SOME OSTROWSKI'S TYPE VECTOR INEQUALITIES FOR FUNCTIONS OF SELFADJOINT OPERATORS IN HILBERT SPACES

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Abstract. Some Ostrowski's type vector inequalities for continuous functions of selfadjoint operators in Hilbert spaces under suitable conditions for the functions and operators involved are given. Applications for particular instances of interest are provided as well.

1. Introduction

The following result is known in the literature as Ostrowski's inequality [27].

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b) with the property that $|f'(t)|\leq M$ for all $t\in(a,b)$. Then*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)M \tag{1.1}$$

for all $x\in[a,b]$. The constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller quantity.

This result has been extended for absolutely continuous functions as follows (see [16] – [18]).

Theorem 2. *Let $f:[a,b]\to\mathbb{R}$ be absolutely continuous on $[a,b]$. Then, for all $x\in[a,b]$, we have:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{1.2}$$
$$\leq\begin{cases}\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\|f'\|_\infty & \text{if } f'\in L_\infty[a,b];\\[2ex] \frac{1}{(p+1)^{\frac{1}{p}}}\left[\left(\frac{x-a}{b-a}\right)^{p+1}+\left(\frac{b-x}{b-a}\right)^{p+1}\right]^{\frac{1}{p}}(b-a)^{\frac{1}{p}}\|f'\|_q & \text{if } f'\in L_q[a,b],\\ & \frac{1}{p}+\frac{1}{q}=1,\ p>1;\\ \left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\|f'\|_1;\end{cases}$$

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where $\|\cdot\|_r$ $(r \in [1,\infty])$ *are the usual Lebesgue norms on* $L_r[a,b]$, *i.e.,*

$$\|g\|_\infty := ess \sup_{t\in[a,b]} |g(t)|$$

and

$$\|g\|_r := \left(\int_a^b |g(t)|^r dt\right)^{\frac{1}{r}}, \ r \in [1,\infty).$$

The constants $\frac{1}{4}$, $\frac{1}{(p+1)^{\frac{1}{p}}}$ *and* $\frac{1}{2}$ *respectively are sharp in the sense presented in Theorem 1.*

The above inequalities can also be obtained from the Fink result in [19] on choosing $n = 1$ and performing some appropriate computations.

If one drops the condition of absolute continuity and assumes that f is Hölder continuous, then one may state the result (see for instance [14] and the references therein for earlier contributions):

Theorem 3. *Let* $f : [a,b] \to \mathbb{R}$ *be of* $r-H-$*Hölder type, i.e.,*

$$|f(x) - f(y)| \leq H|x-y|^r, \text{ for all } x,y \in [a,b], \tag{1.3}$$

where $r \in (0,1]$ *and* $H > 0$ *are fixed. Then, for all* $x \in [a,b]$, *we have the inequality:*

$$\begin{aligned} &\left|f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \\ &\leq \frac{H}{r+1}\left[\left(\frac{b-x}{b-a}\right)^{r+1} + \left(\frac{x-a}{b-a}\right)^{r+1}\right](b-a)^r. \end{aligned} \tag{1.4}$$

The constant $\frac{1}{r+1}$ *is also sharp in the above sense.*

Note that if $r = 1$, i.e., f is Lipschitz continuous, then we get the following version of Ostrowski's inequality for Lipschitzian functions (with L instead of H) (see for instance [8])

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)L. \tag{1.5}$$

Here the constant $\frac{1}{4}$ is also best.

Moreover, if one drops the condition of the continuity of the function, and assumes that it is of bounded variation, then the following result may be stated (see [4]).

Theorem 4. *Assume that* $f : [a,b] \to \mathbb{R}$ *is of bounded variation and denote by* $\bigvee_a^b(f)$ *its total variation. Then*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \left[\frac{1}{2} + \left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\bigvee_a^b(f) \tag{1.6}$$

for all $x \in [a,b]$. *The constant* $\frac{1}{2}$ *is the best possible.*

If we assume more about f, i.e., f is monotonically increasing, then the inequality (1.6) may be improved in the following manner [7] (see also the monograph [15]).

Theorem 5. *Let $f:[a,b]\to\mathbb{R}$ be monotonic nondecreasing. Then for all $x\in[a,b]$, we have the inequality:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{1.7}$$
$$\leq\frac{1}{b-a}\left\{[2x-(a+b)]f(x)+\int_a^b sgn(t-x)f(t)\,dt\right\}$$
$$\leq\frac{1}{b-a}\{(x-a)[f(x)-f(a)]+(b-x)[f(b)-f(x)]\}$$
$$\leq\left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right][f(b)-f(a)].$$

All the inequalities in (1.7) are sharp and the constant $\frac{1}{2}$ is the best possible.

For other scalar inequalities of Ostrowski's type see [1]-[9], [14], [21] and [22].

In order to investigate Ostrowski's type inequalities for continuous functions of selfadjoint operators we need the following concepts and results.

Let A be a selfadjoint linear operator on a complex Hilbert space $(H;\langle .,.\rangle)$. The *Gelfand map* establishes a $*$-isometrically isomorphism Φ between the set $C(Sp(A))$ of all *continuous functions* defined on the *spectrum* of A, denoted $Sp(A)$, and the C^*-algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows (see for instance [20, p. 3]):

For any $f,g\in C(Sp(A))$ and any $\alpha,\beta\in\mathbb{C}$ we have

(i) $\Phi(\alpha f+\beta g)=\alpha\Phi(f)+\beta\Phi(g)$;

(ii) $\Phi(fg)=\Phi(f)\Phi(g)$ and $\Phi(\bar{f})=\Phi(f)^*$;

(iii) $\|\Phi(f)\|=\|f\|:=\sup_{t\in Sp(A)}|f(t)|$;

(iv) $\Phi(f_0)=1_H$ and $\Phi(f_1)=A$, where $f_0(t)=1$ and $f_1(t)=t$, for $t\in Sp(A)$.

With this notation we define

$$f(A):=\Phi(f)\ \text{ for all } f\in C(Sp(A))$$

and we call it the *continuous functional calculus* for a selfadjoint operator A.

If A is a selfadjoint operator and f is a real valued continuous function on $Sp(A)$, then $f(t)\geq 0$ for any $t\in Sp(A)$ implies that $f(A)\geq 0$, *i.e.* $f(A)$ is a *positive operator* on H. Moreover, if both f and g are real valued functions on $Sp(A)$ then the following important property holds:

$$f(t)\geq g(t)\ \text{ for any } t\in Sp(A)\ \text{ implies that } f(A)\geq g(A) \tag{P}$$

in the operator order of $B(H)$.

For a recent monograph devoted to various inequalities for functions of selfadjoint operators, see [20] and the references therein.

For other results see [10], [11], [12], [23], [25], [26] and [28].

In the recent paper [13] we obtained the following Ostrowski type vector inequalities:

Theorem 6. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A)\subseteq[m,M]$ for some real numbers $m<M$. If $f:[m,M]\to\mathbb{R}$ is an*

absolutely continuous function on $[m,M]$*, then we have the vector inequalities*

$$\begin{aligned} &|f(s)\langle x,y\rangle-\langle f(A)x,y\rangle| \\ &\leq \|x\|\|y\| \\ &\times\begin{cases} \left[\frac{1}{2}(M-m)+\left|s-\frac{m+M}{2}\right|\right]\|f'\|_{\infty} & \text{if } f'\in L_{\infty}[m,M] \\ \\ \left[\frac{1}{2}(M-m)+\left|s-\frac{m+M}{2}\right|\right]^{1/q}\|f'\|_{p} & \text{if } f'\in L_{p}[m,M],\ p>1, \\ & \frac{1}{p}+\frac{1}{q}=1, \end{cases} \end{aligned} \tag{1.8}$$

for any $x,y\in H$ *and* $s\in[m,M]$.

In the case when the function $f:[m,M]\to\mathbb{R}$ is $r-H$-Hölder continuous, i.e.,

$$|f(s)-f(t)|\leq H|s-t|^{r} \text{ for any } s,t\in[m,M]$$

where $r\in(0,1]$ and $H>0$ are given, then the following result can be stated as well:

Theorem 7. *Let* A *be a selfadjoint operator in the Hilbert space* H *with the spectrum* $Sp(A)\subseteq[m,M]$ *for some real numbers* $m<M$*. If* $f:[m,M]\to\mathbb{R}$ *is* $r-H$*-Hölder continuous on* $[m,M]$*, then we have the inequality*

$$|f(s)\langle x,y\rangle-\langle f(A)x,y\rangle|\leq H\|x\|\|y\|\left[\frac{1}{2}(M-m)+\left|s-\frac{m+M}{2}\right|\right]^{r} \tag{1.9}$$

for any $x,y\in H$ *and* $s\in[m,M]$.

Motivated by the above results we investigate in this paper some Ostrowski's type vector inequalities for other classes of continuous functions of selfadjoint operators in Hilbert spaces. Applications for some particular functions of interest are provided as well.

2. Some Vector Inequalities for Functions of Bounded Variation

Let U be a selfadjoint operator on the complex Hilbert space $(H,\langle .,.\rangle)$ with the spectrum $Sp(U)$ included in the interval $[m,M]$ for some real numbers $m<M$ and let $\{E_{\lambda}\}_{\lambda}$ be its *spectral family.* Then for any continuous function $f:[m,M]\to\mathbb{R}$, it is well known that we have the following spectral representation in terms of the Riemann-Stieltjes integral:

$$\langle f(U)x,y\rangle=\int_{m-0}^{M}f(\lambda)\,d(\langle E_{\lambda}x,y\rangle), \tag{2.1}$$

for any $x,y\in H$. The function $g_{x,y}(\lambda):=\langle E_{\lambda}x,y\rangle$ is of *bounded variation* on the interval $[m,M]$ and

$$g_{x,y}(m-0)=0 \text{ and } g_{x,y}(M)=\langle x,y\rangle$$

for any $x,y\in H$. It is also well known that $g_{x}(\lambda):=\langle E_{\lambda}x,x\rangle$ is *monotonic nondecreasing* and *right continuous* on $[m,M]$.

The following result holds:

Theorem 8. *Let* A *be a selfadjoint operator in the Hilbert space* H *with the spectrum* $Sp(A)\subseteq[m,M]$ *for some real numbers* $m<M$ *and let* $\{E_{\lambda}\}_{\lambda}$ *be its spectral*

family. If $f:[m,M]\to\mathbb{R}$ is a continuous function of bounded variation on $[m,M]$, then we have the inequality

$$\begin{aligned}
(2.2)\qquad & |f(s)\langle x,y\rangle-\langle f(A)x,y\rangle| \\
&\leq \langle E_s x,x\rangle^{1/2}\langle E_s y,y\rangle^{1/2}\bigvee_m^s(f) \\
&+\langle(1_H-E_s)x,x\rangle^{1/2}\langle(1_H-E_s)y,y\rangle^{1/2}\bigvee_s^M(f) \\
&\leq \|x\|\,\|y\|\left(\frac{1}{2}\bigvee_m^M(f)+\frac{1}{2}\left|\bigvee_m^s(f)-\bigvee_s^M(f)\right|\right)\left(\leq \|x\|\,\|y\|\bigvee_m^M(f)\right)
\end{aligned}$$

for any $x,y\in H$ and for any $s\in[m,M]$.

Proof. We use the following identity for the Riemann-Stieltjes integral established by the author in 2000 in [5] (see also [15, p. 452]):

$$\begin{aligned}
(2.3)\qquad & [u(b)-u(a)]f(s)-\int_a^b f(t)\,du(t) \\
&=\int_a^s[u(t)-u(a)]\,df(t)+\int_s^b[u(t)-u(b)]\,df(t),
\end{aligned}$$

for any $s\in[a,b]$, provided the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ exists.

A simple proof can be done by utilizing the integration by parts formula and starting from the right hand side of (2.3).

If we choose in (2.3) $a=m,b=M$ and $u(t)=\langle E_t x,y\rangle$, then we have the following identity of interest in itself

$$(2.4)\quad f(s)\langle x,y\rangle-\langle f(A)x,y\rangle=\int_{m-0}^s\langle E_t x,y\rangle\,df(t)+\int_s^M\langle(E_t-1_H)x,y\rangle\,df(t)$$

for any $x,y\in H$ and for any $s\in[m,M]$.

It is well known that if $p:[a,b]\to\mathbb{C}$ is a continuous function and $v:[a,b]\to\mathbb{C}$ is of bounded variation, then the Riemann-Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and the following inequality holds

$$\left|\int_a^b p(t)\,dv(t)\right|\leq\max_{t\in[a,b]}|p(t)|\bigvee_a^b(v)$$

where $\bigvee_a^b(v)$ denotes the total variation of v on $[a,b]$.

Utilising this property we have from (2.4) that

$$\begin{aligned}
(2.5)\qquad & |f(s)\langle x,y\rangle-\langle f(A)x,y\rangle| \\
&\leq\left|\int_{m-0}^s\langle E_t x,y\rangle\,df(t)\right|+\left|\int_s^M\langle(E_t-1_H)x,y\rangle\,df(t)\right| \\
&\leq\max_{t\in[m,s]}|\langle E_t x,y\rangle|\bigvee_m^s(f)+\max_{t\in[s,M]}|\langle(E_t-1_H)x,y\rangle|\bigvee_s^M(f):=T
\end{aligned}$$

for any $x,y\in H$ and for any $s\in[m,M]$.

If P is a nonnegative operator on H, i.e., $\langle Px, x\rangle \geq 0$ for any $x \in H$, then the following inequality is a generalization of the Schwarz inequality in H

$$|\langle Px, y\rangle|^2 \leq \langle Px, x\rangle \langle Py, y\rangle \tag{2.6}$$

for any $x, y \in H$.

On applying the inequality (2.6) we have

$$|\langle E_t x, y\rangle| \leq \langle E_t x, x\rangle^{1/2} \langle E_t y, y\rangle^{1/2}$$

and

$$|\langle (1_H - E_t) x, y\rangle| \leq \langle (1_H - E_t) x, x\rangle^{1/2} \langle (1_H - E_t) y, y\rangle^{1/2}$$

for any $x, y \in H$ and $t \in [m, M]$.

Therefore

$$\begin{aligned} T &\leq \max_{t\in[m,s]} \left[\langle E_t x, x\rangle^{1/2} \langle E_t y, y\rangle^{1/2}\right] \bigvee_m^s (f) \\ &+ \max_{t\in[s,M]} \left[\langle (1_H - E_t) x, x\rangle^{1/2} \langle (1_H - E_t) y, y\rangle^{1/2}\right] \bigvee_s^M (f) \\ &\leq \max_{t\in[m,s]} \langle E_t x, x\rangle^{1/2} \max_{t\in[m,s]} \langle E_t y, y\rangle^{1/2} \bigvee_m^s (f) \\ &+ \max_{t\in[s,M]} \langle (1_H - E_t) x, x\rangle^{1/2} \max_{t\in[s,M]} \langle (1_H - E_t) y, y\rangle^{1/2} \bigvee_s^M (f) \\ &= \langle E_s x, x\rangle^{1/2} \langle E_s y, y\rangle^{1/2} \bigvee_m^s (f) \\ &+ \langle (1_H - E_s) x, x\rangle^{1/2} \langle (1_H - E_s) y, y\rangle^{1/2} \bigvee_s^M (f) \\ &:= V \end{aligned} \tag{2.7}$$

for any $x, y \in H$ and for any $s \in [m, M]$, proving the first inequality in (2.2).

Now, observe that

$$\begin{aligned} V &\leq \max\left\{\bigvee_m^s (f), \bigvee_s^M (f)\right\} \\ &\times \left[\langle E_s x, x\rangle^{1/2} \langle E_s y, y\rangle^{1/2} + \langle (1_H - E_s) x, x\rangle^{1/2} \langle (1_H - E_s) y, y\rangle^{1/2}\right]. \end{aligned}$$

Since

$$\max\left\{\bigvee_m^s (f), \bigvee_s^M (f)\right\} = \frac{1}{2}\bigvee_m^M (f) + \frac{1}{2}\left|\bigvee_m^s (f) - \bigvee_s^M (f)\right|$$

and by the Cauchy-Buniakovski-Schwarz inequality for positive real numbers a_1, b_1, a_2, b_2

$$a_1 b_1 + a_2 b_2 \leq \left(a_1^2 + a_2^2\right)^{1/2} \left(b_1^2 + b_2^2\right)^{1/2} \tag{2.8}$$

we have

$$\begin{aligned}&\langle E_s x,x\rangle^{1/2}\langle E_s y,y\rangle^{1/2}+\langle(1_H-E_s)x,x\rangle^{1/2}\langle(1_H-E_s)y,y\rangle^{1/2}\\&\le[\langle E_s x,x\rangle+\langle(1_H-E_s)x,x\rangle]^{1/2}[\langle E_s y,y\rangle+\langle(1_H-E_s)y,y\rangle]^{1/2}\\&=\|x\|\,\|y\|\end{aligned}$$

for any $x,y\in H$ and $s\in[m,M]$, then the last part of (2.2) is proven as well. □

Remark 1. *For the continuous function with bounded variation $f:[m,M]\to\mathbb{R}$ if $p\in[m,M]$ is a point with the property that*

$$\bigvee_m^p(f)=\bigvee_p^M(f)$$

then from (2.2) we get the interesting inequality

$$|f(p)\langle x,y\rangle-\langle f(A)x,y\rangle|\le\frac{1}{2}\|x\|\,\|y\|\bigvee_m^M(f) \tag{2.9}$$

for any $x,y\in H$.

If the continuous function $f:[m,M]\to\mathbb{R}$ is monotonic nondecreasing and therefore of bounded variation, we get from (2.2) the following inequality as well

$$\begin{aligned}&|f(s)\langle x,y\rangle-\langle f(A)x,y\rangle|\\&\le\langle E_s x,x\rangle^{1/2}\langle E_s y,y\rangle^{1/2}[f(s)-f(m)]\\&+\langle(1_H-E_s)x,x\rangle^{1/2}\langle(1_H-E_s)y,y\rangle^{1/2}[f(M)-f(s)]\\&\le\|x\|\,\|y\|\left(\frac{1}{2}(f(M)-f(m))+\left|f(s)-\frac{f(m)+f(M)}{2}\right|\right)\\&\le\|x\|\,\|y\|[f(M)-f(m)]\end{aligned} \tag{2.10}$$

for any $x,y\in H$ and $s\in[m,M]$.

Moreover, if the continuous function $f:[m,M]\to\mathbb{R}$ is nondecreasing on $[m,M]$, then the equation

$$f(s)=\frac{f(m)+f(M)}{2}$$

has got at least a solution in $[m,M]$. In his case we get from (2.10) the following trapezoidal type inequality

$$\left|\frac{f(m)+f(M)}{2}\langle x,y\rangle-\langle f(A)x,y\rangle\right|\le\frac{1}{2}\|x\|\,\|y\|[f(M)-f(m)] \tag{2.11}$$

for any $x,y\in H$.

3. Some Vector Inequalities for Lipschitzian Functions

The following result that incorporates the case of Lipschitzian functions also holds

Theorem 9. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A)\subseteq[m,M]$ for some real numbers $m<M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f:[m,M]\to\mathbb{R}$ is Lipschitzian with the constant $L>0$ on $[m,M]$, i.e.,*

$$|f(s)-f(t)|\le L|s-t| \text{ for any } s,t\in[m,M],$$

then we have the inequality

$$\begin{aligned}
(3.1)\qquad & |f(s)\langle x,y\rangle - \langle f(A)x,y\rangle| \\
& \le L\left[\left(\int_{m-0}^{s}\langle E_t x,x\rangle\,dt\right)^{1/2}\left(\int_{m-0}^{s}\langle E_t y,y\rangle\,dt\right)^{1/2}\right. \\
& \left.+\left(\int_{s}^{M}\langle(1_H-E_t)x,x\rangle\,dt\right)^{1/2}\left(\int_{s}^{M}\langle(1_H-E_t)y,y\rangle\,dt\right)^{1/2}\right] \\
& \le L\langle|A-s1_H|x,x\rangle^{1/2}\langle|A-s1_H|y,y\rangle^{1/2} \\
& \le L\left[D^2(A;x)+\left(s\|x\|^2-\langle Ax,x\rangle\right)^2\right]^{1/4} \\
& \times\left[D^2(A;y)+\left(s\|y\|^2-\langle Ay,y\rangle\right)^2\right]^{1/4}
\end{aligned}$$

for any $x,y\in H$ and $s\in[m,M]$, where $D(A;x)$ is the variance of the selfadjoint operator A in x and is defined by

$$D(A;x) := \left(\|Ax\|^2\|x\|^2-\langle Ax,x\rangle^2\right)^{1/2}.$$

Proof. It is well known that if $p:[a,b]\to\mathbb{C}$ is a Riemann integrable function and $v:[a,b]\to\mathbb{C}$ is Lipschitzian with the constant $L>0$, i.e.,

$$|f(s)-f(t)|\le L|s-t| \text{ for any } t,s\in[a,b],$$

then the Riemann-Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and the following inequality holds

$$\left|\int_a^b p(t)\,dv(t)\right|\le L\int_a^b|p(t)|\,dt.$$

Now, on applying this property of the Riemann-Stieltjes integral, we have from the representation (2.4) that

$$\begin{aligned}
& |f(s)\langle x,y\rangle - \langle f(A)x,y\rangle| \\
& \le\left|\int_{m-0}^{s}\langle E_t x,y\rangle\,df(t)\right|+\left|\int_{s}^{M}\langle(E_t-1_H)x,y\rangle\,df(t)\right| \\
& \le L\left[\int_{m-0}^{s}|\langle E_t x,y\rangle|\,dt+\int_{s}^{M}|\langle(E_t-1_H)x,y\rangle|\,dt\right] := LW
\end{aligned}$$

for any $x,y\in H$ and $s\in[m,M]$.

By utilizing the generalized Schwarz inequality for nonnegative operators (2.6) and the Cauchy-Buniakovski-Schwarz inequality for the Riemann integral we have

$$
\begin{aligned}
W &\leq \int_{m-0}^{s} \langle E_t x, x\rangle^{1/2} \langle E_t y, y\rangle^{1/2} dt \\
&+ \int_{s}^{M} \langle (1_H - E_t) x, x\rangle^{1/2} \langle (1_H - E_t) y, y\rangle^{1/2} dt \\
&\leq \left(\int_{m-0}^{s} \langle E_t x, x\rangle dt\right)^{1/2} \left(\int_{m-0}^{s} \langle E_t y, y\rangle dt\right)^{1/2} \\
&+ \left(\int_{s}^{M} \langle (1_H - E_t) x, x\rangle dt\right)^{1/2} \left(\int_{s}^{M} \langle (1_H - E_t) y, y\rangle dt\right)^{1/2} \\
&:= Z
\end{aligned} \tag{3.2}
$$

for any $x, y \in H$ and $s \in [m, M]$.

On the other hand, by making use of the elementary inequality (2.8) we also have

$$
\begin{aligned}
Z \leq &\left(\int_{m-0}^{s} \langle E_t x, x\rangle dt + \int_{s}^{M} \langle (1_H - E_t) x, x\rangle dt\right)^{1/2} \\
&\times \left(\int_{m-0}^{s} \langle E_t y, y\rangle dt + \int_{s}^{M} \langle (1_H - E_t) y, y\rangle dt\right)^{1/2}
\end{aligned} \tag{3.3}
$$

for any $x, y \in H$ and $s \in [m, M]$.

Now, observe that, by the use of the representation (2.4) for the continuous function $f : [m, M] \to \mathbb{R}$, $f(t) = |t - s|$ where s is fixed in $[m, M]$ we have the following identity that is of interest in itself

$$
\langle |A - s \cdot 1_H| x, y\rangle = \int_{m-0}^{s} \langle E_t x, y\rangle dt + \int_{s}^{M} \langle (1_H - E_t) x, y\rangle dt \tag{3.4}
$$

for any $x, y \in H$.

On utilizing (3.4) for x and then for y we deduce the second part of (3.1).

Finally, by the well known inequality for the modulus of a bounded linear operator

$$
\langle |T| x, x\rangle \leq \|Tx\| \|x\|, x \in H
$$

we have

$$
\begin{aligned}
\langle |A - s \cdot 1_H| x, x\rangle^{1/2} &\leq \|Ax - sx\|^{1/2} \|x\|^{1/2} \\
&= \left(\|Ax\|^2 - 2s \langle Ax, x\rangle + s^2 \|x\|^2\right)^{1/4} \|x\|^{1/2} \\
&= \left[\|Ax\|^2 \|x\|^2 - \langle Ax, x\rangle^2 + \left(s \|x\|^2 - \langle Ax, x\rangle\right)^2\right]^{1/4} \\
&= \left[D^2(A; x) + \left(s \|x\|^2 - \langle Ax, x\rangle\right)^2\right]^{1/4}
\end{aligned}
$$

and a similar relation for y. The proof is thus complete. □

Remark 2. *Since A is a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$, then*

$$\left| A - \frac{m+M}{2} \cdot 1_H \right| \leq \frac{M-m}{2} 1_H$$

giving from (3.1) that

$$\begin{aligned}
(3.5)\qquad & \left| f\left(\frac{m+M}{2}\right) \langle x, y \rangle - \langle f(A) x, y \rangle \right| \\
& \leq L \left[\left(\int_{m-0}^{\frac{m+M}{2}} \langle E_t x, x \rangle \, dt \right)^{1/2} \left(\int_{m-0}^{\frac{m+M}{2}} \langle E_t y, y \rangle \, dt \right)^{1/2} \right. \\
& \left. + \left(\int_{\frac{m+M}{2}}^{M} \langle (1_H - E_t) x, x \rangle \, dt \right)^{1/2} \left(\int_{\frac{m+M}{2}}^{M} \langle (1_H - E_t) y, y \rangle \, dt \right)^{1/2} \right] \\
& \leq L \left\langle \left| A - \frac{m+M}{2} \cdot 1_H \right| x, x \right\rangle^{1/2} \left\langle \left| A - \frac{m+M}{2} \cdot 1_H \right| y, y \right\rangle^{1/2} \\
& \leq \frac{1}{2} L (M - m) \|x\| \|y\|
\end{aligned}$$

for any $x, y \in H$.

The particular case of equal vectors is of interest:

Corollary 1. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$. If $f : [m, M] \to \mathbb{R}$ is Lipschitzian with the constant $L > 0$ on $[m, M]$, then we have the inequality*

$$\begin{aligned}
(3.6)\qquad \left| f(s) \|x\|^2 - \langle f(A) x, x \rangle \right| & \leq L \langle |A - s \cdot 1_H| x, x \rangle \\
& \leq L \left[D^2(A; x) + \left(s \|x\|^2 - \langle Ax, x \rangle \right)^2 \right]^{1/2}
\end{aligned}$$

for any $x \in H$ and $s \in [m, M]$.

Remark 3. *An important particular case that can be obtained from (3.6) is the one when $s = \frac{\langle Ax, x \rangle}{\|x\|^2}, x \neq 0$, giving the inequality*

$$\begin{aligned}
(3.7)\qquad \left| f\left(\frac{\langle Ax, x \rangle}{\|x\|^2} \right) \|x\|^2 - \langle f(A) x, x \rangle \right| & \leq L \left\langle \left| A - \frac{\langle Ax, x \rangle}{\|x\|^2} \cdot 1_H \right| x, x \right\rangle \\
& \leq L D(A; x) \leq \frac{1}{2} L (M - m) \|x\|^2
\end{aligned}$$

for any $x \in H, x \neq 0$.

The following lemma may be stated.

Lemma 1. *Let $u : [a, b] \to \mathbb{R}$ and $\varphi, \Phi \in \mathbb{R}$ with $\Phi > \varphi$. The following statements are equivalent:*

(i) *The function $u - \frac{\varphi + \Phi}{2} \cdot e$, where $e(t) = t, t \in [a, b]$, is $\frac{1}{2}(\Phi - \varphi)$ −Lipschitzian;*
(ii) *We have the inequality:*

$$(3.8)\qquad \varphi \leq \frac{u(t) - u(s)}{t - s} \leq \Phi \quad \text{for each} \quad t, s \in [a, b] \quad \text{with } t \neq s;$$

(iii) *We have the inequality:*

$$\varphi(t-s) \leq u(t) - u(s) \leq \Phi(t-s) \quad \text{for each} \quad t,s \in [a,b] \quad \text{with } t > s. \tag{3.9}$$

Following [22], we can introduce the concept:

Definition 1. *The function $u : [a,b] \to \mathbb{R}$ which satisfies one of the equivalent conditions (i) – (iii) is said to be (φ, Φ) −Lipschitzian on $[a,b]$.*

Notice that in [22], the definition was introduced on utilizing the statement (iii) and only the equivalence (i) ⇔ (iii) was considered.

Utilising *Lagrange's mean value theorem*, we can state the following result that provides practical examples of (φ, Φ) −Lipschitzian functions.

Proposition 1. *Let $u : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). If*

$$-\infty < \gamma := \inf_{t\in(a,b)} u'(t), \qquad \sup_{t\in(a,b)} u'(t) =: \Gamma < \infty \tag{3.10}$$

then u is (γ, Γ) −Lipschitzian on $[a,b]$.

We are able now to provide the following corollary:

Corollary 2. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m,M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f : [m,M] \to \mathbb{R}$ is a (φ, Φ) −Lipschitzian functions on $[m,M]$ with $\Phi > \varphi$, then we have the inequality*

$$\begin{aligned} &\left| \langle f(A)x,y\rangle - \frac{\Phi+\varphi}{2}\langle Ax,y\rangle + \frac{\Phi+\varphi}{2}s\langle x,y\rangle - f(s)\langle x,y\rangle \right| \\ &\leq \frac{1}{2}(\Phi-\varphi)\left[\left(\int_{m-0}^{s}\langle E_t x,x\rangle dt\right)^{1/2}\left(\int_{m-0}^{s}\langle E_t y,y\rangle dt\right)^{1/2}\right. \\ &\left.+\left(\int_{s}^{M}\langle(1_H-E_t)x,x\rangle dt\right)^{1/2}\left(\int_{s}^{M}\langle(1_H-E_t)y,y\rangle dt\right)^{1/2}\right] \\ &\leq \frac{1}{2}(\Phi-\varphi)\langle|A-s1_H|x,x\rangle^{1/2}\langle|A-s1_H|y,y\rangle^{1/2} \\ &\leq \frac{1}{2}(\Phi-\varphi)\left[D^2(A;x)+\left(s\|x\|^2-\langle Ax,x\rangle\right)^2\right]^{1/4} \\ &\times\left[D^2(A;y)+\left(s\|y\|^2-\langle Ay,y\rangle\right)^2\right]^{1/4} \end{aligned} \tag{3.11}$$

for any $x, y \in H$.

Remark 4. *Various particular cases can be stated by utilizing the inequality (3.11), however the details are left to the interested reader.*

4. Some Vector Inequalities for Monotonic Functions

The case of monotonic functions is of interest as well. The corresponding result is incorporated in the following

Theorem 10. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f : [m, M] \to \mathbb{R}$ is a continuous monotonic nondecreasing function on $[m, M]$, then we have the inequality*

$$
\begin{aligned}
(4.1)\qquad & |f(s)\langle x,y\rangle - \langle f(A)x,y\rangle| \\
&\le \left(\int_{m-0}^{s}\langle E_t x,x\rangle\, df(t)\right)^{1/2}\left(\int_{m-0}^{s}\langle E_t y,y\rangle\, df(t)\right)^{1/2} \\
&+ \left(\int_{s}^{M}\langle (1_H - E_t)x,x\rangle\, df(t)\right)^{1/2}\left(\int_{s}^{M}\langle (1_H - E_t)y,y\rangle\, df(t)\right)^{1/2} \\
&\le \langle |f(A) - f(s)1_H|\, x,x\rangle^{1/2}\langle |f(A) - f(s)1_H|\, y,y\rangle^{1/2} \\
&\le \left[D^2(f(A);x) + \left(f(s)\|x\|^2 - \langle f(A)x,x\rangle\right)^2\right]^{1/4} \\
&\times \left[D^2(f(A);y) + \left(f(s)\|y\|^2 - \langle f(A)y,y\rangle\right)^2\right]^{1/4}
\end{aligned}
$$

for any $x, y \in H$ and $s \in [m, M]$, where, as above $D(f(A);x)$ is the variance of the selfadjoint operator $f(A)$ in x.

Proof. From the theory of Riemann-Stieltjes integral is well known that if $p : [a, b] \to \mathbb{C}$ is of bounded variation and $v : [a, b] \to \mathbb{R}$ is continuous and monotonic nondecreasing, then the Riemann-Stieltjes integrals $\int_a^b p(t)\,dv(t)$ and $\int_a^b |p(t)|\,dv(t)$ exist and

$$\left|\int_a^b p(t)\,dv(t)\right| \le \int_a^b |p(t)|\,dv(t).$$

On utilizing this property and the representation (2.4) we have successively

$$
\begin{aligned}
(4.2)\qquad & |f(s)\langle x,y\rangle - \langle f(A)x,y\rangle| \\
&\le \left|\int_{m-0}^{s}\langle E_t x,y\rangle\, df(t)\right| + \left|\int_{s}^{M}\langle (E_t - 1_H)x,y\rangle\, df(t)\right| \\
&\le \int_{m-0}^{s}|\langle E_t x,y\rangle|\, df(t) + \int_{s}^{M}|\langle (E_t - 1_H)x,y\rangle|\, df(t) \\
&\le \int_{m-0}^{s}\langle E_t x,x\rangle^{1/2}\langle E_t y,y\rangle^{1/2}\, df(t) \\
&+ \int_{s}^{M}\langle (1_H - E_t)x,x\rangle^{1/2}\langle (1_H - E_t)y,y\rangle^{1/2}\, df(t) \\
&:= Y,
\end{aligned}
$$

for any $x, y \in H$ and $s \in [m, M]$.

We use now the following version of the Cauchy-Buniakovski-Schwarz inequality for the Riemann-Stieltjes integral with monotonic nondecreasing integrators

$$\left(\int_a^b p(t)\,q(t)\,dv(t)\right)^2 \le \int_a^b p^2(t)\,dv(t)\int_a^b q^2(t)\,dv(t)$$

to get that

$$\int_{m-0}^{s} \langle E_t x, x\rangle^{1/2} \langle E_t y, y\rangle^{1/2} df(t) \le \left(\int_{m-0}^{s} \langle E_t x, x\rangle df(t)\right)^{1/2} \left(\int_{m-0}^{s} \langle E_t y, y\rangle df(t)\right)^{1/2}$$

and

$$\begin{aligned}&\int_{s}^{M} \langle (1_H - E_t) x, x\rangle^{1/2} \langle (1_H - E_t) y, y\rangle^{1/2} df(t)\\ &\le \left(\int_{s}^{M} \langle (1_H - E_t) x, x\rangle df(t)\right)^{1/2} \left(\int_{s}^{M} \langle (1_H - E_t) y, y\rangle df(t)\right)^{1/2}\end{aligned}$$

for any $x, y \in H$ and $s \in [m, M]$.

Therefore

$$\begin{aligned}Y &\le \left(\int_{m-0}^{s} \langle E_t x, x\rangle df(t)\right)^{1/2} \left(\int_{m-0}^{s} \langle E_t y, y\rangle df(t)\right)^{1/2}\\ &+ \left(\int_{s}^{M} \langle (1_H - E_t) x, x\rangle df(t)\right)^{1/2} \left(\int_{s}^{M} \langle (1_H - E_t) y, y\rangle df(t)\right)^{1/2}\\ &\le \left(\int_{m-0}^{s} \langle E_t x, x\rangle df(t) + \int_{s}^{M} \langle (1_H - E_t) x, x\rangle df(t)\right)^{1/2}\\ &\times \left(\int_{m-0}^{s} \langle E_t y, y\rangle df(t) + \int_{s}^{M} \langle (1_H - E_t) y, y\rangle df(t)\right)^{1/2}\end{aligned}$$

for any $x, y \in H$ and $s \in [m, M]$, where, to get the last inequality we have used the elementary inequality (2.8).

Now, since f is monotonic nondecreasing, on applying the representation (2.4) for the function $|f(\cdot) - f(s)|$ with s fixed in $[m, M]$ we deduce the following identity that is of interest in itself as well:

$$\langle |f(A) - f(s)| x, y\rangle = \int_{m-0}^{s} \langle E_t x, y\rangle df(t) + \int_{s}^{M} \langle (1_H - E_t) x, y\rangle df(t) \tag{4.3}$$

for any $x, y \in H$.

The second part of (4.1) follows then by writing (4.3) for x then by y and utilizing the relevant inequalities from above.

The last part is similar to the corresponding one from the proof of Theorem 9 and the details are omitted. □

The following corollary is of interest:

Corollary 3. *With the assumption of Theorem 10 we have the inequalities*

$$\left|\frac{f(m)+f(M)}{2}\langle x,y\rangle-\langle f(A)x,y\rangle\right| \tag{4.4}$$
$$\leq\left\langle\left|f(A)-\frac{f(m)+f(M)}{2}\cdot 1_H\right|x,x\right\rangle^{1/2}$$
$$\times\left\langle\left|f(A)-\frac{f(m)+f(M)}{2}\cdot 1_H\right|y,y\right\rangle^{1/2}$$
$$\leq\frac{1}{2}\left(f(M)-f(m)\right)\|x\|\|y\|,$$

for any $x,y\in H$.

Proof. Since f is monotonic nondecreasing, then $f(u)\in[f(m),f(M)]$ for any $u\in[m,M]$. By the continuity of f it follows that there exists at list one $s\in[m,M]$ such that

$$f(s)=\frac{f(m)+f(M)}{2}.$$

Now, on utilizing the inequality (4.1) for this s we deduce the first inequality in (4.4). The second part follows as above and the details are omitted. □

5. Power Inequalities

We consider the power function $f(t):=t^p$ where $p\in\mathbb{R}\diagdown\{0\}$ and $t>0$. The following mid-point inequalities hold:

Proposition 2. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A)\subseteq[m,M]$ for some real numbers with $0\leq m<M$.*

If $p>0$, then for any $x,y\in H$

$$\left|\left(\frac{m+M}{2}\right)^p\langle x,y\rangle-\langle A^p x,y\rangle\right| \tag{5.1}$$
$$\leq B_p\left\langle\left|A-\frac{m+M}{2}\cdot 1_H\right|x,x\right\rangle^{1/2}\left\langle\left|A-\frac{m+M}{2}\cdot 1_H\right|y,y\right\rangle^{1/2}$$
$$\leq\frac{1}{2}B_p(M-m)\|x\|\|y\|$$

where

$$B_p=p\times\begin{cases}M^{p-1} & \text{if } p\geq 1\\ \\ m^{p-1} & \text{if } 0<p<1, m>0.\end{cases}$$

and

$$\left|\left(\frac{m+M}{2}\right)^{-p}\langle x,y\rangle-\left\langle A^{-p}x,y\right\rangle\right| \tag{5.2}$$
$$\leq C_p\left\langle\left|A-\frac{m+M}{2}\cdot 1_H\right|x,x\right\rangle^{1/2}\left\langle\left|A-\frac{m+M}{2}\cdot 1_H\right|y,y\right\rangle^{1/2}$$
$$\leq\frac{1}{2}C_p(M-m)\|x\|\|y\|$$

where

$$C_p=pm^{-p-1}\text{ and } m>0.$$

The proof follows from (3.5).

We can also state the following trapezoidal type inequalities:

Proposition 3. *With the assumption of Proposition 2 and if $p > 0$ we have the inequalities*

$$\begin{aligned}(5.3)\qquad &\left|\frac{m^p + M^p}{2}\langle x, y\rangle - \langle A^p x, y\rangle\right| \\ &\leq \left\langle \left|A^p - \frac{m^p + M^p}{2}\cdot 1_H\right| x, x\right\rangle^{1/2} \left\langle \left|A^p - \frac{m^p + M^p}{2}\cdot 1_H\right| y, y\right\rangle^{1/2} \\ &\leq \frac{1}{2}\left(M^p - m^p\right)\|x\|\,\|y\|,\end{aligned}$$

and, for $m > 0$,

$$\begin{aligned}(5.4)\qquad &\left|\frac{m^p + M^p}{2m^p M^p}\langle x, y\rangle - \left\langle A^{-p} x, y\right\rangle\right| \\ &\leq \left\langle \left|A^{-p} - \frac{m^p + M^p}{2m^p M^p}\cdot 1_H\right| x, x\right\rangle^{1/2} \left\langle \left|A^{-p} - \frac{m^p + M^p}{2m^p M^p}\cdot 1_H\right| y, y\right\rangle^{1/2} \\ &\leq \frac{1}{2}\left(\frac{M^p - m^p}{M^p m^p}\right)\|x\|\,\|y\|,\end{aligned}$$

for any $x, y \in H$.

The proof follows from Corollary 3.

6. Logarithmic Inequalities

Consider the function $f(t) = \ln t, t > 0$. Denote by $A(a, b) := \frac{a+b}{2}$ the arithmetic mean of $a, b > 0$ and $G(a, b) := \sqrt{ab}$ the geometric mean of these numbers. We have the following result:

Proposition 4. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers with $0 < m < M$. For any $x, y \in H$ we have*

$$\begin{aligned}(6.1)\qquad &\left|\ln A(m, M)\cdot\langle x, y\rangle - \langle \ln A x, y\rangle\right| \\ &\leq \frac{1}{m}\left\langle \left|A - \frac{m + M}{2}\cdot 1_H\right| x, x\right\rangle^{1/2} \left\langle \left|A - \frac{m + M}{2}\cdot 1_H\right| y, y\right\rangle^{1/2} \\ &\leq \frac{1}{2}\left(\frac{M}{m} - 1\right)\|x\|\,\|y\|\end{aligned}$$

and

$$\begin{aligned}(6.2)\qquad &\left|\ln G(m, M)\cdot\langle x, y\rangle - \langle \ln A x, y\rangle\right| \\ &\leq \langle |\ln A - \ln G(m, M)\cdot 1_H|\, x, x\rangle^{1/2}\, \langle |\ln A - \ln G(m, M)\cdot 1_H|\, y, y\rangle^{1/2} \\ &\leq \ln\sqrt{\frac{M}{m}}\cdot\|x\|\,\|y\|.\end{aligned}$$

The proof follows by (3.5) and (4.4).

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