SOME TRAPEZOIDAL VECTOR INEQUALITIES FOR CONTINUOUS FUNCTIONS OF SELFADJOINT OPERATORS IN HILBERT SPACES

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Abstract. On utilising the spectral representation of selfadjoint operators in Hilbert spaces, some trapezoidal inequalities for various classes of continuous functions of such operators are given.

1. Introduction

In Classical Analysis a *trapezoidal type inequality* is an inequality that provides upper and/or lower bounds for the quantity

$$\frac{f(a)+f(b)}{2}(b-a)-\int_a^b f(t)\,dt,$$

that is the error in approximating the integral by a trapezoidal rule, for various classes of integrable functions f defined on the compact interval $[a,b]$.

In order to introduce the reader to some of the well known results and prepare the background for considering a similar problem for functions of selfadjoint operators in Hilbert spaces, we mention the following inequalities.

The case of functions of bounded variation was obtained in [2] (see also [1, p. 68]):

Theorem 1. *Let $f:[a,b]\to\mathbb{C}$ be a function of bounded variation. We have the inequality*

$$\left|\int_a^b f(t)\,dt-\frac{f(a)+f(b)}{2}(b-a)\right|\le\frac{1}{2}(b-a)\bigvee_a^b(f), \tag{1.1}$$

where $\bigvee_a^b(f)$ denotes the total variation of f on the interval $[a,b]$. The constant $\frac{1}{2}$ is the best possible one.

This result may be improved if one assumes the monotonicity of f as follows (see [1, p. 76]):

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Theorem 2. *Let $f:[a,b]\to\mathbb{R}$ be a monotonic nondecreasing function on $[a,b]$. Then we have the inequalities:*

$$\left|\int_a^b f(t)\,dt-\frac{f(a)+f(b)}{2}(b-a)\right| \le \frac{1}{2}(b-a)[f(b)-f(a)]-\int_a^b sgn\left(t-\frac{a+b}{2}\right)f(t)\,dt \le \frac{1}{2}(b-a)[f(b)-f(a)]. \tag{1.2}$$

The above inequalities are sharp.

If the mapping is Lipschitzian, then the following result holds as well [3] (see also [1, p. 82]).

Theorem 3. *Let $f:[a,b]\to\mathbb{C}$ be an $L-$Lipschitzian function on $[a,b]$, i.e., f satisfies the condition:*

$$|f(s)-f(t)|\le L|s-t| \quad \text{for any } s,t\in[a,b] \quad (L>0 \text{ is given}). \tag{L}$$

Then we have the inequality:

$$\left|\int_a^b f(t)\,dt-\frac{f(a)+f(b)}{2}(b-a)\right|\le\frac{1}{4}(b-a)^2L. \tag{1.3}$$

The constant $\frac{1}{4}$ is best in (1.3).

If we would assume absolute continuity for the function f, then the following estimates in terms of the Lebesgue norms of the derivative f' hold [1, p. 93].

Theorem 4. *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous function on $[a,b]$. Then we have*

$$\left|\int_a^b f(t)\,dt-\frac{f(a)+f(b)}{2}(b-a)\right| \le\begin{cases}\frac{1}{4}(b-a)^2\|f'\|_\infty & \text{if } f'\in L_\infty[a,b];\\ \frac{1}{2(q+1)^{\frac{1}{q}}}(b-a)^{1+1/q}\|f'\|_p & \text{if } f'\in L_p[a,b],\\ & \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \frac{1}{2}(b-a)\|f'\|_1, & \end{cases} \tag{1.4}$$

where $\|\cdot\|_p$ $(p\in[1,\infty])$ are the Lebesgue norms, i.e.,

$$\|f'\|_\infty=ess\sup_{s\in[a,b]}|f'(s)|$$

and

$$\|f'\|_p:=\left(\int_a^b|f'(s)|\,ds\right)^{\frac{1}{p}},\ \ p\ge1.$$

The case of convex functions is as follows [4]:

Theorem 5. *Let $f:[a,b]\to\mathbb{R}$ be a convex function on $[a,b]$. Then we have the inequalities*

$$\begin{aligned}\text{(1.5)}\qquad & \frac{1}{8}(b-a)^2\left[f'_+\left(\frac{a+b}{2}\right)-f'_-\left(\frac{a+b}{2}\right)\right] \\ & \le \frac{f(a)+f(b)}{2}(b-a)-\int_a^b f(t)\,dt \\ & \le \frac{1}{8}(b-a)^2\left[f'_-(b)-f'_+(a)\right].\end{aligned}$$

The constant $\frac{1}{8}$ is sharp in both sides of (1.5).

For other scalar trapezoidal type inequalities, see [1].

2. Trapezoidal Operator Inequalities

In order to provide some generalizations for functions of selfadjoint operators of the above trapezoidal inequalities, we need some concepts as results as follows.

Let A be a selfadjoint linear operator on a complex Hilbert space $(H;\langle .,.\rangle)$. The *Gelfand map* establishes a $*$-isometrically isomorphism Φ between the set $C(Sp(A))$ of all *continuous functions* defined on the *spectrum* of A, denoted $Sp(A)$, and the C^*-algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows (see for instance [8, p. 3]):

For any $f,g\in C(Sp(A))$ and any $\alpha,\beta\in\mathbb{C}$ we have

(i) $\Phi(\alpha f+\beta g)=\alpha\Phi(f)+\beta\Phi(g)$;

(ii) $\Phi(fg)=\Phi(f)\Phi(g)$ and $\Phi(\bar{f})=\Phi(f)^*$;

(iii) $\|\Phi(f)\|=\|f\|:=\sup_{t\in Sp(A)}|f(t)|$;

(iv) $\Phi(f_0)=1_H$ and $\Phi(f_1)=A$, where $f_0(t)=1$ and $f_1(t)=t$, for $t\in Sp(A)$.

With this notation we define

$$f(A):=\Phi(f) \text{ for all } f\in C(Sp(A))$$

and we call it the *continuous functional calculus* for a selfadjoint operator A.

If A is a selfadjoint operator and f is a real valued continuous function on $Sp(A)$, then $f(t)\ge 0$ for any $t\in Sp(A)$ implies that $f(A)\ge 0$, *i.e.* $f(A)$ is a *positive operator* on H. Moreover, if both f and g are real valued functions on $Sp(A)$ then the following important property holds:

$$\text{(P)}\qquad f(t)\ge g(t) \text{ for any } t\in Sp(A) \text{ implies that } f(A)\ge g(A)$$

in the operator order of $B(H)$.

For a recent monograph devoted to various inequalities for continuous functions of selfadjoint operators, see [8] and the references therein.

For other recent results see [5], [6], [7], [9], [10], [11] and [12].

Let U be a selfadjoint operator on the complex Hilbert space $(H,\langle .,.\rangle)$ with the spectrum $Sp(U)$ included in the interval $[m,M]$ for some real numbers $m<M$ and let $\{E_\lambda\}_\lambda$ be its *spectral family.* Then for any continuous function $f:[m,M]\to\mathbb{C}$, it is well known that we have the following *spectral representation in terms of the Riemann-Stieltjes integral*:

$$\langle f(U)x,y\rangle=\int_{m-0}^{M} f(\lambda)\,d(\langle E_\lambda x,y\rangle), \tag{2.1}$$

for any $x, y \in H$. The function $g_{x,y}(\lambda) := \langle E_\lambda x, y \rangle$ is of *bounded variation* on the interval $[m, M]$ and

$$g_{x,y}(m-0) = 0 \text{ and } g_{x,y}(M) = \langle x, y \rangle$$

for any $x, y \in H$. It is also well known that $g_x(\lambda) := \langle E_\lambda x, x \rangle$ is *monotonic nondecreasing* and *right continuous* on $[m, M]$.

With the notations introduced above, we consider in this paper the problem of bounding the error

$$\frac{f(M) + f(m)}{2} \cdot \langle x, y \rangle - \langle f(A) x, y \rangle$$

in approximating $\langle f(A) x, y \rangle$ by the trapezoidal type formula $\frac{f(M)+f(m)}{2} \cdot \langle x, y \rangle$, where x, y are vectors in the Hilbert space H, f is a continuous functions of the selfadjoint operator A with the spectrum in the compact interval of real numbers $[m, M]$. Applications for some particular elementary functions are also provided.

3. Some Trapezoidal Vector Inequalities

The following result holds:

Theorem 6. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f : [m, M] \to \mathbb{C}$ is a continuous function of bounded variation on $[m, M]$, then we have the inequality*

$$\begin{aligned} & \left| \frac{f(M) + f(m)}{2} \cdot \langle x, y \rangle - \langle f(A) x, y \rangle \right| \\ & \leq \frac{1}{2} \max_{\lambda \in [m, M]} \Big[\langle E_\lambda x, x \rangle^{1/2} \langle E_\lambda y, y \rangle^{1/2} \\ & + \langle (1_H - E_\lambda) x, x \rangle^{1/2} \langle (1_H - E_\lambda) y, y \rangle^{1/2} \Big] \bigvee_m^M (f) \\ & \leq \frac{1}{2} \|x\| \|y\| \bigvee_m^M (f) \end{aligned} \tag{3.1}$$

for any $x, y \in H$.

Proof. If $f, u : [m, M] \to \mathbb{C}$ are such that the Riemann-Stieltjes integral $\int_a^b f(t)\, du(t)$ exists, then a simple integration by parts reveals the identity

$$\begin{aligned} \int_a^b f(t)\, du(t) = & \frac{f(a) + f(b)}{2} [u(b) - u(a)] \\ & - \int_a^b \left[u(t) - \frac{u(a) + u(b)}{2} \right] df(t). \end{aligned} \tag{3.2}$$

If we write the identity (3.2) for $u(\lambda) = \langle E_\lambda x, y \rangle$, then we get

$$\begin{aligned} \int_{m-0}^M f(\lambda)\, d(\langle E_\lambda x, y \rangle) = & \frac{f(m) + f(M)}{2} \cdot \langle x, y \rangle \\ & - \int_{m-0}^M \left(\langle E_\lambda x, y \rangle - \frac{1}{2} \langle x, y \rangle \right) df(\lambda) \end{aligned}$$

which, by (2.1), gives the following identity of interest in itself

$$\frac{f(m)+f(M)}{2}\cdot\langle x,y\rangle-\langle f(A)x,y\rangle = \frac{1}{2}\int_{m-0}^{M}\left[\langle E_\lambda x,y\rangle+\langle (E_\lambda-1_H)x,y\rangle\right]df(\lambda), \tag{3.3}$$

for any $x,y\in H$.

It is well known that if $p:[a,b]\to\mathbb{C}$ is a continuous function and $v:[a,b]\to\mathbb{C}$ is of bounded variation, then the Riemann-Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and the following inequality holds

$$\left|\int_a^b p(t)\,dv(t)\right|\le\max_{t\in[a,b]}|p(t)|\bigvee_a^b(v) \tag{3.4}$$

where $\bigvee_a^b(v)$ denotes the total variation of v on $[a,b]$.

Utilising the property (3.4), we have from (3.3) that

$$\begin{aligned}&\left|\frac{f(m)+f(M)}{2}\cdot\langle x,y\rangle-\langle f(A)x,y\rangle\right|\\ &\le\frac{1}{2}\max_{\lambda\in[m,M]}|\langle E_\lambda x,y\rangle+\langle (E_\lambda-1_H)x,y\rangle|\bigvee_m^M(f)\\ &\le\frac{1}{2}\left[\max_{\lambda\in[m,M]}\left[|\langle E_\lambda x,y\rangle|+|\langle (1_H-E_\lambda)x,y\rangle|\right]\right]\bigvee_m^M(f).\end{aligned} \tag{3.5}$$

If P is a nonnegative operator on H, i.e., $\langle Px,x\rangle\ge 0$ for any $x\in H$, then the following inequality is a generalization of the Schwarz inequality in the Hilbert space H

$$|\langle Px,y\rangle|^2\le\langle Px,x\rangle\langle Py,y\rangle, \tag{3.6}$$

for any $x,y\in H$.

On applying the inequality (3.6) we have

$$|\langle E_\lambda x,y\rangle|\le\langle E_\lambda x,x\rangle^{1/2}\langle E_\lambda y,y\rangle^{1/2}$$

and

$$|\langle (1_H-E_\lambda)x,y\rangle|\le\langle (1_H-E_\lambda)x,x\rangle^{1/2}\langle (1_H-E_\lambda)y,y\rangle^{1/2},$$

which, together with the elementary inequality for $a,b,c,d\ge 0$

$$ab+cd\le\left(a^2+c^2\right)^{1/2}\left(b^2+d^2\right)^{1/2}$$

produce the inequalities

$$\begin{aligned}&|\langle E_\lambda x,y\rangle|+|\langle (1_H-E_\lambda)x,y\rangle|\\ &\le\langle E_\lambda x,x\rangle^{1/2}\langle E_\lambda y,y\rangle^{1/2}+\langle (1_H-E_\lambda)x,x\rangle^{1/2}\langle (1_H-E_\lambda)y,y\rangle^{1/2}\\ &\le(\langle E_\lambda x,x\rangle+\langle (1_H-E_\lambda)x,x\rangle)(\langle E_\lambda y,y\rangle+\langle (1_H-E_\lambda)y,y\rangle)\\ &=\|x\|\,\|y\|\end{aligned} \tag{3.7}$$

for any $x,y\in H$.

On utilizing (3.5) and taking the maximum in (3.7) we deduce the desired result (3.1). □

The case of Lipschitzian functions may be useful for applications:

Theorem 7. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f : [m, M] \to \mathbb{C}$ is Lipschitzian with the constant $L > 0$ on $[m, M]$, then we have the inequality*

$$
\begin{aligned}
(3.8)\qquad & \left| \frac{f(M) + f(m)}{2} \cdot \langle x, y \rangle - \langle f(A)x, y \rangle \right| \\
& \leq \frac{1}{2} L \int_{m-0}^{M} \left[\langle E_\lambda x, x \rangle^{1/2} \langle E_\lambda y, y \rangle^{1/2} \right. \\
& \left. + \langle (1_H - E_\lambda)x, x \rangle^{1/2} \langle (1_H - E_\lambda)y, y \rangle^{1/2} \right] d\lambda \\
& \leq \frac{1}{2} (M - m) L \|x\| \|y\|
\end{aligned}
$$

for any $x, y \in H$.

Proof. It is well known that if $p : [a, b] \to \mathbb{C}$ is a Riemann integrable function and $v : [a, b] \to \mathbb{C}$ is Lipschitzian with the constant $L > 0$, i.e.,

$$
|f(s) - f(t)| \leq L |s - t| \text{ for any } t, s \in [a, b],
$$

then the Riemann-Stieltjes integral $\int_a^b p(t)\, dv(t)$ exists and the following inequality holds

$$
\left| \int_a^b p(t)\, dv(t) \right| \leq L \int_a^b |p(t)|\, dt.
$$

Now, on applying this property of the Riemann-Stieltjes integral, we have from the representation (3.3) that

$$
\begin{aligned}
(3.9)\qquad & \left| \frac{f(m) + f(M)}{2} \cdot \langle x, y \rangle - \langle f(A)x, y \rangle \right| \\
& \leq \frac{1}{2} L \int_{m-0}^{M} |\langle E_\lambda x, y \rangle + \langle (E_\lambda - 1_H)x, y \rangle|\, d\lambda, \\
& \leq \frac{1}{2} L \int_{m-0}^{M} [|\langle E_\lambda x, y \rangle| + |\langle (1_H - E_\lambda)x, y \rangle|]\, d\lambda,
\end{aligned}
$$

for any $x, y \in H$.

Further, integrating (3.7) on $[m, M]$ we have

$$
\begin{aligned}
(3.10)\qquad & \int_{m-0}^{M} [|\langle E_\lambda x, y \rangle| + |\langle (1_H - E_\lambda)x, y \rangle|]\, d\lambda \\
& \leq \int_{m-0}^{M} \left[\langle E_\lambda x, x \rangle^{1/2} \langle E_\lambda y, y \rangle^{1/2} \right. \\
& \left. + \langle (1_H - E_\lambda)x, x \rangle^{1/2} \langle (1_H - E_\lambda)y, y \rangle^{1/2} \right] d\lambda \\
& \leq (M - m) \|x\| \|y\|
\end{aligned}
$$

which together with (3.9) produces the desired result (3.8). □

4. Other Trapezoidal Vector Inequalities

The following result provides a different perspective in bounding the error in the trapezoidal approximation:

Theorem 8. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. Assume that $f : [m, M] \to \mathbb{C}$ is a continuous function on $[m, M]$. Then we have the inequalities*

$$\left| \frac{f(M) + f(m)}{2} \cdot \langle x, y \rangle - \langle f(A) x, y \rangle \right| \tag{4.1}$$

$$\leq \begin{cases} \max_{\lambda \in [m,M]} \left| \left\langle E_\lambda x - \frac{1}{2} x, y \right\rangle \right| \bigvee_m^M (f) & \text{if } f \text{ is of bounded variation} \\ L \int_{m-0}^M \left| \left\langle E_\lambda x - \frac{1}{2} x, y \right\rangle \right| d\lambda & \text{if } f \text{ is } L \text{ Lipschitzian} \\ \int_{m-0}^M \left| \left\langle E_\lambda x - \frac{1}{2} x, y \right\rangle \right| df(\lambda) & \text{if } f \text{ is nondecreasing} \end{cases}$$

$$\leq \frac{1}{2} \|x\| \|y\| \begin{cases} \bigvee_m^M (f) & \text{if } f \text{ is of bounded variation} \\ L(M - m) & \text{if } f \text{ is } L \text{ Lipschitzian} \\ (f(M) - f(m)) & \text{if } f \text{ is nondecreasing} \end{cases}$$

for any $x, y \in H$.

Proof. From (3.5) we have that

$$\left| \frac{f(m) + f(M)}{2} \cdot \langle x, y \rangle - \langle f(A) x, y \rangle \right| \tag{4.2}$$

$$\leq \frac{1}{2} \max_{\lambda \in [m,M]} |\langle E_\lambda x, y \rangle + \langle (E_\lambda - 1_H) x, y \rangle| \bigvee_m^M (f)$$

$$= \max_{\lambda \in [m,M]} \left| \left\langle E_\lambda x - \frac{1}{2} x, y \right\rangle \right| \bigvee_m^M (f)$$

for any $x, y \in H$.

Utilising the Schwarz inequality in H and the fact that E_λ are projectors we have successively

$$\left| \left\langle E_\lambda x - \frac{1}{2} x, y \right\rangle \right| \leq \left\| E_\lambda x - \frac{1}{2} x \right\| \|y\| \tag{4.3}$$

$$= \left[\langle E_\lambda x, E_\lambda x \rangle - \langle E_\lambda x, x \rangle + \frac{1}{4} \|x\|^2 \right]^{1/2} \|y\|$$

$$= \frac{1}{2} \|x\| \|y\|$$

for any $x, y \in H$, which proves the first branch in (4.1).

The second inequality follows from (3.9).

From the theory of Riemann-Stieltjes integral is well known that if $p : [a, b] \to \mathbb{C}$ is of bounded variation and $v : [a, b] \to \mathbb{R}$ is continuous and monotonic nondecreasing, then the Riemann-Stieltjes integrals $\int_a^b p(t)\, dv(t)$ and $\int_a^b |p(t)|\, dv(t)$ exist and

$$\left| \int_a^b p(t)\, dv(t) \right| \le \int_a^b |p(t)|\, dv(t). \tag{4.4}$$

From the representation (3.3) we then have

$$\begin{aligned} &\left| \frac{f(m) + f(M)}{2} \cdot \langle x, y \rangle - \langle f(A) x, y \rangle \right| \\ &\le \frac{1}{2} \int_{m-0}^{M} |\langle E_\lambda x, y \rangle + \langle (E_\lambda - 1_H) x, y \rangle|\, df(\lambda) \\ &= \int_{m-0}^{M} \left| \left\langle E_\lambda x - \frac{1}{2} x, y \right\rangle \right| df(\lambda) \end{aligned} \tag{4.5}$$

for any $x, y \in H$, from which we obtain the last branch in (4.1). $\square$

We recall that a function $f : [a, b] \to \mathbb{C}$ is called $r - H$-Hölder continuous with fixed $r \in (0, 1]$ and $H > 0$ if

$$|f(t) - f(s)| \le H |t - s|^r \text{ for any } t, s \in [a, b].$$

We have the following result concerning this class of functions.

Theorem 9. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f : [m, M] \to \mathbb{C}$ is $r - H$-Hölder continuous on $[m, M]$, then we have the inequality*

$$\begin{aligned} \left| \frac{f(m) + f(M)}{2} \cdot \langle x, y \rangle - \langle f(A) x, y \rangle \right| &\le \frac{1}{2^r} H (M - m)^r \bigvee_m^M \left(\left\langle E_{(\cdot)} x, y \right\rangle \right) \\ &\le \frac{1}{2^r} H (M - m)^r \|x\| \|y\| \end{aligned} \tag{4.6}$$

for any $x, y \in H$.

Proof. We start with the equality

$$\begin{aligned} &\frac{f(M) + f(m)}{2} \cdot \langle x, y \rangle - \langle f(A) x, y \rangle \\ &= \int_{m-0}^{M} \left[\frac{f(M) + f(m)}{2} - f(\lambda) \right] d(\langle E_\lambda x, y \rangle) \end{aligned} \tag{4.7}$$

for any $x, y \in H$, that follows from the spectral representation (2.1).

Since the function $\left\langle E_{(\cdot)} x, y \right\rangle$ is of bounded variation for any vector $x, y \in H$, by applying the inequality (3.4) we conclude that

$$\begin{aligned} &\left| \frac{f(m) + f(M)}{2} \cdot \langle x, y \rangle - \langle f(A) x, y \rangle \right| \\ &\le \max_{\lambda \in [m, M]} \left| \frac{f(M) + f(m)}{2} - f(\lambda) \right| \bigvee_m^M \left(\left\langle E_{(\cdot)} x, y \right\rangle \right) \end{aligned} \tag{4.8}$$

for any $x, y \in H$.

As $f:[m,M]\to\mathbb{C}$ is $r-H$-Hölder continuous on $[m,M]$, then we have

$$\left|\frac{f(M)+f(m)}{2}-f(\lambda)\right|\le\frac{1}{2}|f(M)-f(\lambda)|+\frac{1}{2}|f(\lambda)-f(m)| \tag{4.9}$$
$$\le\frac{1}{2}H\left[(M-\lambda)^r+(\lambda-m)^r\right]$$

for any $\lambda\in[m,M]$.

Since, obviously, the function $g_r(\lambda):=(M-\lambda)^r+(\lambda-m)^r$, $r\in(0,1)$ has the property that

$$\max_{\lambda\in[m,M]}g_r(\lambda)=g_r\left(\frac{m+M}{2}\right)=2^{1-r}(M-m)^r,$$

then by (4.8) we deduce the first part of (4.6).

Now, if $d:m=t_0<t_1<...<t_{n-1}<t_n=M$ is an arbitrary partition of the interval $[m,M]$, then we have by the Schwarz inequality for nonnegative operators that

$$\begin{aligned}
&\bigvee_m^M\left(\langle E_{(\cdot)}x,y\rangle\right)\\
&=\sup_d\left\{\sum_{i=0}^{n-1}\left|\left\langle\left(E_{t_{i+1}}-E_{t_i}\right)x,y\right\rangle\right|\right\}\\
&\le\sup_d\left\{\sum_{i=0}^{n-1}\left[\left\langle\left(E_{t_{i+1}}-E_{t_i}\right)x,x\right\rangle^{1/2}\left\langle\left(E_{t_{i+1}}-E_{t_i}\right)y,y\right\rangle^{1/2}\right]\right\}:=I.
\end{aligned}$$

By the Cauchy-Buniakovski-Schwarz inequality for sequences of real numbers we also have that

$$\begin{aligned}
I&\le\sup_d\left\{\left[\sum_{i=0}^{n-1}\left\langle\left(E_{t_{i+1}}-E_{t_i}\right)x,x\right\rangle\right]^{1/2}\left[\sum_{i=0}^{n-1}\left\langle\left(E_{t_{i+1}}-E_{t_i}\right)y,y\right\rangle\right]^{1/2}\right\}\\
&\le\sup_d\left\{\left[\sum_{i=0}^{n-1}\left\langle\left(E_{t_{i+1}}-E_{t_i}\right)x,x\right\rangle\right]^{1/2}\sup_d\left[\sum_{i=0}^{n-1}\left\langle\left(E_{t_{i+1}}-E_{t_i}\right)y,y\right\rangle\right]^{1/2}\right\}\\
&=\left[\bigvee_m^M\left(\langle E_{(\cdot)}x,x\rangle\right)\right]^{1/2}\left[\bigvee_m^M\left(\langle E_{(\cdot)}y,y\rangle\right)\right]^{1/2}=\|x\|\,\|y\|
\end{aligned}$$

for any $x,y\in H$. These prove the last part of (4.6). □

5. Applications for Some Particular Functions

It is obvious that the results established above can be applied for various particular functions of selfadjoint operators. We restrict ourselves here to only two examples, namely the logarithm and the power functions.

1. If we consider the logarithmic function $f:(0,\infty)\to\mathbb{R}$, $f(t)=\ln t$, then we can state the following result:

Proposition 1. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A)\subseteq[m,M]$ for some real numbers with $0<m<M$ and let $\{E_\lambda\}_\lambda$*

be its spectral family. Then for any $x, y \in H$ we have

$$\begin{aligned}
(5.1)\qquad & \left|\langle x,y\rangle \ln\sqrt{mM} - \langle \ln Ax, y\rangle\right| \\
& \le \ln\left(\frac{M}{m}\right) \times \begin{cases} \frac{1}{2}\max_{\lambda\in[m,M]}\left[\langle E_\lambda x,x\rangle^{1/2}\langle E_\lambda y,y\rangle^{1/2} \right. \\ \quad \left. + \langle (1_H - E_\lambda)x,x\rangle^{1/2}\langle (1_H - E_\lambda)y,y\rangle^{1/2}\right] \\ \\ \max_{\lambda\in[m,M]}\left|\left\langle E_\lambda x - \frac{1}{2}x, y\right\rangle\right| \end{cases} \\
& \le \frac{1}{2}\|x\|\,\|y\|\ln\left(\frac{M}{m}\right)
\end{aligned}$$

and

$$\begin{aligned}
(5.2)\qquad & \left|\langle x,y\rangle \ln\sqrt{mM} - \langle \ln Ax, y\rangle\right| \\
& \le \frac{1}{m} \times \begin{cases} \frac{1}{2}\int_{m-0}^{M}\left[\langle E_\lambda x,x\rangle^{1/2}\langle E_\lambda y,y\rangle^{1/2} \right. \\ \quad \left. + \langle (1_H - E_\lambda)x,x\rangle^{1/2}\langle (1_H - E_\lambda)y,y\rangle^{1/2}\right] d\lambda \\ \\ \int_{m-0}^{M}\left|\left\langle E_\lambda x - \frac{1}{2}x, y\right\rangle\right| d\lambda \end{cases} \\
& \le \frac{1}{2}\|x\|\,\|y\|\left(\frac{M}{m} - 1\right)
\end{aligned}$$

and

$$\begin{aligned}
(5.3)\qquad \left|\langle x,y\rangle \ln\sqrt{mM} - \langle \ln Ax, y\rangle\right| & \le \int_{m-0}^{M}\left|\left\langle E_\lambda x - \frac{1}{2}x, y\right\rangle\right|\lambda^{-1}d\lambda \\
& \le \frac{1}{2}\|x\|\,\|y\|\ln\left(\frac{M}{m}\right)
\end{aligned}$$

respectively.

The proof is obvious from Theorems 6, 7 and 8 applied for the logarithmic function. The details are omitted.

2. Consider now the power function $f : (0,\infty) \to \mathbb{R}$, $f(t) = t^p$ with $p \in (-\infty, 0) \cup (0, \infty)$. In the case when $p \in (0,1)$, the function is $p - H$-Hölder continuous with $H = 1$ on any subinterval $[m, M]$ of $[0,\infty)$. By making use of Theorem 9 we can state the following result:

Proposition 2. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers with $0 \le m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. Then for $p \in (0,1)$ we have*

$$\begin{aligned}
(5.4)\qquad \left|\frac{m^p + M^p}{2}\cdot\langle x,y\rangle - \langle A^p x, y\rangle\right| & \le \frac{1}{2^p}(M-m)^p \bigvee_m^M \left(\langle E_{(\cdot)}x, y\rangle\right) \\
& \le \frac{1}{2^p}(M-m)^p\|x\|\,\|y\|,
\end{aligned}$$

for any $x, y \in H$.

The case of powers $p \ge 1$ is embodied in the following:

Proposition 3. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers with $0 \leq m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. Then for $p \geq 1$ and for any $x, y \in H$ we have*

$$\left|\frac{m^p + M^p}{2} \cdot \langle x, y\rangle - \langle A^p x, y\rangle\right| \leq (M^p - m^p) \times \begin{cases} \frac{1}{2}\max_{\lambda\in[m,M]}\left[\langle E_\lambda x, x\rangle^{1/2}\langle E_\lambda y, y\rangle^{1/2} \right. \\ \left. \quad + \langle (1_H - E_\lambda)x, x\rangle^{1/2}\langle (1_H - E_\lambda)y, y\rangle^{1/2}\right] \\ \\ \max_{\lambda\in[m,M]}\left|\left\langle E_\lambda x - \frac{1}{2}x, y\right\rangle\right| \end{cases} \leq \frac{1}{2}\|x\|\|y\|(M^p - m^p) \tag{5.5}$$

and

$$\left|\frac{m^p + M^p}{2} \cdot \langle x, y\rangle - \langle A^p x, y\rangle\right| \leq pM^{p-1} \times \begin{cases} \frac{1}{2}\int_{m-0}^{M}\left[\langle E_\lambda x, x\rangle^{1/2}\langle E_\lambda y, y\rangle^{1/2} \right. \\ \left. \quad + \langle (1_H - E_\lambda)x, x\rangle^{1/2}\langle (1_H - E_\lambda)y, y\rangle^{1/2}\right] d\lambda \\ \\ \int_{m-0}^{M}\left|\left\langle E_\lambda x - \frac{1}{2}x, y\right\rangle\right| d\lambda \end{cases} \leq \frac{1}{2}p\|x\|\|y\| M^{p-1} \tag{5.6}$$

and

$$\left|\frac{m^p + M^p}{2} \cdot \langle x, y\rangle - \langle A^p x, y\rangle\right| \leq p\int_{m-0}^{M}\left|\left\langle E_\lambda x - \frac{1}{2}x, y\right\rangle\right|\lambda^{p-1}d\lambda \leq \frac{1}{2}\|x\|\|y\|(M^p - m^p) \tag{5.7}$$

respectively.

The proof is obvious from Theorems 6, 7 and 8 applied for the power function $f : (0, \infty) \to \mathbb{R}$, $f(t) = t^p$ with $p \geq 1$. The details are omitted.

The case of negative powers is similar. The details are left to the interested reader.

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