4] S.S. Dragomir, On the Ostrowski's inequality for mappings of bounded variation and applications, *Math. Ineq. & Appl.,* **4**(1) (2001), 33-40.

[5] S.S. Dragomir, On the Ostrowski's inequality for Riemann-Stieltjes integral, *Korean J. Appl. Math.,* **7** (2000), 477-485.

[6] S.S. Dragomir, On the Ostrowski inequality for Riemann-Stieltjes integral $\int_a^b f(t)\, du(t)$ where f is of Hölder type and u is of bounded variation and applications, *J. KSIAM,* **5**(1) (2001), 35-45.

[7] S.S. Dragomir, Ostrowski's inequality for monotonous mappings and applications, *J. KSIAM,* **3**(1) (1999), 127-135.

[8] S.S. Dragomir, The Ostrowski's integral inequality for Lipschitzian mappings and applications, *Comp. and Math. with Appl.,* **38** (1999), 33-37.

[9] S.S. Dragomir, Ostrowski type inequalities for isotonic linear functionals, *J. Inequal. Pure & Appl. Math.,* **3**(5) (2002), Art. 68. [ONLINE `http://jipam.vu.edu.au/article.php?sid=220`].

[10] S.S. Dragomir, Čebyšev's type inequalities for functions of selfadjoint operators in Hilbert spaces, Preprint *RGMIA Res. Rep. Coll.*, **11**(e) (2008), Art. 9. [ONLINE: http://www.staff.vu.edu.au/RGMIA/v11(E).asp]

[11] S.S. Dragomir, Grüss' type inequalities for functions of selfadjoint operators in Hilbert spaces, Preprint *RGMIA Res. Rep. Coll.*, **11**(e) (2008), Art. 11. [ONLINE: http://www.staff.vu.edu.au/RGMIA/v11(E).asp]

[12] S.S. Dragomir, Inequalities for the Čebyšev functional of two functions of selfadjoint operators in Hilbert spaces, *RGMIA Res. Rep. Coll.*, **11**(e) (2008), Art. . [ONLINE: http://www.staff.vu.edu.au/RGMIA/v11(E).asp]

[13] S.S. Dragomir, P. Cerone, J. Roumeliotis and S. Wang, A weighted version of Ostrowski inequality for mappings of Hölder type and applications in numerical analysis, *Bull. Math. Soc. Sci. Math. Romanic,* **42(90)** (4) (1999), 301-314.

[14] S.S. Dragomir and Th. M. Rassias (Eds), Ostrowski Type Inequalities and Applications in Numerical Integration, Kluwer Academic Publisher, 2002.

[15] S.S. Dragomir and S. Wang, A new inequality of Ostrowski's type in L_1-norm and applications to some special means and to some numerical quadrature rules, *Tamkang J. of Math.*, **28** (1997), 239-244.

[16] S.S. Dragomir and S. Wang, A new inequality of Ostrowski's type in L_p-norm and applications to some special means and to some numerical quadrature rules, *Indian J. of Math.,* **40**(3) (1998), 245-304.

[17] S.S. Dragomir and S. Wang, Applications of Ostrowski's inequality to the estimation of error bounds for some special means and some numerical quadrature rules, *Appl. Math. Lett.*, **11** (1998), 105-109.
[18] A.M. Fink, Bounds on the deviation of a function from its averages, *Czechoslovak Math. J.*, **42**(117) (1992), No. 2, 298-310.
[19] T. Furuta, J. Mićić Hot, J. Pečarić and Y. Seo, *Mond-Pečarić Method in Operator Inequalities. Inequalities for Bounded Selfadjoint Operators on a Hilbert Space*, Element, Zagreb, 2005.
[20] P. Kumar, The Ostrowski type moments integral inequalities and moment bounds for continuous random variables, *Comp. and Math. with Appl.,* **49**(11-12) (2005), 1929-1940.
[21] Z. Liu, Refinement of an inequality of Grüss type for Riemann-Stieltjes integral, *Soochow J. Math.,* **30**(4) (2004), 483-489.
[22] Z. Liu, Refinement of an inequality of Grüss type for Riemann-Stieltjes integral, *Soochow J. Math.,* **30**(4) (2004), 483-489.
[23] A. Matković, J. Pečarić and I. Perić, A variant of Jensen's inequality of Mercer's type for operators with applications. *Linear Algebra Appl.* **418** (2006), No. 2-3, 551–564.
[24] C.A. McCarthy, c_p, *Israel J. Math.*, **5**(1967), 249-271.
[25] B. Mond and J. Pečarić, Convex inequalities in Hilbert spaces, *Houston J. Math.*, **19**(1993), 405-420.
[26] B. Mond and J. Pečarić, Classical inequalities for matrix functions, *Utilitas Math.*, **46**(1994), 155-166.
[27] A. Ostrowski, Über die Absolutabweichung einer differentienbaren Funktionen von ihren Integralmittelwert, *Comment. Math. Hel,* **10** (1938), 226-227.
[28] J. Pečarić, J. Mićić and Y. Seo, Inequalities between operator means based on the Mond-Pečarić method. *Houston J. Math.* **30** (2004), no. 1, 191–207.

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