# OSTROWSKI'S TYPE INEQUALITIES FOR HÖLDER CONTINUOUS FUNCTIONS OF SELFADJOINT OPERATORS IN HILBERT SPACES

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Abstract. Some Ostrowski's type inequalities for Hölder continuous functions of selfadjoint operators in Hilbert spaces, under suitable assumptions for the involved operators, are given.

## 1. Introduction

Let $U$ be a selfadjoint operator on the Hilbert space $(H, \langle .,. \rangle)$ with the spectrum $Sp(U)$ included in the interval $[m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ be its *spectral family.* Then for any continuous function $f : [m, M] \to \mathbb{C}$, it is well known that we have the following *spectral representation* in terms of the *Riemann-Stieltjes integral*:

$$\langle f(U)x, x\rangle = \int_{m-0}^{M} f(\lambda)\, d(\langle E_\lambda x, x\rangle), \tag{1.1}$$

for any $x \in H$ with $\|x\| = 1$. The function $g_x(\lambda) := \langle E_\lambda x, x\rangle$ is *monotonic nondecreasing* on the interval $[m, M]$ and

$$g_x(m-0) = 0 \text{ and } g_x(M) = 1 \tag{1.2}$$

for any $x \in H$ with $\|x\| = 1$.

Utilising the representation (1.1) and the following Ostrowski's type inequality for the Riemann-Stieltjes integral obtained by the author in [1]:

$$\left| f(s)[u(b) - u(a)] - \int_a^b f(t)\, du(t) \right| \le L\left[\frac{1}{2}(b-a) + \left| s - \frac{a+b}{2} \right|\right]^r \bigvee_a^b (u) \tag{1.3}$$

for any $s \in [a, b]$, provided that $f$ is of $r - L-$Hölder type on $[a, b]$ (see (1.4) below), $u$ is of *bounded variation* on $[a, b]$ and $\bigvee_a^b (u)$ denotes the *total variation* of $u$ on $[a, b]$, we obtained the following inequality of Ostrowski type for selfadjoint operators:

**Theorem 1** (Dragomir, 2008, [4]). *Let $A$ and $B$ be selfadjoint operators with $Sp(A)$, $Sp(B) \subseteq [m, M]$ for some real numbers $m < M$. If $f : [m, M] \longrightarrow \mathbb{R}$*

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*is of* $r-L-$*Hölder type, i.e., for a given* $r\in(0,1]$ *and* $L>0$ *we have*

$$|f(s)-f(t)|\leq L|s-t|^{r} \text{ for any } s,t\in[m,M], \tag{1.4}$$

*then we have the inequality:*

$$|f(s)-\langle f(A)x,x\rangle|\leq L\left[\frac{1}{2}(M-m)+\left|s-\frac{m+M}{2}\right|\right]^{r}, \tag{1.5}$$

*for any* $s\in[m,M]$ *and any* $x\in H$ *with* $\|x\|=1$*.*

*Moreover, we have*

$$\begin{aligned}&|\langle f(B)y,y\rangle-\langle f(A)x,x\rangle|\\&\leq\langle|f(B)-\langle f(A)x,x\rangle\cdot 1_H|y,y\rangle\\&\leq L\left[\frac{1}{2}(M-m)+\left\langle\left|B-\frac{m+M}{2}\cdot 1_H\right|y,y\right\rangle\right]^{r},\end{aligned} \tag{1.6}$$

*for any* $x,y\in H$ *with* $\|x\|=\|y\|=1$*.*

With the above assumptions for $f,A$ and $B$ we have the following particular inequalities of interest:

$$\left|f\left(\frac{m+M}{2}\right)-\langle f(A)x,x\rangle\right|\leq\frac{1}{2^{r}}L(M-m)^{r} \tag{1.7}$$

and

$$|f(\langle Ax,x\rangle)-\langle f(A)x,x\rangle|\leq L\left[\frac{1}{2}(M-m)+\left|\langle Ax,x\rangle-\frac{m+M}{2}\right|\right]^{r}, \tag{1.8}$$

for any $x\in H$ with $\|x\|=1$.

We also have the inequalities:

$$\begin{aligned}&|\langle f(A)y,y\rangle-\langle f(A)x,x\rangle|\\&\leq\langle|f(A)-\langle f(A)x,x\rangle\cdot 1_H|y,y\rangle\\&\leq L\left[\frac{1}{2}(M-m)+\left\langle\left|A-\frac{m+M}{2}\cdot 1_H\right|y,y\right\rangle\right]^{r},\end{aligned} \tag{1.9}$$

for any $x,y\in H$ with $\|x\|=\|y\|=1$,

$$\begin{aligned}&|\langle[f(B)-f(A)]x,x\rangle|\\&\leq\langle|f(B)-\langle f(A)x,x\rangle\cdot 1_H|x,x\rangle\\&\leq L\left[\frac{1}{2}(M-m)+\left\langle\left|B-\frac{m+M}{2}\cdot 1_H\right|x,x\right\rangle\right]^{r}\end{aligned} \tag{1.10}$$

and, more particularly,

$$\begin{aligned}&\langle|f(A)-\langle f(A)x,x\rangle\cdot 1_H|x,x\rangle\\&\leq L\left[\frac{1}{2}(M-m)+\left\langle\left|A-\frac{m+M}{2}\cdot 1_H\right|x,x\right\rangle\right]^{r},\end{aligned} \tag{1.11}$$

for any $x\in H$ with $\|x\|=1$.

We also have the norm inequality

$$\|f(B)-f(A)\|\leq L\left[\frac{1}{2}(M-m)+\left\|B-\frac{m+M}{2}\cdot 1_H\right\|\right]^{r}. \tag{1.12}$$

For various generalizations, extensions and related Ostrowski type inequalities for functions of one or several variables see the monograph [5] and the references therein.

Let $A$ be a selfadjoint linear operator on a complex Hilbert space $(H;\langle.,.\rangle)$. The *Gelfand map* establishes a $*$-isometrically isomorphism $\Phi$ between the set $C(Sp(A))$ of all *continuous functions* defined on the *spectrum* of $A$, denoted $Sp(A)$, and the $C^*$-algebra $C^*(A)$ generated by $A$ and the identity operator $1_H$ on $H$ as follows (see for instance [6, p. 3]):

For any $f,g \in C(Sp(A))$ and any $\alpha,\beta \in \mathbb{C}$ we have

(i) $\Phi(\alpha f+\beta g)=\alpha\Phi(f)+\beta\Phi(g)$;

(ii) $\Phi(fg)=\Phi(f)\Phi(g)$ and $\Phi(\bar{f})=\Phi(f)^*$;

(iii) $\|\Phi(f)\|=\|f\|:=\sup_{t\in Sp(A)}|f(t)|$;

(iv) $\Phi(f_0)=1_H$ and $\Phi(f_1)=A$, where $f_0(t)=1$ and $f_1(t)=t$, for $t\in Sp(A)$.

With this notation we define

$$f(A):=\Phi(f) \text{ for all } f\in C(Sp(A))$$

and we call it the *continuous functional calculus* for a selfadjoint operator $A$.

If $A$ is a selfadjoint operator and $f$ is a real valued continuous function on $Sp(A)$, then $f(t)\geq 0$ for any $t\in Sp(A)$ implies that $f(A)\geq 0$, *i.e.* $f(A)$ is a *positive operator* on $H$. Moreover, if both $f$ and $g$ are real valued functions on $Sp(A)$ then the following important property holds:

$$\text{(P)} \qquad f(t)\geq g(t) \text{ for any } t\in Sp(A) \text{ implies that } f(A)\geq g(A)$$

in the operator order of $B(H)$.

For a recent monograph devoted to various inequalities for functions of selfadjoint operators, see [6] and the references therein.

For other results see [2], [3], [4], [8], [10], [11] and [12].

Motivated by the above results we investigate in this paper other inequalities of Ostrowski type for Hölder continuous functions and compare them with the ones above to decide which are better. Applications in relation with the Jensen inequality for selfadjoint operators obtained by Mond and Pečarić in [10] (see also the monograph [6]) are provided as well.

## 2. More Inequalities of Ostrowski's Type

The following result holds:

**Theorem 2.** *Let $A$ be a selfadjoint operator with $Sp(A)\subseteq[m,M]$ for some real numbers $m<M$. If $f:[m,M]\longrightarrow\mathbb{R}$ is of $r-L-$Hölder type with $r\in(0,1]$, then we have the inequality:*

$$|f(s)-\langle f(A)x,x\rangle|\leq L\langle|s\cdot 1_H-A|x,x\rangle^r \leq L\left[(s-\langle Ax,x\rangle)^2+D^2(A;x)\right]^{r/2}, \tag{2.1}$$

*for any $s\in[m,M]$ and any $x\in H$ with $\|x\|=1$, where $D(A;x)$ is the variance of the selfadjoint operator $A$ in $x$ and is defined by*

$$D(A;x):=\left(\|Ax\|^2-\langle Ax,x\rangle^2\right)^{1/2},$$

*where $x\in H$ with $\|x\|=1$.*

*Proof.* First of all, by the Jensen inequality for convex functions of selfadjoint operators (see for instance [6, p. 5]) applied for the modulus, we can state that

$$|\langle h(A)x,x\rangle| \le \langle |h(A)|x,x\rangle \tag{M}$$

for any $x \in H$ with $\|x\| = 1$, where $h$ is a continuous function on $[m,M]$.

Utilising the property (M) we then get

$$|f(s) - \langle f(A)x,x\rangle| = |\langle f(s)\cdot 1_H - f(A)x,x\rangle| \le \langle |f(s)\cdot 1_H - f(A)|x,x\rangle \tag{2.2}$$

for any $x \in H$ with $\|x\| = 1$ and any $s \in [m,M]$.

Since $f$ is of $r-L-$Hölder type, then for any $t,s \in [m,M]$ we have

$$|f(s) - f(t)| \le L|s-t|^r. \tag{2.3}$$

If we fix $s \in [m,M]$ and apply the property (P) for the inequality (2.3) and the operator $A$ we get

$$\langle |f(s)\cdot 1_H - f(A)|x,x\rangle \le L\langle |s\cdot 1_H - A|^r x,x\rangle \le L\langle |s\cdot 1_H - A|x,x\rangle^r \tag{2.4}$$

for any $x \in H$ with $\|x\| = 1$ and any $s \in [m,M]$, where, for the last inequality we have used the fact that if $P$ is a positive operator and $r \in (0,1)$ then, by the Hölder-McCarthy inequality [9],

$$\langle P^r x,x\rangle \le \langle Px,x\rangle^r \tag{HM}$$

for any $x \in H$ with $\|x\| = 1$. This proves the fist inequality in (2.1).

Now, observe that for any bounded linear operator $T$ we have

$$\langle |T|x,x\rangle = \left\langle (T^*T)^{1/2}x,x\right\rangle \le \langle (T^*T)x,x\rangle^{1/2} = \|Tx\|$$

for any $x \in H$ with $\|x\| = 1$ which implies that

$$\begin{aligned} \langle |s\cdot 1_H - A|x,x\rangle^r &\le \|sx - Ax\|^r = \left(s^2 - 2s\langle Ax,x\rangle + \|Ax\|^2\right)^{r/2} \\ &= \left[(s - \langle Ax,x\rangle)^2 + \|Ax\|^2 - \langle Ax,x\rangle^2\right]^{r/2} \end{aligned} \tag{2.5}$$

for any $x \in H$ with $\|x\| = 1$ and any $s \in [m,M]$.

Finally, on making use of (2.2), (2.4) and (2.5) we deduce the desired result (2.1). □

**Remark 1.** *If we choose in (2.1) $s = \frac{m+M}{2}$, then we get the sequence of inequalities*

$$\begin{aligned} \left|f\left(\frac{m+M}{2}\right) - \langle f(A)x,x\rangle\right| &\le L\left\langle \left|\frac{m+M}{2}\cdot 1_H - A\right|x,x\right\rangle^r \\ &\le L\left[\left(\frac{m+M}{2} - \langle Ax,x\rangle\right)^2 + D^2(A;x)\right]^{r/2} \\ &\le L\left[\frac{1}{4}(M-m)^2 + D^2(A;x)\right]^{r/2} \\ &\le \frac{1}{2^r}L(M-m)^r \end{aligned} \tag{2.6}$$

*for any $x \in H$ with $\|x\| = 1$, since, obviously,*

$$\left(\frac{m+M}{2} - \langle Ax,x\rangle\right)^2 \le \frac{1}{4}(M-m)^2$$

*and*

$$D^2(A;x) \le \frac{1}{4}(M-m)^2$$

*for any $x \in H$ with $\|x\| = 1$.*

*We notice that the inequality (2.6) provides a refinement for the result (1.7) above.*

The best inequality we can get from (2.1) is incorporated in the following:

**Corollary 1.** *Let $A$ be a selfadjoint operator with $Sp(A) \subseteq [m,M]$ for some real numbers $m < M$. If $f : [m,M] \longrightarrow \mathbb{R}$ is of $r-L-$Hölder type with $r \in (0,1]$, then we have the inequality*

$$|f(\langle Ax,x\rangle) - \langle f(A)x,x\rangle| \le L\langle |\langle Ax,x\rangle \cdot 1_H - A|\,x,x\rangle^r \le LD^r(A;x), \tag{2.7}$$

*for any $x \in H$ with $\|x\| = 1$.*

The inequality (2.1) may be used to obtain other inequalities for two selfadjoint operators as follows:

**Corollary 2.** *Let $A$ and $B$ be selfadjoint operators with $Sp(A), Sp(B) \subseteq [m,M]$ for some real numbers $m < M$. If $f : [m,M] \longrightarrow \mathbb{R}$ is of $r-L-$Hölder type with $r \in (0,1]$, then we have the inequality*

$$\begin{aligned} &|\langle f(B)y,y\rangle - \langle f(A)x,x\rangle| \\ &\le L\left[(\langle By,y\rangle - \langle Ax,x\rangle)^2 + D^2(A;x) + D^2(B;y)\right]^{r/2} \end{aligned} \tag{2.8}$$

*for any $x,y \in H$ with $\|x\| = \|y\| = 1$.*

*Proof.* If we apply the property (P) to the inequality (2.1) and for the operator $B$, then we get

$$\begin{aligned} &\langle |f(B) - \langle f(A)x,x\rangle \cdot 1_H|\,y,y\rangle \\ &\le L\left\langle \left[(B - \langle Ax,x\rangle \cdot 1_H)^2 + D^2(A;x)\cdot 1_H\right]^{r/2} y,y\right\rangle \end{aligned} \tag{2.9}$$

for any $x,y \in H$ with $\|x\| = \|y\| = 1$.

Utilising the inequality (M) we also have that

$$|f(\langle By,y\rangle) - \langle f(A)x,x\rangle| \le \langle |f(B) - \langle f(A)x,x\rangle \cdot 1_H|\,y,y\rangle \tag{2.10}$$

for any $x,y \in H$ with $\|x\| = \|y\| = 1$.

Now, by the Hölder-McCarthy inequality (HM) we also have

$$\begin{aligned} &\left\langle \left[(B - \langle Ax,x\rangle \cdot 1_H)^2 + D^2(A;x)\cdot 1_H\right]^{r/2} y,y\right\rangle \\ &\le \left\langle \left[(B - \langle Ax,x\rangle \cdot 1_H)^2 + D^2(A;x)\cdot 1_H\right] y,y\right\rangle^{r/2} \\ &= \left((\langle By,y\rangle - \langle Ax,x\rangle)^2 + D^2(A;x) + D^2(B;y)\right)^{r/2} \end{aligned} \tag{2.11}$$

for any $x,y \in H$ with $\|x\| = \|y\| = 1$.

On making use of (2.9)-(2.11) we deduce the desired result (2.8). □

**Remark 2.** *Since*

$$D^2(A;x) \le \frac{1}{4}(M-m)^2, \tag{2.12}$$

*then we obtain from (2.8) the following vector inequalities*

$$\begin{aligned} &|\langle f(A)y,y\rangle - \langle f(A)x,x\rangle| \\ &\le L\left[(\langle Ay,y\rangle - \langle Ax,x\rangle)^2 + D^2(A;x) + D^2(A;y)\right]^{r/2} \\ &\le L\left[(\langle Ay,y\rangle - \langle Ax,x\rangle)^2 + \frac{1}{2}(M-m)^2\right]^{r/2}, \end{aligned} \tag{2.13}$$

*and*

$$\begin{aligned} &|\langle [f(B) - f(A)]x,x\rangle| \\ &\le L\left[\langle (B-A)x,x\rangle^2 + D^2(A;x) + D^2(B;x)\right]^{r/2} \\ &\le L\left[\langle (B-A)x,x\rangle^2 + \frac{1}{2}(M-m)^2\right]^{r/2}. \end{aligned} \tag{2.14}$$

*In particular, we have the norm inequality*

$$\|f(B) - f(A)\| \le L\left[\|B-A\|^2 + \frac{1}{2}(M-m)^2\right]^{r/2}. \tag{2.15}$$

The following result provides convenient examples for applications:

**Corollary 3.** *Let $A$ be a selfadjoint operator with $Sp(A) \subseteq [m,M]$ for some real numbers $m < M$. If $f : [m,M] \longrightarrow \mathbb{R}$ is absolutely continuous on $[m,M]$, then we have the inequality:*

$$\begin{aligned} &|f(s) - \langle f(A)x,x\rangle| \\ &\le \begin{cases} \langle |s\cdot 1_H - A|x,x\rangle \|f'\|_{[m,M],\infty} & \text{if } f' \in L_\infty[m,M], \\ \langle |s\cdot 1_H - A|x,x\rangle^{1/q} \|f'\|_{[m,M],p} & \text{if } f' \in L_p[m,M], \\ & p>1, \frac{1}{p}+\frac{1}{q}=1, \end{cases} \\ &\le \begin{cases} \left[(s-\langle Ax,x\rangle)^2 + D^2(A;x)\right]^{1/2} \|f'\|_{[m,M],\infty} & \text{if } f' \in L_\infty[m,M], \\ \left[(s-\langle Ax,x\rangle)^2 + D^2(A;x)\right]^{\frac{1}{2q}} \|f'\|_{[m,M],p} & \text{if } f' \in L_p[m,M], \\ & p>1, \frac{1}{p}+\frac{1}{q}=1, \end{cases} \end{aligned} \tag{2.16}$$

*for any $s \in [m,M]$ and any $x \in H$ with $\|x\| = 1$, where $\|f'\|_{[m,M],\ell}$ are the Lebesgue norms, i.e.,*

$$\|f'\|_{[m,M],\ell} := \begin{cases} ess\sup_{t\in[m,M]} |f'(t)| & \text{if } \ell = \infty \\ \left(\int_m^M |f'(t)|^p dt\right)^{1/p} & \text{if } \ell = p \ge 1. \end{cases}$$

*Proof.* Follows from Theorem 2 and on tacking into account that if $f:[m,M]\longrightarrow\mathbb{R}$ is absolutely continuous on $[m,M]$, then for any $s,t\in[m,M]$ we have

$$|f(s)-f(t)|=\left|\int_t^s f'(u)\,du\right| \leq \begin{cases} |s-t|\,ess\sup_{t\in[m,M]}|f'(t)| & \text{if } f'\in L_\infty[m,M] \\ |s-t|^{1/q}\left(\int_m^M |f'(t)|^p\,dt\right)^{1/p} & \text{if } f'\in L_p[m,M],\, p>1,\, \frac{1}{p}+\frac{1}{q}=1. \end{cases}$$

□

**Remark 3.** *It is clear that all the inequalities from Corollaries 1, 2 and Remark 2 may be stated for absolutely continuous functions. However, we mention here only one, namely*

$$\begin{aligned} &|f(\langle Ax,x\rangle)-\langle f(A)x,x\rangle| \\ &\leq \begin{cases} \langle |\langle Ax,x\rangle\cdot 1_H-A|\,x,x\rangle\,\|f'\|_{[m,M],\infty} & \text{if } f'\in L_\infty[m,M] \\ \langle |\langle Ax,x\rangle\cdot 1_H-A|\,x,x\rangle^{1/q}\,\|f'\|_{[m,M],p} & \text{if } f'\in L_p[m,M],\ p>1,\, \frac{1}{p}+\frac{1}{q}=1, \end{cases} \\ &\leq \begin{cases} D(A;x)\,\|f'\|_{[m,M],\infty} & \text{if } f'\in L_\infty[m,M] \\ D^{1/q}(A;x)\,\|f'\|_{[m,M],p} & \text{if } f'\in L_p[m,M],\ p>1,\, \frac{1}{p}+\frac{1}{q}=1. \end{cases} \end{aligned} \tag{2.17}$$

## 3. The Case of $(\varphi,\Phi)$ −Lipschitzian Functions

The following lemma may be stated.

**Lemma 1.** *Let $u:[a,b]\to\mathbb{R}$ and $\varphi,\Phi\in\mathbb{R}$ be such that $\Phi>\varphi$. The following statements are equivalent:*

(i) *The function $u-\frac{\varphi+\Phi}{2}\cdot e$, where $e(t)=t$, $t\in[a,b]$, is $\frac{1}{2}(\Phi-\varphi)$ −Lipschitzian;*

(ii) *We have the inequality:*

$$\varphi\leq\frac{u(t)-u(s)}{t-s}\leq\Phi \quad \text{for each} \quad t,s\in[a,b] \quad \text{with } t\neq s; \tag{3.1}$$

(iii) *We have the inequality:*

$$\varphi(t-s)\leq u(t)-u(s)\leq\Phi(t-s) \quad \text{for each} \quad t,s\in[a,b] \quad \text{with } t>s. \tag{3.2}$$

We can introduce the following class of functions, see also [7]:

**Definition 1.** *The function $u:[a,b]\to\mathbb{R}$ which satisfies one of the equivalent conditions (i) – (iii) is said to be $(\varphi,\Phi)$ −Lipschitzian on $[a,b]$.*

Utilising *Lagrange's mean value theorem*, we can state the following result that provides practical examples of $(\varphi,\Phi)$ −Lipschitzian functions.

**Proposition 1.** *Let $u:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on $(a,b)$. If*

$$-\infty<\gamma:=\inf_{t\in(a,b)}u'(t), \qquad \sup_{t\in(a,b)}u'(t)=:\Gamma<\infty \tag{3.3}$$

*then $u$ is $(\gamma,\Gamma)$ −Lipschitzian on $[a,b]$.*

The following result can be stated:

**Proposition 2.** *Let $A$ be a selfadjoint operator with $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$. If $f : [m, M] \longrightarrow \mathbb{R}$ is $(\gamma, \Gamma)$−Lipschitzian on $[m, M]$, then we have the inequality*

$$|f(\langle Ax, x\rangle) - \langle f(A)x, x\rangle| \leq \frac{1}{2}(\Gamma - \gamma)\langle|\langle Ax, x\rangle \cdot 1_H - A|x, x\rangle \leq \frac{1}{2}(\Gamma - \gamma)D(A; x), \tag{3.4}$$

*for any $x \in H$ with $\|x\| = 1$.*

*Proof.* Follows by Corollary 1 on taking into account that in this case we have $r = 1$ and $L = \frac{1}{2}(\Gamma - \gamma)$. □

We can use the result (3.4) for the particular case of convex functions to provide an interesting reverse inequality for the Jensen's type operator inequality due to Mond and Pečarić [10] (see also [6, p. 5]):

**Theorem 3** (Mond-Pečarić, 1993, [10])**.** *Let $A$ be a selfadjoint operator on the Hilbert space $H$ and assume that $Sp(A) \subseteq [m, M]$ for some scalars $m, M$ with $m < M$. If $f$ is a convex function on $[m, M]$, then*

$$f(\langle Ax, x\rangle) \leq \langle f(A)x, x\rangle \tag{MP}$$

*for each $x \in H$ with $\|x\| = 1$.*

**Corollary 4.** *With the assumptions of Theorem 3 we have the inequality*

$$\begin{aligned}(0 \leq) &\langle f(A)x, x\rangle - f(\langle Ax, x\rangle) \\ &\leq \frac{1}{2}\left(f'_-(M) - f'_+(m)\right)\langle|\langle Ax, x\rangle \cdot 1_H - A|x, x\rangle \\ &\leq \frac{1}{2}\left(f'_-(M) - f'_+(m)\right)D(A; x) \leq \frac{1}{4}\left(f'_-(M) - f'_+(m)\right)(M - m)\end{aligned} \tag{3.5}$$

*for each $x \in H$ with $\|x\| = 1$.*

*Proof.* Follows by Proposition 2 on taking into account that

$$f'_+(m)(t - s) \leq f(t) - f(s) \leq f'_-(M)(t - s)$$

for each $s, t$ with the property that $M > t > s > m$. □

The following result may be stated as well:

**Proposition 3.** *Let $A$ be a selfadjoint operator with $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$. If $f : [m, M] \longrightarrow \mathbb{R}$ is $(\gamma, \Gamma)$−Lipschitzian on $[m, M]$, then we have the inequality*

$$\begin{aligned}&|f(\langle Ax, x\rangle) - \langle f(A)x, x\rangle| \\ &\leq \frac{1}{2}(\Gamma - \gamma)\left[\frac{1}{2}(M - m) + \left|\langle Ax, x\rangle - \frac{m + M}{2}\right|\right]\end{aligned} \tag{3.6}$$

*for any $x \in H$ with $\|x\| = 1$.*

The following particular case for convex functions holds:

**Corollary 5.** *With the assumptions of Theorem 3 we have the inequality*

$$(0 \leq) \langle f(A)x,x\rangle - f(\langle Ax,x\rangle) \leq \frac{1}{2}\left(f'_-(M) - f'_+(m)\right)\left[\frac{1}{2}(M-m) + \left|\langle Ax,x\rangle - \frac{m+M}{2}\right|\right] \tag{3.7}$$

*for each $x \in H$ with $\|x\| = 1$.*

## 4. Related Results

In the previous sections we have compared amongst other the following quantities

$$f\left(\frac{m+M}{2}\right) \text{ and } f(\langle Ax,x\rangle)$$

with $\langle f(A)x,x\rangle$ for a selfadjoint operator $A$ on the Hilbert space $H$ with $Sp(A) \subseteq [m,M]$ for some real numbers $m < M$, $f : [m,M] \longrightarrow \mathbb{R}$ a function of $r-L-$Hölder type with $r \in (0,1]$ and $x \in H$ with $\|x\| = 1$.

Since, obviously,

$$m \leq \frac{1}{M-m}\int_m^M f(t)\,dt \leq M,$$

then is also natural to compare $\frac{1}{M-m}\int_m^M f(t)\,dt$ with $\langle f(A)x,x\rangle$ under the same assumptions for $f, A$ and $x$.

The following result holds:

**Theorem 4.** *Let $A$ be a selfadjoint operator with $Sp(A) \subseteq [m,M]$ for some real numbers $m < M$. If $f : [m,M] \longrightarrow \mathbb{R}$ is of $r-L-$Hölder type with $r \in (0,1]$, then we have the inequality:*

$$\begin{aligned} &\left|\frac{1}{M-m}\int_m^M f(s)\,dt - \langle f(A)x,x\rangle\right| \\ &\leq \frac{1}{r+1}L(M-m)^r \\ &\times\left[\left\langle\left(\frac{M\cdot 1_H - A}{M-m}\right)^{r+1}x,x\right\rangle + \left\langle\left(\frac{A - m\cdot 1_H}{M-m}\right)^{r+1}x,x\right\rangle\right] \\ &\leq \frac{1}{r+1}L(M-m)^r, \end{aligned} \tag{4.1}$$

*for any $x \in H$ with $\|x\| = 1$.*

*In particular, if $f : [m,M] \longrightarrow \mathbb{R}$ is Lipschitzian with a constant $K$, then*

$$\begin{aligned} &\left|\frac{1}{M-m}\int_m^M f(s)\,dt - \langle f(A)x,x\rangle\right| \\ &\leq K(M-m)\left[\frac{1}{4} + \frac{1}{(M-m)^2}\left(D^2(A;x) + \left(\langle Ax,x\rangle - \frac{m+M}{2}\right)^2\right)\right] \\ &\leq \frac{1}{2}K(M-m) \end{aligned} \tag{4.2}$$

*for any $x \in H$ with $\|x\| = 1$.*

*Proof.* We use the following Ostrowski's type result (see for instance [5, p. 3]) written for the function $f$ that is of $r-L-$Hölder type on the interval $[m,M]$ :

$$\left|\frac{1}{M-m}\int_m^M f(s)\,dt - f(t)\right| \le \frac{L}{r+1}(M-m)^r\left[\left(\frac{M-t}{M-m}\right)^{r+1}+\left(\frac{t-m}{M-m}\right)^{r+1}\right] \tag{4.3}$$

for any $t\in[m,M]$.

If we apply the properties (P) and (M) then we have successively

$$\begin{aligned}&\left|\frac{1}{M-m}\int_m^M f(s)\,dt-\langle f(A)x,x\rangle\right| \\ &\le\left\langle\left|\frac{1}{M-m}\int_m^M f(s)\,dt-f(A)\right|x,x\right\rangle \\ &\le\frac{L}{r+1}(M-m)^r \\ &\times\left[\left\langle\left(\frac{M\cdot 1_H-A}{M-m}\right)^{r+1}x,x\right\rangle+\left\langle\left(\frac{A-m\cdot 1_H}{M-m}\right)^{r+1}x,x\right\rangle\right]\end{aligned} \tag{4.4}$$

which proves the first inequality in (4.1).

Utilising the Lah-Ribarić inequality version for selfadjoint operators $A$ with $Sp(A)\subseteq[m,M]$ for some real numbers $m<M$ and convex functions $g:[m,M]\to\mathbb{R}$, namely (see for instance [6, p. 57]):

$$\langle g(A)x,x\rangle\le\frac{M-\langle Ax,x\rangle}{M-m}g(m)+\frac{\langle Ax,x\rangle-m}{M-m}g(M)$$

for any $x\in H$ with $\|x\|=1$, then we get for the convex function $g(t):=\left(\frac{M-t}{M-m}\right)^{r+1}$,

$$\left\langle\left(\frac{M\cdot 1_H-A}{M-m}\right)^{r+1}x,x\right\rangle\le\frac{M-\langle Ax,x\rangle}{M-m}$$

and for the convex function $g(t):=\left(\frac{t-m}{M-m}\right)^{r+1}$,

$$\left\langle\left(\frac{A-m\cdot 1_H}{M-m}\right)^{r+1}x,x\right\rangle\le\frac{\langle Ax,x\rangle-m}{M-m}$$

for any $x\in H$ with $\|x\|=1$.

Now, on making use of the last two inequalities, we deduce the second part of (4.1).

Since

$$\begin{aligned}&\frac{1}{2}\left\langle\left(\frac{M\cdot 1_H-A}{M-m}\right)^{2}x,x\right\rangle+\left\langle\left(\frac{A-m\cdot 1_H}{M-m}\right)^{2}x,x\right\rangle \\ &=\frac{1}{4}+\frac{1}{(M-m)^2}\left(D^2(A;x)+\left(\langle Ax,x\rangle-\frac{m+M}{2}\right)^2\right)\end{aligned}$$

for any $x \in H$ with $\|x\| = 1$, then on choosing $r = 1$ in (4.1) we deduce the desired result (4.2). □

**Remark 4.** *We should notice from the proof of the above theorem, we also have the following inequalities in the operator order of* $B(H)$

$$\left| f(A) - \left( \frac{1}{M-m} \int_m^M f(s)\,dt \right) \cdot 1_H \right| \tag{4.5}$$
$$\leq \frac{L}{r+1}(M-m)^r \left[ \left( \frac{M \cdot 1_H - A}{M-m} \right)^{r+1} + \left( \frac{A - m \cdot 1_H}{M-m} \right)^{r+1} \right]$$
$$\leq \frac{1}{r+1} L (M-m)^r \cdot 1_H .$$

The following particular case is of interest:

**Corollary 6.** *Let $A$ be a selfadjoint operator with $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$. If $f : [m, M] \longrightarrow \mathbb{R}$ is $(\gamma, \Gamma)$ −Lipschitzian on $[m, M]$, then we have the inequality*

$$\left| \langle f(A)x, x \rangle - \frac{\Gamma + \gamma}{2} - \frac{1}{M-m} \int_m^M f(s)\,dt + \frac{\Gamma + \gamma}{2} \cdot \frac{m+M}{2} \right| \tag{4.6}$$
$$\leq \frac{1}{2}(\Gamma - \gamma)(M-m)$$
$$\times \left[ \frac{1}{4} + \frac{1}{(M-m)^2} \left( D^2(A;x) + \left( \langle Ax, x \rangle - \frac{m+M}{2} \right)^2 \right) \right]$$
$$\leq \frac{1}{4}(\Gamma - \gamma)(M-m).$$

*Proof.* Follows by (4.2) applied for the $\frac{1}{2}(\Gamma - \gamma)$-Lipshitzian function $f - \frac{\Gamma+\gamma}{2} \cdot e$. □

## 5. Applications for Some Particular Functions

**1.** We have the following important inequality in Operator Theory that is well known as the Hölder-McCarthy inequality:

**Theorem 5** (Hölder-McCarthy, 1967, [9])**.** *Let $A$ be a selfadjoint positive operator on a Hilbert space $H$. Then*

*(i)* $\langle A^r x, x \rangle \geq \langle Ax, x \rangle^r$ *for all* $r > 1$ *and* $x \in H$ *with* $\|x\| = 1$;

*(ii)* $\langle A^r x, x \rangle \leq \langle Ax, x \rangle^r$ *for all* $0 < r < 1$ *and* $x \in H$ *with* $\|x\| = 1$;

*(iii) If $A$ is invertible, then* $\langle A^{-r} x, x \rangle \geq \langle Ax, x \rangle^{-r}$ *for all* $r > 0$ *and* $x \in H$ *with* $\|x\| = 1$.

We can provide the following reverse inequalities:

**Proposition 4.** *Let $A$ be a selfadjoint positive operator on a Hilbert space $H$ and $0 < r < 1$. Then*

$$(0 \leq) \langle Ax, x \rangle^r - \langle A^r x, x \rangle \leq \langle |\langle Ax, x \rangle \cdot 1_H - A| \, x, x \rangle^r \leq D^r(A;x) \tag{5.1}$$

*for all* $x \in H$ *with* $\|x\| = 1$.

*Proof.* Follows from Corollary 1 by taking into account that the function $f(t) = t^r$ is of $r - L-$Hölder type with $L = 1$ on any compact interval of $(0, \infty)$. □

On making use of Corollary 4 we can state the following result as well:

**Proposition 5.** *Let $A$ be a selfadjoint positive operator on a Hilbert space $H$. Assume that $Sp(A) \subseteq [m, M] \subseteq [0, \infty)$.*

*(i) We have*

$$0 \leq \langle A^r x, x\rangle - \langle Ax, x\rangle^r \leq \frac{1}{2} r \left(M^{r-1} - m^{r-1}\right) \langle |\langle Ax, x\rangle \cdot 1_H - A| x, x\rangle \leq \frac{1}{2} r \left(M^{r-1} - m^{r-1}\right) D(A; x) \left(\leq \frac{1}{2} r \left(M^{r-1} - m^{r-1}\right)(M - m)\right) \tag{5.2}$$

*for all $r > 1$ and $x \in H$ with $\|x\| = 1$;*

*(ii) We also have*

$$0 \leq \langle Ax, x\rangle^r - \langle A^r x, x\rangle \leq \frac{1}{2} r \left(\frac{M^{1-r} - m^{1-r}}{m^{1-r} M^{1-r}}\right) \langle |\langle Ax, x\rangle \cdot 1_H - A| x, x\rangle \leq \frac{1}{2} r \left(\frac{M^{1-r} - m^{1-r}}{m^{1-r} M^{1-r}}\right) D(A; x) \left(\leq \frac{1}{2} r \left(\frac{M^{1-r} - m^{1-r}}{m^{1-r} M^{1-r}}\right)(M - m)\right) \tag{5.3}$$

*for all $0 < r < 1$ and $x \in H$ with $\|x\| = 1$;*

*(iii) If $A$ is invertible, then*

$$0 \leq \left\langle A^{-r} x, x\right\rangle - \langle Ax, x\rangle^{-r} \leq \frac{1}{2} r \left(\frac{M^{r+1} - m^{r+1}}{M^{r+1} m^{r+1}}\right) \langle |\langle Ax, x\rangle \cdot 1_H - A| x, x\rangle \leq \frac{1}{2} r \left(\frac{M^{r+1} - m^{r+1}}{M^{r+1} m^{r+1}}\right) D(A; x) \left(\leq \frac{1}{2} r \left(\frac{M^{r+1} - m^{r+1}}{M^{r+1} m^{r+1}}\right)(M - m)\right) \tag{5.4}$$

*for all $r > 0$ and $x \in H$ with $\|x\| = 1$.*

**2.** Consider the convex function $f : (0, \infty) \to \mathbb{R}$, $f(x) = -\ln x$. On utilizing the inequality (3.5), we can state the following result:

**Proposition 6.** *For any positive definite operator $A$ on the Hilbert space $H$ with $Sp(A) \subseteq [m, M] \subseteq [0, \infty)$ we have the inequality*

$$(0 \leq) \ln(\langle Ax, x\rangle) - \langle \ln(A) x, x\rangle \leq \frac{1}{2} \cdot \frac{M - m}{mM} \langle |\langle Ax, x\rangle \cdot 1_H - A| x, x\rangle \leq \frac{1}{2} \cdot \frac{M - m}{mM} D(A; x) \left(\leq \frac{1}{4} \cdot \frac{(M - m)^2}{mM}\right) \tag{5.5}$$

*for any $x \in H$ with $\|x\| = 1$.*

Finally, the following result for logarithms also holds:

**Proposition 7.** *Under the assumptions of Proposition 6 we have the inequality*

$$(0 \leq) \langle A \ln(A) x, x\rangle - \langle Ax, x\rangle \ln(\langle Ax, x\rangle) \leq \ln \sqrt{\frac{M}{m}} \langle |\langle Ax, x\rangle \cdot 1_H - A| x, x\rangle \leq \ln \sqrt{\frac{M}{m}} \cdot D(A; x) \left(\leq \frac{1}{2}(M - m) \ln \sqrt{\frac{M}{m}}\right) \tag{5.6}$$

*for any $x \in H$ with $\|x\| = 1$.*

**Remark 5.** *On utilizing the results from the previous sections for other convex functions of interest such as* $f(x) = \ln[(1-x)/x]$, $x \in (0, 1/2)$ *or* $f(x) = \ln(1 + \exp x)$, $x \in (-\infty, \infty)$ *we can get other interesting operator inequalities. However, the details are left to the interested reader.*

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