APPROXIMATING n-TIME DIFFERENTIABLE FUNCTIONS OF SELFADJOINT OPERATORS IN HILBERT SPACES BY TWO POINT TAYLOR'S TYPE EXPANSION

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Abstract. On utilizing the spectral representation of selfadjoint operators in Hilbert spaces, some approximations for the n-time differentiable functions of selfadjoint operators in Hilbert spaces by two point Taylor's type expansions are given.

1. Introduction

Let U be a selfadjoint operator on the complex Hilbert space $(H, \langle .,. \rangle)$ with the spectrum $Sp(U)$ included in the interval $[m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its *spectral family.* Then for any continuous function $f : [m, M] \to \mathbb{C}$, it is well known that we have the following *spectral representation in terms of the Riemann-Stieltjes integral*:

$$f(U) = \int_{m-0}^{M} f(\lambda)\, dE_\lambda, \tag{1.1}$$

which in terms of vectors can be written as

$$\langle f(U) x, y \rangle = \int_{m-0}^{M} f(\lambda)\, d\langle E_\lambda x, y \rangle, \tag{1.2}$$

for any $x, y \in H$. The function $g_{x,y}(\lambda) := \langle E_\lambda x, y \rangle$ is of *bounded variation* on the interval $[m, M]$ and

$$g_{x,y}(m-0) = 0 \text{ and } g_{x,y}(M) = \langle x, y \rangle$$

for any $x, y \in H$. It is also well known that $g_x(\lambda) := \langle E_\lambda x, x \rangle$ is *monotonic nondecreasing* and *right continuous* on $[m, M]$.

For a recent monograph devoted to various inequalities for continuous functions of selfadjoint operators, see [11] and the references therein.

For other recent results see [1], [2], [5]-[9], [13], [14], [15] and [16].

The following result provides a Taylor's type representation for a function of selfadjoint operators in Hilbert spaces with integral remainder:

Theorem 1 (Dragomir, 2010, [10])**.** *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$, $\{E_\lambda\}_\lambda$ be its spectral family, I be a closed subinterval on $\mathbb{R}$ with $[m, M] \subset \mathring{I}$ (the interior of I) and let n be an integer with $n \geq 1$. If $f : I \to \mathbb{C}$ is such that the n-th derivative*

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$f^{(n)}$ is of bounded variation on the interval $[m,M]$, then for any $c \in [m,M]$ we have the equalities

$$f(A) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(c)(A - c1_H)^k + R_n(f,c,m,M) \tag{1.3}$$

where

$$R_n(f,c,m,M) = \frac{1}{n!}\int_{m-0}^{M}\left(\int_c^{\lambda}(\lambda - t)^n d\left(f^{(n)}(t)\right)\right) dE_\lambda. \tag{1.4}$$

This representation provides the following vectorial error bounds:

Theorem 2 (Dragomir, 2010, [10])**.** *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m,M]$ for some real numbers $m < M$, $\{E_\lambda\}_\lambda$ be its spectral family, I be a closed subinterval on $\mathbb{R}$ with $[m,M] \subset \mathring{I}$ (the interior of I) and let n be an integer with $n \geq 1$. If $f : I \to \mathbb{C}$ is such that the n-th derivative $f^{(n)}$ is of bounded variation on the interval $[m,M]$, then for any $c \in [m,M]$ we have the inequality*

$$\begin{aligned} &\left|\langle f(A)x,y\rangle - \sum_{k=0}^{n}\frac{1}{k!}f^{(k)}(c)\left\langle (A-c1_H)^k x,y\right\rangle\right| \\ &\leq \frac{1}{n!}\left[(c-m)^n \bigvee_m^c \left(f^{(n)}\right)\bigvee_m^c\left(\langle E_{(\cdot)}x,y\rangle\right)\right. \\ &\left.+(M-c)^n \bigvee_c^M \left(f^{(n)}\right)\bigvee_c^M\left(\langle E_{(\cdot)}x,y\rangle\right)\right] \\ &\leq \frac{1}{n!}\max\left\{(M-c)^n\bigvee_c^M\left(f^{(n)}\right),(c-m)^n\bigvee_c^M\left(f^{(n)}\right)\right\}\bigvee_m^M\left(\langle E_{(\cdot)}x,y\rangle\right) \\ &\leq \frac{1}{n!}\left(\frac{1}{2}(M-m)+\left|c-\frac{m+M}{2}\right|\right)^n \bigvee_m^M\left(f^{(n)}\right)\bigvee_m^M\left(\langle E_{(\cdot)}x,y\rangle\right), \end{aligned} \tag{1.5}$$

for any $x,y \in H$.for any $x,y \in H$.

For other error bounds in the case when the n-th derivative $f^{(n)}$ is Lipschitzian and some applications for particular functions including the exponential and logarithmic function see [10].

As one can see, by choosing in (1.5) either $c = m, c = M$ or $c = \frac{m+M}{2}$, that one can obtain some Taylor's like expansions in terms of the function and the derivatives values in that specific point. The error estimation is best when c is taken in the midle of the interval $[m,M]$ where the spectrum of the operator is located.

In the present paper however we develop a Taylor's type expansion in terms of the function and the derivatives values in both exteremal points m and M. Applications for some elementary functions of interest including the logarithmic and exponential functions are also provided.

2. Representation Results

We start with the following identity that has been obtained in [4]. For the sake of completeness we give here a short proof as well.

Lemma 1. *Let I be a closed subinterval on $\mathbb{R}$, let $a,b\in I$ with $a<b$ and let n be a nonnegative integer. If $f:I\to\mathbb{R}$ is such that the n-th derivative $f^{(n)}$ is of bounded variation on the interval $[a,b]$, then, for any $x\in[a,b]$ we have the representation*

$$\begin{aligned} f(x) &= \frac{1}{b-a}\left[(b-x)f(a)+(x-a)f(b)\right] \\ &+\frac{(b-x)(x-a)}{b-a} \\ &\times\sum_{k=1}^{n}\frac{1}{k!}\left\{(x-a)^{k-1}f^{(k)}(a)+(-1)^{k}(b-x)^{k-1}f^{(k)}(b)\right\} \\ &+\frac{1}{b-a}\int_a^b S_n(x,t)\,d\left(f^{(n)}(t)\right), \end{aligned} \tag{2.1}$$

where the kernel $S_n:[a,b]^2\to\mathbb{R}$ is given by

$$S_n(x,t)=\frac{1}{n!}\times\begin{cases}(x-t)^n(b-x) & \text{if } a\le t\le x;\\ (-1)^{n+1}(t-x)^n(x-a) & \text{if } x<t\le b\end{cases} \tag{2.2}$$

and the integral in the remainder is taken in the Riemann-Stieltjes sense.

Proof. We utilize the following Taylor's representation formula for functions $f:I\to\mathbb{R}$ such that the n-th derivatives $f^{(n)}$ are of locally bounded variation on the interval I,

$$f(x)=\sum_{k=0}^{n}\frac{1}{k!}(x-c)^k f^{(k)}(c)+\frac{1}{n!}\int_c^x (x-t)^n\,d\left(f^{(n)}(t)\right), \tag{2.3}$$

where x and c are in I and the integral in the remainder is taken in the Riemann-Stieltjes sense.

Choosing $c=a$ and then $c=b$ in (2.3) we can write that

$$f(x)=\sum_{k=0}^{n}\frac{1}{k!}(x-a)^k f^{(k)}(a)+\frac{1}{n!}\int_a^x (x-t)^n\,d\left(f^{(n)}(t)\right), \tag{2.4}$$

and

$$f(x)=\sum_{k=0}^{n}\frac{(-1)^k}{k!}(b-x)^k f^{(k)}(b)+\frac{(-1)^{n+1}}{n!}\int_x^b (t-x)^n\,d\left(f^{(n)}(t)\right), \tag{2.5}$$

for any $x\in[a,b]$.

Now, by multiplying (2.4) with $(b-x)$ and (2.5) with $(x-a)$ we get

$$\begin{aligned}(b-x)f(x) &= (b-x)f(a)+(b-x)(x-a)\sum_{k=1}^{n}\frac{1}{k!}(x-a)^{k-1}f^{(k)}(a) \\ &+\frac{1}{n!}(b-x)\int_a^x (x-t)^n\,d\left(f^{(n)}(t)\right)\end{aligned} \tag{2.6}$$

and

$$(x-a) f(x) = (x-a) f(b) + (b-x)(x-a) \sum_{k=1}^{n} \frac{(-1)^k}{k!} (b-x)^{k-1} f^{(k)}(b) + \frac{(-1)^{n+1}}{n!} (x-a) \int_x^b (t-x)^n \, d\left(f^{(n)}(t)\right) \tag{2.7}$$

respectively.

Finally, by adding the equalities (2.6) and (2.7) and dividing the sum with $(b-a)$, we obtain the desired representation (2.2). □

Remark 1. *The case $n=0$ provides the representation*

$$f(x) = \frac{1}{b-a}[(b-x) f(a) + (x-a) f(b)] + \frac{1}{b-a} \int_a^b S(x,t) \, d(f(t)) \tag{2.8}$$

for any $x \in [a,b]$, where

$$S(x,t) = \begin{cases} b-x & \text{if } a \le t \le x, \\ a-x & \text{if } x < t \le b, \end{cases}$$

and f is of bounded variation on $[a,b]$. This result was obtained by a different approach in [3].

The case $n=1$ provides the representation

$$f(x) = \frac{1}{b-a}[(b-x) f(a) + (x-a) f(b)] + \frac{1}{b-a} \int_a^b Q(x,t) \, d(f'(t)), \tag{2.9}$$

where

$$Q(x,t) = \begin{cases} (a-t)(b-x) & \text{if } a \le t \le x, \\ (t-b)(x-a) & \text{if } x \le t \le b. \end{cases}$$

Notice that the representation (2.9) was obtained by a different approach in [3].

Theorem 3. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m,M]$ for some real numbers $m < M$, $\{E_\lambda\}_\lambda$ be its spectral family, I be a closed subinterval on $\mathbb{R}$ with $[m,M] \subset \mathring{I}$ and let n be an integer with $n \ge 1$. If $f : I \to \mathbb{C}$ is such that the n-th derivative $f^{(n)}$ is of bounded variation on the interval $[m,M]$, then we have the representation*

$$\begin{aligned} f(A) &= \frac{1}{M-m}[f(m)(M1_H - A) + f(M)(A - m1_H)] \\ &\quad + \frac{(M1_H - A)(A - m1_H)}{M-m} \\ &\quad \times \sum_{k=1}^{n} \frac{1}{k!} \left\{ f^{(k)}(m)(A - m1_H)^{k-1} + (-1)^k f^{(k)}(M)(M1_H - A)^{k-1} \right\} \\ &\quad + T_n(f,m,M), \end{aligned} \tag{2.10}$$

where the remainder $T_n(f,m,M)$ is given by

$$T_n(f,m,M) := \frac{1}{(M-m)\, n!} \int_{m-0}^{M} K_n(m,M,f;\lambda) \, dE_\lambda \tag{2.11}$$

and the kernel $K_n(m,M,f;\cdot)$ *has the representation*

$$K_n(m,M,f;\lambda) := (M-\lambda)\left(\int_m^\lambda (\lambda-t)^n d\left(f^{(n)}(t)\right)\right) + (-1)^{n+1}(\lambda-m)\left(\int_\lambda^M (t-\lambda)^n d\left(f^{(n)}(t)\right)\right) \tag{2.12}$$

for $\lambda\in[m,M]$.

Proof. Utilising Lemma 1 we have the representation

$$\begin{aligned} f(\lambda) &= \frac{1}{M-m}\left[(M-\lambda)f(m)+(\lambda-m)f(M)\right] \\ &+ \frac{(M-\lambda)(\lambda-m)}{M-m} \\ &\times \sum_{k=1}^{n}\frac{1}{k!}\left\{(\lambda-m)^{k-1}f^{(k)}(m)+(-1)^k(M-\lambda)^{k-1}f^{(k)}(M)\right\} \\ &+ \frac{1}{(M-m)\,n!}\left[(M-\lambda)\int_m^\lambda(\lambda-t)^n d\left(f^{(n)}(t)\right)\right. \\ &\left.+(-1)^{n+1}(\lambda-m)\int_\lambda^M(t-\lambda)^n d\left(f^{(n)}(t)\right)\right], \end{aligned} \tag{2.13}$$

for any $\lambda\in[m,M]$.

If we integrate (2.13) in the Riemann-Stieltjes sense on the interval $[m,M]$ with the integrator E_λ, then we get

$$\begin{aligned} &\int_{m-0}^{M} f(\lambda)\,dE_\lambda \\ &= \frac{1}{M-m}\int_{m-0}^{M}\left[(M-\lambda)f(m)+(\lambda-m)f(M)\right]dE_\lambda \\ &+ \int_{m-0}^{M}\frac{(M-\lambda)(\lambda-m)}{M-m}\sum_{k=1}^{n}\frac{1}{k!}\left\{(\lambda-m)^{k-1}f^{(k)}(m)\right. \\ &\left.+(-1)^k(M-\lambda)^{k-1}f^{(k)}(M)\right\}dE_\lambda + \frac{1}{(M-m)\,n!} \\ &\times\left[\int_{m-0}^{M}(M-\lambda)\left(\int_m^\lambda(\lambda-t)^n d\left(f^{(n)}(t)\right)\right)dE_\lambda\right. \\ &\left.+(-1)^{n+1}\int_{m-0}^{M}(\lambda-m)\left(\int_\lambda^M(t-\lambda)^n d\left(f^{(n)}(t)\right)\right)dE_\lambda\right]. \end{aligned} \tag{2.14}$$

Now, on making use of the spectral representation (1.1) we deduce from (2.14) the equality (2.1) with the remainder representation (2.2). □

Remark 2. *Let* A *be a selfadjoint operator in the Hilbert space* H *with the spectrum* $Sp(A)\subseteq[m,M]$ *for some real numbers* $m<M$, $\{E_\lambda\}_\lambda$ *be its spectral family. In the case when the function* f *is continuous and of bounded variation on* $[m,M]$,

then we get the representation

$$f(A)=\frac{1}{M-m}\left[f(m)(M1_H-A)+f(M)(A-m1_H)\right] + \frac{1}{(M-m)}\left[\int_{m-0}^{M}(M-\lambda)\left[f(\lambda)-f(m)\right]dE_\lambda - \int_{m-0}^{M}(\lambda-m)\left[f(M)-f(\lambda)\right]dE_\lambda\right]. \tag{2.15}$$

Also, if the derivative f' is of bounded variation, then we have the representation

$$f(A) = \frac{1}{M-m}\left[f(m)(M1_H-A)+f(M)(A-m1_H)\right] + \frac{1}{(M-m)}\left[\int_{m-0}^{M}(M-\lambda)\left(\int_m^\lambda(\lambda-t)\,d(f'(t))\right)dE_\lambda + \int_{m-0}^{M}(\lambda-m)\left(\int_\lambda^M(t-\lambda)\,d(f'(t))\right)dE_\lambda\right]. \tag{2.16}$$

Example 1. *a. Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A)\subseteq[m,M]$ for some real numbers $m<M$ and $\{E_\lambda\}_\lambda$ be its spectral family. If we consider the exponential function, then we get from (2.10) and (2.11) that*

$$e^A=\frac{1}{M-m}\left[e^m(M1_H-A)+e^M(A-m1_H)\right] + \frac{(M1_H-A)(A-m1_H)}{M-m} \times\sum_{k=1}^{n}\frac{1}{k!}\left\{e^m(A-m1_H)^{k-1}+(-1)^k e^M(M1_H-A)^{k-1}\right\} + \frac{1}{(M-m)\,n!}\times\left[\int_{m-0}^{M}(M-\lambda)\left(\int_m^\lambda(\lambda-t)^n e^t dt\right)dE_\lambda + (-1)^{n+1}\int_{m-0}^{M}(\lambda-m)\left(\int_\lambda^M(t-\lambda)^n e^t dt\right)dE_\lambda\right]. \tag{2.17}$$

b. If A is a positive definite selfadjoint operator with the spectrum $Sp(A) \subseteq [m,M] \subset (0,\infty)$ and $\{E_\lambda\}_\lambda$ is its spectral family, then we have the representation

$$\begin{aligned}
\ln A &= \frac{1}{M-m}\left[(M1_H - A)\ln m + (A - m1_H)\ln M\right] \\
&+ \frac{(M1_H - A)(A - m1_H)}{M-m} \\
&\times \sum_{k=1}^{n} \frac{1}{k}\left\{(-1)^{k-1}\frac{(A-m1_H)^{k-1}}{m^k} - \frac{(M1_H - A)^{k-1}}{M^k}\right\} \\
&+ \frac{1}{(M-m)}\left[(-1)^n \int_{m-0}^{M} (M-\lambda)\left(\int_m^\lambda \frac{(\lambda-t)^n}{t^{n+1}}dt\right) dE_\lambda\right. \\
&\left. - \int_{m-0}^{M} (\lambda - m)\left(\int_\lambda^M \frac{(t-\lambda)^n}{t^{n+1}}dt\right) dE_\lambda\right].
\end{aligned} \tag{2.18}$$

The case of functions for which the n-th derivative $f^{(n)}$ is absolutely continuous is of interest for applications. In this case the remainder can be represented as follows:

Theorem 4. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m,M]$ for some real numbers $m < M$, $\{E_\lambda\}_\lambda$ be its spectral family, I be a closed subinterval on $\mathbb{R}$ with $[m,M] \subset \mathring{I}$ and let n be an integer with $n \geq 1$. If $f : I \to \mathbb{C}$ is such that the n-th derivative $f^{(n)}$ is absolutely continuous on the interval $[m,M]$, then we have the representation (2.10) where the remainder $T_n(f,m,M)$ is given by*

$$T_n(f,m,M) := \frac{1}{(M-m)\,n!}\int_{m-0}^{M} W_n(m,M,f;\lambda)\, E_\lambda d\lambda \tag{2.19}$$

and the kernel $W_n(m,M,f;\cdot)$ has the representation

$$\begin{aligned}
W_n(m,M,f;\lambda) &:= (-1)^n \int_m^\lambda (\lambda - t)^{n-1}\left[nM + t - (n+1)\lambda\right] f^{(n+1)}(t)\,dt \\
&\quad - \int_\lambda^M (t-\lambda)^{n-1}\left[t + nm - (n+1)\lambda\right] f^{(n+1)}(t)\,dt
\end{aligned} \tag{2.20}$$

for $\lambda \in [m,M]$.

Proof. Observe that, by Leibnitz's rule for differentiation under the integral sign, we have

$$\begin{aligned}
&\frac{d}{d\lambda}\left[(M-\lambda)\left(\int_m^\lambda (\lambda-t)^n f^{(n+1)}(t)\,dt\right)\right] \\
&= -\int_m^\lambda (\lambda-t)^n f^{(n+1)}(t)\,dt + (M-\lambda)\frac{d}{d\lambda}\left(\int_m^\lambda (\lambda-t)^n f^{(n+1)}(t)\,dt\right) \\
&= -\int_m^\lambda (\lambda-t)^n f^{(n+1)}(t)\,dt + n(M-\lambda)\int_m^\lambda (\lambda-t)^{n-1} f^{(n+1)}(t)\,dt \\
&= \int_m^\lambda (\lambda-t)^{n-1}\left[nM + t - (n+1)\lambda\right] f^{(n+1)}(t)\,dt
\end{aligned} \tag{2.21}$$

for any $\lambda \in [m, M]$.

Integrating by parts in the Riemann-Stieltjes integral we have

$$
\begin{aligned}
(2.22)\quad & \int_{m-0}^{M} (M-\lambda) \left(\int_{m}^{\lambda} (\lambda - t)^{n} f^{(n+1)}(t)\, dt \right) dE_{\lambda} \\
&= (M-\lambda) \left(\int_{m}^{\lambda} (\lambda - t)^{n} d\left(f^{(n)}(t) \right) \right) E_{\lambda} \Big|_{m-0}^{M} \\
&\quad - \int_{m-0}^{M} \left(\int_{m}^{\lambda} (\lambda - t)^{n-1} \left[nM + t - (n+1)\lambda \right] f^{(n+1)}(t)\, dt \right) E_{\lambda} d\lambda \\
&= - \int_{m-0}^{M} \left(\int_{m}^{\lambda} (\lambda - t)^{n-1} \left[nM + t - (n+1)\lambda \right] f^{(n+1)}(t)\, dt \right) E_{\lambda} d\lambda .
\end{aligned}
$$

By Leibnitz's rule we also have

$$
\begin{aligned}
(2.23)\quad & \frac{d}{d\lambda} \left[(\lambda - m) \left(\int_{\lambda}^{M} (t-\lambda)^{n} f^{(n+1)}(t)\, dt \right) \right] \\
&= \int_{\lambda}^{M} (t-\lambda)^{n} f^{(n+1)}(t)\, dt + (\lambda - m) \frac{d}{d\lambda} \left(\int_{\lambda}^{M} (t-\lambda)^{n} f^{(n+1)}(t)\, dt \right) \\
&= \int_{\lambda}^{M} (t-\lambda)^{n} f^{(n+1)}(t)\, dt - n(\lambda - m) \int_{\lambda}^{M} (t-\lambda)^{n-1} f^{(n+1)}(t)\, dt \\
&= \int_{\lambda}^{M} (t-\lambda)^{n-1} \left[t + nm - (n+1)\lambda \right] f^{(n+1)}(t)\, dt
\end{aligned}
$$

for any $\lambda \in [m, M]$.

Utilising the integration by parts and (2.24) we get

$$
\begin{aligned}
(2.24)\quad & \int_{m-0}^{M} (\lambda - m) \left(\int_{\lambda}^{M} (t-\lambda)^{n} f^{(n+1)}(t)\, dt \right) dE_{\lambda} \\
&= (\lambda - m) \left(\int_{\lambda}^{M} (t-\lambda)^{n} f^{(n+1)}(t)\, dt \right) E_{\lambda} \Big|_{m-0}^{M} \\
&\quad - \int_{m-0}^{M} \left(\int_{\lambda}^{M} (t-\lambda)^{n-1} \left[t + nm - (n+1)\lambda \right] f^{(n+1)}(t)\, dt \right) E_{\lambda} d\lambda \\
&= - \int_{m-0}^{M} \left(\int_{\lambda}^{M} (t-\lambda)^{n-1} \left[t + nm - (n+1)\lambda \right] f^{(n+1)}(t)\, dt \right) E_{\lambda} d\lambda .
\end{aligned}
$$

Finally, on utilizing the representation (2.11) for the remainder $T_n(f, m, M)$ and the equalities (2.22) and (2.24) we deduce (2.19). The details are omitted. □

Remark 3. *The case $n = 1$ provides the following equality*

$$
\begin{aligned}
(2.25)\quad f(A) &= \frac{1}{M-m} \left[f(m)(M1_H - A) + f(M)(A - m1_H) \right] \\
&\quad + \frac{1}{(M-m)} \int_{m-0}^{M} W_1(m, M, f; \lambda) E_{\lambda} d\lambda ,
\end{aligned}
$$

where

$$W_1(m,M,f;\lambda) := \int_m^\lambda (2\lambda - M - t) f''(t)\,dt + \int_\lambda^M (2\lambda - t - m) f''(t)\,dt \tag{2.26}$$

for $\lambda \in [m,M]$.

3. Error Bounds for $f^{(n)}$ of Bonded Variation

The following result that provides bounds for the absolute value of the kernel $K_n(m,M,f;\cdot)$ holds:

Lemma 2. *Let I be a closed subinterval on $\mathbb{R}$ with $[m,M] \subset \mathring{I}$, let n be an integer with $n \geq 1$ and assume that $f : I \to \mathbb{C}$ is such that the n-th derivative $f^{(n)}$ exists on the interval $[m,M]$.*

1. If $f^{(n)}$ is of bounded variation on $[m,M]$, then

$$\begin{aligned} &|K_n(m,M,f;\lambda)| \\ &\leq (M-\lambda)(\lambda-m)^n \bigvee_m^\lambda \left(f^{(n)}\right) + (\lambda-m)(M-\lambda)^n \bigvee_\lambda^M \left(f^{(n)}\right) \\ &\leq \frac{1}{4}(M-m)^2 \left[(\lambda-m)^{n-1} \bigvee_m^\lambda \left(f^{(n)}\right) + (M-\lambda)^{n-1} \bigvee_\lambda^M \left(f^{(n)}\right)\right] \\ &\leq \frac{1}{4}(M-m)^2 J_n(m,M;\lambda) \end{aligned} \tag{3.1}$$

where

$$J_n(m,M;\lambda) := \begin{cases} \left[\frac{1}{2}(M-m) + \left|\lambda - \frac{m+M}{2}\right|\right]^{n-1} \bigvee_m^M \left(f^{(n)}\right); \\ \left[(\lambda-m)^{p(n-1)} + (M-\lambda)^{p(n-1)}\right]^{1/p} \\ \times \left[\left(\bigvee_m^\lambda \left(f^{(n)}\right)\right)^q + \left(\bigvee_\lambda^M \left(f^{(n)}\right)\right)^q\right]^{1/q} \\ \text{if } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \left[\frac{1}{2} \bigvee_m^M \left(f^{(n)}\right) + \frac{1}{2}\left|\bigvee_m^\lambda \left(f^{(n)}\right) - \bigvee_\lambda^M \left(f^{(n)}\right)\right|\right] \\ \times \left[(\lambda-m)^{n-1} + (M-\lambda)^{n-1}\right] \end{cases} \tag{3.2}$$

and $\lambda \in [m,M]$.

2. If $\lambda \in (m, M)$ and $f^{(n)}$ is $L_{n,1,\lambda}$-Lipschitzian on $[m, \lambda]$ and $L_{n,2,\lambda}$-Lipschitzian on $[\lambda, M]$, then

$$
\begin{aligned}
(3.3)\quad & |K_n(m, M, f; \lambda)| \\
& \leq \frac{1}{n+1}\left[L_{n,1,\lambda}(M-\lambda)(\lambda-m)^{n+1} + L_{n,2,\lambda}(\lambda-m)(M-\lambda)^{n+1}\right] \\
& \leq \frac{1}{4(n+1)}\left[L_{n,1,\lambda}(\lambda-m)^{n} + L_{n,2,\lambda}(M-\lambda)^{n}\right] \\
& \leq \frac{1}{4(n+1)} \\
& \times \begin{cases}
\left[(\lambda-m)^{n} + (M-\lambda)^{n}\right]\max\left\{L_{n,1,\lambda}, L_{n,2,\lambda}\right\} \\[2ex]
\left[(\lambda-m)^{pn} + (M-\lambda)^{pn}\right]^{1/p}\left(L_{n,1,\lambda}^{q} + L_{n,2,\lambda}^{q}\right)^{1/q} \\
\text{if } p>1, \frac{1}{p}+\frac{1}{q}=1; \\
\left[\frac{1}{2}(M-m) + \left|\lambda - \frac{m+M}{2}\right|\right]^{n}\left(L_{n,1,\lambda} + L_{n,2,\lambda}\right)
\end{cases}
\end{aligned}
$$

and $\lambda \in [m, M]$.

In particular, if $f^{(n)}$ is L_n-Lipschitzian on $[m, M]$, then

$$
\begin{aligned}
(3.4)\quad & |K_n(m, M, f; \lambda)| \\
& \leq \frac{L_n}{n+1}\left[(M-\lambda)(\lambda-m)^{n+1} + (\lambda-m)(M-\lambda)^{n+1}\right] \\
& \leq \frac{L_n(M-m)^2}{4(n+1)}\left[(\lambda-m)^{n} + (M-\lambda)^{n}\right]
\end{aligned}
$$

and $\lambda \in [m, M]$.

3. If the function $f^{(n)}$ is monotonic nondecreasing on $[m, M]$, then

$$
\begin{aligned}
(3.5)\quad & |K_n(m, M, f; \lambda)| \\
& \leq (M-\lambda)\left[n\int_m^{\lambda}(\lambda-t)^{n-1} f^{(n)}(t)\,dt - (\lambda-m)^{n} f^{(n)}(m)\right] \\
& + (\lambda-m)\left[(M-\lambda)^{n} f^{(n)}(M) - n\int_{\lambda}^{M}(t-\lambda)^{n-1} f^{(n)}(t)\,dt\right] \\
& \leq (M-\lambda)(\lambda-m) \\
& \times\left[(\lambda-m)^{n-1}\left[f^{(n)}(\lambda) - f^{(n)}(m)\right] + (M-\lambda)^{n-1}\left[f^{(n)}(M) - f^{(n)}(\lambda)\right]\right] \\
& \leq \frac{1}{4}(M-m)^2 \\
& \times\left[(\lambda-m)^{n-1}\left[f^{(n)}(\lambda) - f^{(n)}(m)\right] + (M-\lambda)^{n-1}\left[f^{(n)}(M) - f^{(n)}(\lambda)\right]\right] \\
& \leq \frac{1}{4}(M-m)^2 T_n(m, M; \lambda)
\end{aligned}
$$

where

$$T_n(m,M;\lambda) \tag{3.6}$$
$$:= \begin{cases} \left[\frac{1}{2}(M-m)+\left|\lambda-\frac{m+M}{2}\right|\right]^{n-1}\left[f^{(n)}(M)-f^{(n)}(m)\right]; \\ \left[(\lambda-m)^{p(n-1)}+(M-\lambda)^{p(n-1)}\right]^{1/p} \\ \times\left[\left(f^{(n)}(M)-f^{(n)}(\lambda)\right)^q+\left(f^{(n)}(\lambda)-f^{(n)}(m)\right)^q\right]^{1/q} \\ \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \left[\frac{1}{2}\left[f^{(n)}(M)-f^{(n)}(m)\right]+\left|f^{(n)}(\lambda)-\frac{f^{(n)}(M)+f^{(n)}(m)}{2}\right|\right] \\ \times\left[(\lambda-m)^{n-1}+(M-\lambda)^{n-1}\right]. \end{cases}$$

Proof. 1. It is well known that if $p:[a,b]\to\mathbb{C}$ is a continuous function, $v:[a,b]\to\mathbb{C}$ is of bounded variation then the Riemann-Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and the following inequality holds

$$\left|\int_a^b p(t)\,dv(t)\right|\le\max_{t\in[a,b]}|p(t)|\bigvee_a^b(v), \tag{3.7}$$

where $\bigvee_a^b(v)$ denotes the total variation of v on $[a,b]$.

Utilising the representation (2.12) and the property (3.7) we have successively

$$\begin{aligned} &|K_n(m,M,f;\lambda)| \\ &\le(M-\lambda)\left|\int_m^\lambda(\lambda-t)^n\,d\left(f^{(n)}(t)\right)\right|+(\lambda-m)\left|\int_\lambda^M(t-\lambda)^n\,d\left(f^{(n)}(t)\right)\right| \\ &\le(M-\lambda)(\lambda-m)^n\bigvee_m^\lambda\left(f^{(n)}\right)+(\lambda-m)(M-\lambda)^n\bigvee_\lambda^M\left(f^{(n)}\right) \\ &=(M-\lambda)(\lambda-m)\left[(\lambda-m)^{n-1}\bigvee_m^\lambda\left(f^{(n)}\right)+(M-\lambda)^{n-1}\bigvee_\lambda^M\left(f^{(n)}\right)\right] \\ &\le\frac{1}{4}(M-m)^2\left[(\lambda-m)^{n-1}\bigvee_m^\lambda\left(f^{(n)}\right)+(M-\lambda)^{n-1}\bigvee_\lambda^M\left(f^{(n)}\right)\right] \\ &\le\frac{1}{4}(M-m)^2 I_n(m,M;\lambda) \end{aligned} \tag{3.8}$$

for any $\lambda\in[m,M]$.

By Hölder's inequality we also have

$$I_n(m,M;\lambda) \tag{3.9}$$
$$\leq \begin{cases} \left[\frac{1}{2}(M-m)+\left|\lambda-\frac{m+M}{2}\right|\right]^{n-1}\bigvee_m^M\left(f^{(n)}\right); \\ \left[(\lambda-m)^{p(n-1)}+(M-\lambda)^{p(n-1)}\right]^{1/p} \\ \times\left[\left(\bigvee_m^\lambda\left(f^{(n)}\right)\right)^q+\left(\bigvee_\lambda^M\left(f^{(n)}\right)\right)^q\right]^{1/q} \\ \text{if } p>1, \frac{1}{p}+\frac{1}{q}=1; \\ \left[\frac{1}{2}\bigvee_m^M\left(f^{(n)}\right)+\frac{1}{2}\left|\bigvee_m^\lambda\left(f^{(n)}\right)-\bigvee_\lambda^M\left(f^{(n)}\right)\right|\right] \\ \times\left[(\lambda-m)^{n-1}+(M-\lambda)^{n-1}\right]. \end{cases}$$

for any $\lambda\in[m,M]$.

On making use of (3.8) and (3.9) we deduce (3.1).

2. We recall that if $p:[a,b]\to\mathbb{C}$ is a Riemann integrable function and $v:[a,b]\to\mathbb{C}$ is Lipschitzian with the constant $L>0$, i.e.,

$$|f(s)-f(t)|\leq L|s-t| \text{ for any } t,s\in[a,b],$$

then the Riemann-Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and the following inequality holds

$$\left|\int_a^b p(t)\,dv(t)\right|\leq L\int_a^b|p(t)|\,dt.$$

Now, on applying this property of the Riemann-Stieltjes integral we have

$$|K_n(m,M,f;\lambda)| \tag{3.10}$$
$$\leq(M-\lambda)\left|\int_m^\lambda(\lambda-t)^n\,d\left(f^{(n)}(t)\right)\right|+(\lambda-m)\left|\int_\lambda^M(t-\lambda)^n\,d\left(f^{(n)}(t)\right)\right|$$
$$\leq\frac{1}{n+1}\left[L_{n,1,\lambda}(M-\lambda)(\lambda-m)^{n+1}+L_{n,2,\lambda}(\lambda-m)(M-\lambda)^{n+1}\right]$$
$$=\frac{(M-\lambda)(\lambda-m)}{n+1}\left[L_{n,1,\lambda}(\lambda-m)^n+L_{n,2,\lambda}(M-\lambda)^n\right]$$
$$\leq\frac{(M-m)^2}{4(n+1)}\left[L_{n,1,\lambda}(\lambda-m)^n+L_{n,2,\lambda}(M-\lambda)^n\right]$$
$$\leq\frac{(M-m)^2}{4(n+1)}$$
$$\times\begin{cases}\left[(\lambda-m)^n+(M-\lambda)^n\right]\max\{L_{n,1,\lambda},L_{n,2,\lambda}\} \\ \left[(\lambda-m)^{pn}+(M-\lambda)^{pn}\right]^{1/p}\left(L_{n,1,\lambda}^q+L_{n,2,\lambda}^q\right)^{1/q} \\ \text{if } p>1, \frac{1}{p}+\frac{1}{q}=1; \\ \left[\frac{1}{2}(M-m)+\left|\lambda-\frac{m+M}{2}\right|\right]^n\left(L_{n,1,\lambda}+L_{n,2,\lambda}\right)\end{cases}$$

which prove the desired result (3.4).

3. From the theory of Riemann-Stieltjes integral is well known that if $p:[a,b]\to\mathbb{C}$ is continuous and $v:[a,b]\to\mathbb{R}$ is monotonic nondecreasing, then the Riemann-Stieltjes integrals $\int_a^b p(t)\,dv(t)$ and $\int_a^b |p(t)|\,dv(t)$ exist and

$$\left|\int_a^b p(t)\,dv(t)\right| \le \int_a^b |p(t)|\,dv(t) \le \max_{t\in[a,b]}|p(t)|\,[v(b)-v(a)]. \tag{3.11}$$

By utilizing this property, we have

$$\begin{aligned} &|K_n(m,M,f;\lambda)| \\ &\le (M-\lambda)\left|\int_m^\lambda (\lambda-t)^n\,d\left(f^{(n)}(t)\right)\right| + (\lambda-m)\left|\int_\lambda^M (t-\lambda)^n\,d\left(f^{(n)}(t)\right)\right| \\ &\le (M-\lambda)\int_m^\lambda (\lambda-t)^n\,d\left(f^{(n)}(t)\right) + (\lambda-m)\int_\lambda^M (t-\lambda)^n\,d\left(f^{(n)}(t)\right) \\ &= H_n(m,M;\lambda) \end{aligned} \tag{3.12}$$

By the second part of (3.11) we also have that

$$\begin{aligned} &H_n(m,M;\lambda) \\ &\le (M-\lambda)(\lambda-m)^n\left[f^{(n)}(\lambda)-f^{(n)}(m)\right] \\ &+ (\lambda-m)(M-\lambda)^n\left[f^{(n)}(M)-f^{(n)}(\lambda)\right] \\ &= (M-\lambda)(\lambda-m) \\ &\times\left[(\lambda-m)^{n-1}\left[f^{(n)}(\lambda)-f^{(n)}(m)\right]+(M-\lambda)^{n-1}\left[f^{(n)}(M)-f^{(n)}(\lambda)\right]\right] \\ &\le \frac{1}{4}(M-m)^2 \\ &\times\left[(\lambda-m)^{n-1}\left[f^{(n)}(\lambda)-f^{(n)}(m)\right]+(M-\lambda)^{n-1}\left[f^{(n)}(M)-f^{(n)}(\lambda)\right]\right] \\ &= \frac{1}{4}(M-m)^2 L_n(m,M;\lambda) \end{aligned} \tag{3.13}$$

with

$$L_n(m,M;\lambda) \le \begin{cases} \left[\frac{1}{2}(M-m)+\left|\lambda-\frac{m+M}{2}\right|\right]^{n-1}\left[f^{(n)}(M)-f^{(n)}(m)\right]; \\ \\ \left[(\lambda-m)^{p(n-1)}+(M-\lambda)^{p(n-1)}\right]^{1/p} \\ \times\left[\left(f^{(n)}(M)-f^{(n)}(\lambda)\right)^q+\left(f^{(n)}(\lambda)-f^{(n)}(m)\right)^q\right]^{1/q} \\ \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \left[\frac{1}{2}\left[f^{(n)}(M)-f^{(n)}(m)\right]+\left|f^{(n)}(\lambda)-\frac{f^{(n)}(M)+f^{(n)}(m)}{2}\right|\right] \\ \times\left[(\lambda-m)^{n-1}+(M-\lambda)^{n-1}\right]. \end{cases} \tag{3.14}$$

Integrating by parts we have

$$
\begin{aligned}
&H_n(m,M;\lambda) \qquad (3.15)\\
&=(M-\lambda)\int_m^\lambda(\lambda-t)^n\,d\left(f^{(n)}(t)\right)+(\lambda-m)\int_\lambda^M(t-\lambda)^n\,d\left(f^{(n)}(t)\right)\\
&=(M-\lambda)\left[(\lambda-t)^n f^{(n)}(t)\Big|_m^\lambda+n\int_m^\lambda(\lambda-t)^{n-1}f^{(n)}(t)\,dt\right]\\
&+(\lambda-m)\left[(t-\lambda)^n f^{(n)}(t)\Big|_\lambda^M-n\int_\lambda^M(t-\lambda)^{n-1}f^{(n)}(t)\,dt\right]\\
&=(M-\lambda)\left[n\int_m^\lambda(\lambda-t)^{n-1}f^{(n)}(t)\,dt-(\lambda-m)^n f^{(n)}(m)\right]\\
&+(\lambda-m)\left[(M-\lambda)^n f^{(n)}(M)-n\int_\lambda^M(t-\lambda)^{n-1}f^{(n)}(t)\,dt\right].
\end{aligned}
$$

On making use of (3.12)-(3.15) we deduce the desired result (3.5). □

On making use of the bounds for the kernel $K_n(m,M,f;\cdot)$ provided above, we can establish the following error estimates for the remainder $T_n(f,m,M)$ in the representation formula (2.10).

Theorem 5. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A)\subseteq[m,M]$ for some real numbers $m<M$, $\{E_\lambda\}_\lambda$ be its spectral family, I be a closed subinterval on $\mathbb{R}$ with $[m,M]\subset\mathring{I}$ and let n be an integer with $n\geq 1$. If $f:I\to\mathbb{C}$ is such that the n-th derivative $f^{(n)}$ is of bounded variation on the interval $[m,M]$, then we have the representation*

$$
\begin{aligned}
\langle f(A)x,y\rangle &= \frac{1}{M-m}\left[f(m)\langle(M1_H-A)x,y\rangle+f(M)\langle(A-m1_H)x,y\rangle\right] \qquad (3.16)\\
&+\frac{1}{M-m}\\
&\times\left\{\sum_{k=1}^n\frac{1}{k!}f^{(k)}(m)\left\langle(M1_H-A)(A-m1_H)^k x,y\right\rangle\right.\\
&\left.+\sum_{k=1}^n\frac{1}{k!}(-1)^k f^{(k)}(M)\left\langle(A-m1_H)(M1_H-A)^k x,y\right\rangle\right\}\\
&+T_n(f,m,M;x,y),
\end{aligned}
$$

where the remainder $T_n(f,m,M;x,y)$ is given by

$$
T_n(f,m,M;x,y):=\frac{1}{(M-m)\,n!}\int_{m-0}^M K_n(m,M,f;\lambda)\,d\langle E_\lambda x,y\rangle \qquad (3.17)
$$

and the kernel $K_n(m,M,f;\cdot)$ has the representation (2.12).

Moreover, we have the error estimate

$$\begin{aligned}
(3.18)\qquad & |T_n(f,m,M;x,y)| \\
& \leq \frac{1}{4n!}(M-m)\bigvee_m^M(\langle E_{(\cdot)}x,y\rangle) \\
& \times \max_{\lambda\in[m,M]}\left[(\lambda-m)^{n-1}\bigvee_m^{\lambda}\left(f^{(n)}\right)+(M-\lambda)^{n-1}\bigvee_{\lambda}^{M}\left(f^{(n)}\right)\right] \\
& \leq \frac{1}{4n!}(M-m)^n\bigvee_m^M\left(f^{(n)}\right)\bigvee_m^M(\langle E_{(\cdot)}x,y\rangle) \\
& \leq \frac{1}{4n!}(M-m)^n\bigvee_m^M\left(f^{(n)}\right)\|x\|\,\|y\|
\end{aligned}$$

for any $x,y\in H$.

Proof. The identity (3.16) with the remainder representation (3.17) follows from (2.10) and (2.11).

Now, on utilizing the property (3.7) for the Riemann-Stieltjes integral we deduce from (3.17) that

$$(3.19)\quad |T_n(f,m,M;x,y)|\leq\frac{1}{(M-m)\,n!}\max_{\lambda\in[m,M]}|K_n(m,M,f;\lambda)|\bigvee_m^M(\langle E_{(\cdot)}x,y\rangle)$$

for any $x,y\in H$.

Further, by (3.1) and (3.2) we have the bounds

$$\begin{aligned}
(3.20)\qquad & |K_n(m,M,f;\lambda)| \\
& \leq\frac{1}{4}(M-m)^2\left[(\lambda-m)^{n-1}\bigvee_m^{\lambda}\left(f^{(n)}\right)+(M-\lambda)^{n-1}\bigvee_{\lambda}^{M}\left(f^{(n)}\right)\right] \\
& \leq\frac{1}{4}(M-m)^2\left[\frac{1}{2}(M-m)+\left|\lambda-\frac{m+M}{2}\right|\right]^{n-1}\bigvee_m^M\left(f^{(n)}\right);
\end{aligned}$$

for any $\lambda\in[m,M]$.

Taking the maximum over $\lambda\in[m,M]$ in (3.20) we deduce the first and the second inequalities in (3.18).

To prove the last part of (3.18), we notice that if P is a nonnegative operator on H, i.e., $\langle Px,x\rangle\geq 0$ for any $x\in H$, then the following inequality is a generalization of the Schwarz inequality in H

$$(3.21)\qquad |\langle Px,y\rangle|^2\leq\langle Px,x\rangle\langle Py,y\rangle$$

for any $x,y\in H$.

Now, if $d: m=t_0<t_1<...<t_{n-1}<t_n=M$ is an arbitrary partition of the interval $[m,M]$, then we have by Schwarz's inequality for nonnegative operators

(3.21) that

$$\begin{aligned}
&\bigvee_m^M \left(\left\langle E_{(\cdot)}x, y\right\rangle\right) \\
&= \sup_d \left\{ \sum_{i=0}^{n-1} \left| \left\langle \left(E_{t_{i+1}} - E_{t_i}\right) x, y \right\rangle \right| \right\} \\
&\le \sup_d \left\{ \sum_{i=0}^{n-1} \left[\left\langle \left(E_{t_{i+1}} - E_{t_i}\right) x, x \right\rangle^{1/2} \left\langle \left(E_{t_{i+1}} - E_{t_i}\right) y, y \right\rangle^{1/2} \right] \right\} := B.
\end{aligned}$$

By the Cauchy-Buniakovski-Schwarz inequality for sequences of real numbers we also have that

$$\begin{aligned}
B &\le \sup_d \left\{ \left[\sum_{i=0}^{n-1} \left\langle \left(E_{t_{i+1}} - E_{t_i}\right) x, x \right\rangle \right]^{1/2} \left[\sum_{i=0}^{n-1} \left\langle \left(E_{t_{i+1}} - E_{t_i}\right) y, y \right\rangle \right]^{1/2} \right\} \\
&\le \sup_d \left\{ \left[\sum_{i=0}^{n-1} \left\langle \left(E_{t_{i+1}} - E_{t_i}\right) x, x \right\rangle \right]^{1/2} \sup_d \left[\sum_{i=0}^{n-1} \left\langle \left(E_{t_{i+1}} - E_{t_i}\right) y, y \right\rangle \right]^{1/2} \right\} \\
&= \left[\bigvee_m^M \left(\left\langle E_{(\cdot)}x, x\right\rangle\right) \right]^{1/2} \left[\bigvee_m^M \left(\left\langle E_{(\cdot)}y, y\right\rangle\right) \right]^{1/2} = \|x\| \, \|y\|
\end{aligned}$$

for any $x, y \in H$, and the proof is complete. □

Corollary 1. *With the assumptions from Theorem 5 and if $f^{(n)}$ is L_n-Lipschitzian on $[m, M]$, then*

$$\begin{aligned}
&|T_n(f, m, M; x, y)| \\
&\le \frac{1}{(n+1)!\,(M-m)} L_n \bigvee_m^M \left(\left\langle E_{(\cdot)}x, y\right\rangle\right) \\
&\quad \times \max_{\lambda \in [m,M]} \left[(M-\lambda)(\lambda-m)^{n+1} + (\lambda-m)(M-\lambda)^{n+1} \right] \\
&\le \frac{1}{4(n+1)!} (M-m)^{n+1} L_n \bigvee_m^M \left(\left\langle E_{(\cdot)}x, y\right\rangle\right) \\
&\le \frac{1}{4(n+1)!} (M-m)^{n+1} L_n \|x\| \, \|y\|
\end{aligned} \tag{3.22}$$

for any $x, y \in H$.

4. Error Bounds for $f^{(n)}$ Absolutely Continuous

The following result that provides bounds for the absolute value of the kernel $W_n(m, M, f; \cdot)$ holds:

Lemma 3. *Let I be a closed subinterval on $\mathbb{R}$ with $[m, M] \subset \mathring{I}$, let n be an integer with $n \ge 1$ and assume that $f : I \to \mathbb{C}$ is such that the n-th derivative $f^{(n)}$ is absolutely continuous on the interval $[m, M]$. Then we have the bound*

$$|W_n(m, M, f; \lambda)| \le \sum_{i=1}^{4} B_n^{(i)}(m, M, f; \lambda) \tag{4.1}$$

where

$$B_n^{(1)}(m,M,f;\lambda) := n(M-\lambda)\int_m^\lambda (\lambda-t)^{n-1}\left|f^{(n+1)}(t)\right|dt \leq n(M-\lambda) \times \begin{cases} \frac{1}{n}(\lambda-m)^n \left\|f^{(n+1)}\right\|_{[m,\lambda],\infty} & \text{if } f^{(n+1)}\in L_\infty[m,\lambda]; \\ \frac{1}{[(n-1)p_1+1]^{1/p_1}}(\lambda-m)^{n-1+1/p_1}\left\|f^{(n+1)}\right\|_{[m,\lambda],q_1} & \text{if } f^{(n+1)}\in L_{q_1}[m,\lambda], p_1>1, \frac{1}{p_1}+\frac{1}{q_1}=1; \\ (\lambda-m)^{n-1}\left\|f^{(n+1)}\right\|_{[m,\lambda],1}; \end{cases} \tag{4.2}$$

$$B_n^{(2)}(m,M,f;\lambda) := \int_m^\lambda (\lambda-t)^{n}\left|f^{(n+1)}(t)\right|dt \leq \begin{cases} \frac{1}{n+1}(\lambda-m)^{n+1} \left\|f^{(n+1)}\right\|_{[m,\lambda],\infty} & \text{if } f^{(n+1)}\in L_\infty[m,\lambda]; \\ \frac{1}{(np_2+1)^{1/p_2}}(\lambda-m)^{n+1/p_2}\left\|f^{(n+1)}\right\|_{[m,\lambda],q_2} & \text{if } f^{(n+1)}\in L_{q_2}[m,\lambda], \\ p_2>1, \frac{1}{p_2}+\frac{1}{q_2}=1; \\ (\lambda-m)^{n}\left\|f^{(n+1)}\right\|_{[m,\lambda],1} \end{cases} \tag{4.3}$$

$$B_n^{(3)}(m,M,f;\lambda) := \int_\lambda^M (t-\lambda)^{n}\left|f^{(n+1)}(t)\right|dt \leq \begin{cases} \frac{1}{n+1}(M-\lambda)^{n+1} \left\|f^{(n+1)}\right\|_{[\lambda,M],\infty} & \text{if } f^{(n+1)}\in L_\infty[\lambda,M]; \\ \frac{1}{(np_3+1)^{1/p_3}}(M-\lambda)^{n+1/p_3}\left\|f^{(n+1)}\right\|_{[\lambda,M],q_3} & \text{if } f^{(n+1)}\in L_{q_3}[\lambda,M], \\ p_3>1, \frac{1}{p_3}+\frac{1}{q_3}=1; \\ (M-\lambda)^{n}\left\|f^{(n+1)}\right\|_{[\lambda,M],1} \end{cases} \tag{4.4}$$

and

$$B_n^{(4)}(m,M,f;\lambda) := n(\lambda-m)\int_\lambda^M (t-\lambda)^{n-1}\left|f^{(n+1)}(t)\right|dt \leq n(\lambda-m) \times \begin{cases} \frac{1}{n}(M-\lambda)^n \left\|f^{(n+1)}\right\|_{[\lambda,M],\infty} & \text{if } f^{(n+1)}\in L_\infty[\lambda,M]; \\ \frac{1}{[(n-1)p_4+1]^{1/p_4}}(M-\lambda)^{n-1+1/p_4}\left\|f^{(n+1)}\right\|_{[\lambda,M],q_4} & \text{if } f^{(n+1)}\in L_{q_1}[\lambda,M], p_4>1, \frac{1}{p4}+\frac{1}{q_4}=1; \\ (M-\lambda)^{n-1}\left\|f^{(n+1)}\right\|_{[m,\lambda],1}; \end{cases} \tag{4.5}$$

for any $\lambda \in [m,M]$, *where the Lebesgue norms* $\|\cdot\|_{[a,b],p}$ *are defined by*

$$\|g\|_{[a,b],p} := \begin{cases} \left(\int_a^b |g(t)|^p dt\right)^{1/p} & \text{if } g \in L_p[a,b], p \geq 1 \\ \\ ess\sup_{t\in[a,b]} |g(t)| & \text{if } g \in L_\infty[a,b]. \end{cases}$$

Proof. From (2.20) we have

$$\begin{aligned} &|W_n(m,M,f;\lambda)| \\ &\leq \left|\int_m^\lambda (\lambda-t)^{n-1}[nM+t-(n+1)\lambda] f^{(n+1)}(t)\,dt\right| \\ &+ \left|\int_\lambda^M (t-\lambda)^{n-1}[t+nm-(n+1)\lambda] f^{(n+1)}(t)\,dt\right| \\ &\leq \int_m^\lambda (\lambda-t)^{n-1}|nM+t-(n+1)\lambda| \left|f^{(n+1)}(t)\right| dt \\ &+ \int_\lambda^M (t-\lambda)^{n-1}|t+nm-(n+1)\lambda| \left|f^{(n+1)}(t)\right| dt \\ &\leq \int_m^\lambda (\lambda-t)^{n-1}[n(M-\lambda)+(\lambda-t)] \left|f^{(n+1)}(t)\right| dt \\ &+ \int_\lambda^M (t-\lambda)^{n-1}[(t-\lambda)+n(\lambda-m)] \left|f^{(n+1)}(t)\right| dt \\ &= \sum_{i=1}^4 B_n^{(i)}(m,M,f;\lambda) \end{aligned} \tag{4.6}$$

for any $\lambda \in [m,M]$, which proves (4.1).

The other bounds follows by Hölder's integral inequality and the details are omitted. □

Remark 4. *It is obvious that the inequalities (4.1)-(4.5) can produce 12 different bounds for* $|W_n(m,M,f;\lambda)|$. *However, we mention here only the case when* $f^{(n+1)} \in L_\infty[\lambda,M]$, *namely*

$$\begin{aligned} &|W_n(m,M,f;\lambda)| \\ &\leq (M-\lambda)(\lambda-m)^n \left\|f^{(n+1)}\right\|_{[m,\lambda],\infty} + \frac{1}{n+1}(\lambda-m)^{n+1}\left\|f^{(n+1)}\right\|_{[m,\lambda],\infty} \\ &+ \frac{1}{n+1}(M-\lambda)^{n+1}\left\|f^{(n+1)}\right\|_{[\lambda,M],\infty} + (\lambda-m)(M-\lambda)^n\left\|f^{(n+1)}\right\|_{[\lambda,M],\infty} \\ &\leq \Big[(M-\lambda)(\lambda-m)^n + (\lambda-m)(M-\lambda)^n \\ &+ \frac{1}{n+1}(\lambda-m)^{n+1} + \frac{1}{n+1}(M-\lambda)^{n+1}\Big]\left\|f^{(n+1)}\right\|_{[m,M],\infty} \end{aligned} \tag{4.7}$$

for any $\lambda \in [m,M]$.

Finally, we can state the following result as well:

Theorem 6. *Let* A *be a selfadjoint operator in the Hilbert space* H *with the spectrum* $Sp(A) \subseteq [m,M]$ *for some real numbers* $m < M$, $\{E_\lambda\}_\lambda$ *be its spectral family,*

I be a closed subinterval on $\mathbb{R}$ with $[m,M]\subset \mathring{I}$ and let n be an integer with $n\geq 1$. If $f:I\rightarrow \mathbb{C}$ is such that the n-th derivative $f^{(n)}$ is absolutely continuous on the interval $[m,M]$, then we have the representation (3.16) where the remainder $T_n(f,m,M;x,y)$ is given by

$$T_n(f,m,M;x,y):=\frac{1}{(M-m)\,n!}\int_{m-0}^{M}W_n(m,M,f;\lambda)\langle E_\lambda x,y\rangle d\lambda \tag{4.8}$$

and the kernel $W_n(m,M,f;\cdot)$ has the representation (2.20).

We also have the error bounds

$$\begin{aligned}&|T_n(f,m,M;x,y)|\\&\leq \frac{1}{(M-m)\,n!}\int_{m-0}^{M}|W_n(m,M,f;\lambda)|\,|\langle E_\lambda x,y\rangle|\,d\lambda\\&\leq \frac{1}{(M-m)\,n!}\int_{m-0}^{M}|W_n(m,M,f;\lambda)|\,\langle E_\lambda x,x\rangle^{1/2}\langle E_\lambda y,y\rangle^{1/2}d\lambda\\&\leq \frac{1}{(M-m)\,n!}\|x\|\,\|y\|\int_{m}^{M}|W_n(m,M,f;\lambda)|\,d\lambda\end{aligned} \tag{4.9}$$

for any $x,y\in H$.

Remark 5. *On making use of Lemma 3 one can produce further bounds. However, the details are left to the interested reader.*

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