Received 05/01/11

BOUNDS FOR THE DIFFERENCE BETWEEN FUNCTIONS OF SELFADJOINT OPERATORS IN HILBERT SPACES AND INTEGRAL MEANS

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Abstract. Bounds for the difference between functions of selfadjoint operators in Hilbert spaces and integral mean under suitable conditions for the functions and operators involved are given. Applications for particular instances of interest are provided as well.

1. Introduction

Let U be a selfadjoint operator on the complex Hilbert space $(H, \langle .,. \rangle)$ with the spectrum $Sp(U)$ included in the interval $[m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its *spectral family*. Then for any continuous function $f : [m, M] \to \mathbb{R}$, it is well known that we have the following *spectral representation* in terms of the Riemann-Stieltjes integral:

$$\langle f(U) x, y \rangle = \int_{m-0}^{M} f(\lambda) \, d(\langle E_\lambda x, y \rangle), \tag{1.1}$$

for any $x, y \in H$. The function $g_{x,y}(\lambda) := \langle E_\lambda x, y \rangle$ is of *bounded variation* on the interval $[m, M]$ and

$$g_{x,y}(m-0) = 0 \text{ and } g_{x,y}(M) = \langle x, y \rangle$$

for any $x, y \in H$. It is also well known that $g_x(\lambda) := \langle E_\lambda x, x \rangle$ is *monotonic nondecreasing* and *right continuous* on $[m, M]$.

The following result concerning bounds involving the spectral family holds, see [16]:

Theorem 1. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f : [m, M] \to \mathbb{R}$ is a continuous function of bounded variation on $[m, M]$,*

1991 *Mathematics Subject Classification.* 47A63; 47A99.

Key words and phrases. Selfadjoint operators, Functions of Selfadjoint operators, Ostrowski's type inequalities, Vector inequalities.

then we have the inequality

$$
\begin{aligned}
&|f(s)\langle x,y\rangle-\langle f(A)x,y\rangle| \\
&\leq \langle E_s x,x\rangle^{1/2}\langle E_s y,y\rangle^{1/2}\bigvee_m^s(f) \\
&+\langle(1_H-E_s)x,x\rangle^{1/2}\langle(1_H-E_s)y,y\rangle^{1/2}\bigvee_s^M(f) \\
&\leq \|x\|\|y\|\left(\frac{1}{2}\bigvee_m^M(f)+\frac{1}{2}\left|\bigvee_m^s(f)-\bigvee_s^M(f)\right|\right)\left(\leq \|x\|\|y\|\bigvee_m^M(f)\right)
\end{aligned} \tag{1.2}
$$

for any $x,y\in H$ and for any $s\in[m,M]$.

Remark 1. *For the continuous function with bounded variation $f:[m,M]\to\mathbb{R}$ if $p\in[m,M]$ is a point with the property that*

$$\bigvee_m^p(f)=\bigvee_p^M(f),$$

then from (1.2) we get the interesting inequality

$$|f(p)\langle x,y\rangle-\langle f(A)x,y\rangle|\leq\frac{1}{2}\|x\|\|y\|\bigvee_m^M(f) \tag{1.3}$$

for any $x,y\in H$.

If the continuous function $f:[m,M]\to\mathbb{R}$ is monotonic nondecreasing and therefore of bounded variation, then we get from (2.2) the following inequality as well

$$
\begin{aligned}
&|f(s)\langle x,y\rangle-\langle f(A)x,y\rangle| \\
&\leq \langle E_s x,x\rangle^{1/2}\langle E_s y,y\rangle^{1/2}(f(s)-f(m)) \\
&+\langle(1_H-E_s)x,x\rangle^{1/2}\langle(1_H-E_s)y,y\rangle^{1/2}(f(M)-f(s)) \\
&\leq \|x\|\|y\|\left(\frac{1}{2}(f(M)-f(m))+\left|f(s)-\frac{f(m)+f(M)}{2}\right|\right) \\
&\leq \|x\|\|y\|[f(M)-f(m)]
\end{aligned} \tag{1.4}
$$

for any $x,y\in H$ and $s\in[m,M]$.

Moreover, if the continuous function $f:[m,M]\to\mathbb{R}$ is nondecreasing on $[m,M]$, then the equation

$$f(s)=\frac{f(m)+f(M)}{2}$$

has got at least a solution in $[m,M]$. In his case we get from (1.4) the following trapezoidal type inequality

$$\left|\frac{f(m)+f(M)}{2}\langle x,y\rangle-\langle f(A)x,y\rangle\right|\leq\frac{1}{2}\|x\|\|y\|(f(M)-f(m)) \tag{1.5}$$

for any $x,y\in H$.

In the following, we consider the problem of bounding the absolute value of the difference between the quantities $\langle f(A)x,y\rangle$ and $\frac{1}{M-m}\int_m^M f(t)\,dt$ where A is a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A)\subseteq[m,M]$ for some real numbers $m<M, x,y\in H$ and $f:[m,M]\to\mathbb{C}$ is continuous on the interval $[m,M]$. Applications for some particular functions are also given.

The following result holds, see [17]:

Theorem 2. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A)\subseteq[m,M]$ for some real numbers $m<M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f:[m,M]\to\mathbb{C}$ is a continuous function of bounded variation on $[m,M]$, then we have the inequality*

$$\left|\langle x,y\rangle\frac{1}{M-m}\int_m^M f(s)\,ds-\langle f(A)x,y\rangle\right| \tag{1.6}$$
$$\le\frac{1}{M-m}\bigvee_m^M(f)\max_{t\in[m,M]}\Big[(M-t)\langle E_tx,x\rangle^{1/2}\langle E_ty,y\rangle^{1/2}$$
$$+(t-m)\langle(1_H-E_t)x,x\rangle^{1/2}\langle(1_H-E_t)y,y\rangle^{1/2}\Big]$$
$$\le\|x\|\,\|y\|\bigvee_m^M(f)$$

for any $x,y\in H$.

For other similar results, see [17].

In this paper, motivated by the above results, we establish other bounds for the magnitude of the difference

$$\langle x,y\rangle\frac{1}{M-m}\int_m^M f(s)\,ds-\langle f(A)x,y\rangle$$

where $x,y\in H$. Some applications for the power and logarithmic functions are provided as well.

2. Vector Inequalities Via Ostrowski's Type Bounds

The following result holds:

Theorem 3. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A)\subseteq[m,M]$ for some real numbers $m<M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f:[m,M]\to\mathbb{R}$ is a continuous function on $[m,M]$, then we have the inequality*

$$\left|\langle f(A)x,y\rangle-\langle x,y\rangle\frac{1}{M-m}\int_m^M f(s)\,ds\right| \tag{2.1}$$
$$\le\max_{t\in[m,M]}\left|f(t)-\frac{1}{M-m}\int_m^M f(s)\,ds\right|\bigvee_m^M\left(\langle E_{(\cdot)}x,y\rangle\right)$$
$$\le\max_{t\in[m,M]}\left|f(t)-\frac{1}{M-m}\int_m^M f(s)\,ds\right|\|x\|\,\|y\|$$

for any $x,y\in H$.

Proof. Utilising the spectral representation (1.1) we have the following equality of interest

$$\begin{aligned}(2.2)\qquad &\langle f(A)x,y\rangle-\langle x,y\rangle\frac{1}{M-m}\int_m^M f(s)\,ds\\&=\int_{m-0}^M\left[f(t)-\frac{1}{M-m}\int_m^M f(s)\,ds\right]d(\langle E_t x,y\rangle)\end{aligned}$$

for any $x,y\in H$.

It is well known that if $p:[a,b]\to\mathbb{C}$ is a continuous function and $v:[a,b]\to\mathbb{C}$ is of bounded variation, then the Riemann-Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and the following inequality holds

$$(2.3)\qquad \left|\int_a^b p(t)\,dv(t)\right|\le\max_{t\in[a,b]}|p(t)|\bigvee_a^b(v),$$

where $\bigvee_a^b(v)$ denotes the total variation of v on $[a,b]$.

Applying this property and the equality (2.2) we deduce the first inequality in (2.1).

If P is a nonnegative operator on H, i.e., $\langle Px,x\rangle\ge 0$ for any $x\in H$, then the following inequality is a generalization of the Schwarz inequality in H

$$(2.4)\qquad |\langle Px,y\rangle|^2\le\langle Px,x\rangle\langle Py,y\rangle,$$

for any $x,y\in H$.

Now, if $d:m=t_0<t_1<...<t_{n-1}<t_n=M$ is an arbitrary partition of the interval $[m,M]$, then we have by Schwarz's inequality for nonnegative operators that

$$\begin{aligned}(2.5)\qquad &\bigvee_m^M(\langle E_{(\cdot)}x,y\rangle)\\&=\sup_d\left\{\sum_{i=0}^{n-1}\left|\left\langle\left(E_{t_{i+1}}-E_{t_i}\right)x,y\right\rangle\right|\right\}\\&\le\sup_d\left\{\sum_{i=0}^{n-1}\left[\left\langle\left(E_{t_{i+1}}-E_{t_i}\right)x,x\right\rangle^{1/2}\left\langle\left(E_{t_{i+1}}-E_{t_i}\right)y,y\right\rangle^{1/2}\right]\right\}:=I.\end{aligned}$$

By the Cauchy-Buniakovski-Schwarz inequality for sequences of real numbers we also have that

$$\begin{aligned}(2.6)\qquad I&\le\sup_d\left\{\left[\sum_{i=0}^{n-1}\left\langle\left(E_{t_{i+1}}-E_{t_i}\right)x,x\right\rangle\right]^{1/2}\left[\sum_{i=0}^{n-1}\left\langle\left(E_{t_{i+1}}-E_{t_i}\right)y,y\right\rangle\right]^{1/2}\right\}\\&\le\sup_d\left\{\left[\sum_{i=0}^{n-1}\left\langle\left(E_{t_{i+1}}-E_{t_i}\right)x,x\right\rangle\right]^{1/2}\sup_d\left[\sum_{i=0}^{n-1}\left\langle\left(E_{t_{i+1}}-E_{t_i}\right)y,y\right\rangle\right]^{1/2}\right\}\\&=\left[\bigvee_m^M(\langle E_{(\cdot)}x,x\rangle)\right]^{1/2}\left[\bigvee_m^M(\langle E_{(\cdot)}y,y\rangle)\right]^{1/2}=\|x\|\,\|y\|\end{aligned}$$

for any $x, y \in H$.

Making use of (2.5) and (2.6) we deduce then the last part of (2.1). □

For particular classes of continuous functions $f : [m, M] \to \mathbb{C}$ we are able to provide simpler bounds as incorporated in the following corollary:

Corollary 1. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$, $\{E_\lambda\}_\lambda$ be its spectral family and $f : [m, M] \to \mathbb{C}$ a continuous function on $[m, M]$.*

1. If f is of bounded variation on $[m, M]$, then

$$\left|\langle f(A)x, y\rangle - \langle x, y\rangle \frac{1}{M-m}\int_m^M f(s)\,ds\right| \leq \bigvee_m^M (f) \bigvee_m^M \left(\langle E_{(\cdot)}x, y\rangle\right) \leq \|x\|\,\|y\| \bigvee_m^M (f) \tag{2.7}$$

for any $x, y \in H$.

2. If $f : [m, M] \longrightarrow \mathbb{C}$ is of $r-H-$Hölder type, i.e., for a given $r \in (0, 1]$ and $H > 0$ we have

$$|f(s) - f(t)| \leq H|s-t|^r \text{ for any } s, t \in [m, M], \tag{2.8}$$

then we have the inequality:

$$\left|\langle f(A)x, y\rangle - \langle x, y\rangle \frac{1}{M-m}\int_m^M f(s)\,ds\right| \leq \frac{1}{r+1}H(M-m)^r \bigvee_m^M \left(\langle E_{(\cdot)}x, y\rangle\right) \leq \frac{1}{r+1}H(M-m)^r \|x\|\,\|y\| \tag{2.9}$$

for any $x, y \in H$.

In particular, if $f : [m, M] \longrightarrow \mathbb{C}$ is Lipschitzian with the constant $L > 0$, then

$$\left|\langle f(A)x, y\rangle - \langle x, y\rangle \frac{1}{M-m}\int_m^M f(s)\,ds\right| \leq \frac{1}{2}L(M-m) \bigvee_m^M \left(\langle E_{(\cdot)}x, y\rangle\right) \leq \frac{1}{2}L(M-m)\|x\|\,\|y\| \tag{2.10}$$

for any $x, y \in H$.

3. If $f:[m,M]\longrightarrow\mathbb{C}$ is absolutely continuous, then

$$\left|\langle f(A)x,y\rangle-\langle x,y\rangle\frac{1}{M-m}\int_m^M f(s)\,ds\right| \tag{2.11}$$
$$\leq \bigvee_m^M\left(\langle E_{(\cdot)}x,y\rangle\right)\times\begin{cases}\frac{1}{2}(M-m)\|f'\|_\infty & \text{if } f'\in L_\infty[m,M]\\ \frac{1}{(q+1)^{1/q}}(M-m)^{1/q}\|f'\|_p & \text{if } f'\in L_p[m,M]\\ & p>1,1/p+1/q=1;\\ \|f'\|_1 & \end{cases}$$
$$\leq \|x\|\,\|y\|\times\begin{cases}\frac{1}{2}(M-m)\|f'\|_\infty & \text{if } f'\in L_\infty[m,M]\\ \frac{1}{(q+1)^{1/q}}(M-m)^{1/q}\|f'\|_p & \text{if } f'\in L_p[m,M]\\ & p>1,1/p+1/q=1;\\ \|f'\|_1 & \end{cases}$$

for any $x,y\in H$, where $\|f'\|_p$ are the Lebesgue norms, i.e., we recall that

$$\|f'\|_p:=\begin{cases}ess\sup_{s\in[m,M]}|f'(s)| & \text{if } p=\infty;\\ \left(\int_m^M|f(s)|^p\,ds\right)^{1/p} & \text{if } p\geq 1.\end{cases}$$

Proof. We use the Ostrowski type inequalities in order to provide upper bounds for the quantity

$$\max_{t\in[m,M]}\left|f(t)-\frac{1}{M-m}\int_m^M f(s)\,ds\right|$$

where $f:[m,M]\longrightarrow\mathbb{C}$ is a continuous function.

The following result may be stated (see [4]) for functions of bounded variation:

Lemma 1. *Assume that $f:[m,M]\to\mathbb{C}$ is of bounded variation and denote by $\bigvee_m^M(f)$ its total variation. Then*

$$\left|f(t)-\frac{1}{M-m}\int_m^M f(s)\,ds\right|\leq\left[\frac{1}{2}+\left|\frac{t-\frac{m+M}{2}}{M-m}\right|\right]\bigvee_m^M(f) \tag{2.12}$$

for all $t\in[m,M]$. The constant $\frac{1}{2}$ is the best possible.

Now, taking the maximum over $x\in[m,M]$ in (2.12) we deduce (2.7).

If f is Hölder continuous, then one may state the result (see for instance [18] and the references therein for earlier contributions):

Lemma 2. *Let $f:[m,M]\to\mathbb{C}$ be of $r-H-$Hölder type, where $r\in(0,1]$ and $H>0$ are fixed, then, for all $x\in[m,M]$, we have the inequality:*

$$\left|f(t)-\frac{1}{M-m}\int_m^M f(s)\,ds\right| \leq \frac{H}{r+1}\left[\left(\frac{M-t}{M-m}\right)^{r+1}+\left(\frac{t-m}{M-m}\right)^{r+1}\right](M-m)^r. \tag{2.13}$$

The constant $\frac{1}{r+1}$ is also sharp in the above sense.

Note that if $r=1$, i.e., f is Lipschitz continuous, then we get the following version of Ostrowski's inequality for Lipschitzian functions (with L instead of H) (see for instance [8])

$$\left|f(t)-\frac{1}{M-m}\int_m^M f(s)\,ds\right|\leq\left[\frac{1}{4}+\left(\frac{t-\frac{m+M}{2}}{M-m}\right)^2\right](M-m)L, \tag{2.14}$$

for any $x\in[m,M]$. Here the constant $\frac{1}{4}$ is also best.

Taking the maximum over $x\in[m,M]$ in (2.13) we deduce (2.9) and the second part of the corollary is proved.

The following Ostrowski type result for absolutely continuous functions holds (see [22] – [24]).

Lemma 3. *Let $f:[a,b]\to\mathbb{R}$ be absolutely continuous on $[a,b]$. Then, for all $t\in[a,b]$, we have:*

$$\left|f(t)-\frac{1}{M-m}\int_m^M f(s)\,ds\right| \tag{2.15}$$

$$\leq\begin{cases}\left[\frac{1}{4}+\left(\frac{t-\frac{m+M}{2}}{M-m}\right)^2\right](M-m)\|f'\|_\infty & \text{if } f'\in L_\infty[m,M];\\[2ex] \frac{1}{(q+1)^{\frac{1}{q}}}\left[\left(\frac{t-m}{M-m}\right)^{q+1}+\left(\frac{M-t}{M-m}\right)^{q+1}\right]^{\frac{1}{q}}(M-m)^{\frac{1}{q}}\|f'\|_p & \text{if } f'\in L_p[m,M],\\ & \frac{1}{p}+\frac{1}{q}=1,\ p>1;\\[2ex] \left[\frac{1}{2}+\left|\frac{t-\frac{m+M}{2}}{M-m}\right|\right]\|f'\|_1.\end{cases}$$

The constants $\frac{1}{4}$, $\frac{1}{(p+1)^{\frac{1}{p}}}$ and $\frac{1}{2}$ respectively are sharp in the sense presented above.

The above inequalities can also be obtained from the Fink result in [25] on choosing $n=1$ and performing some appropriate computations.

Taking the maximum in these inequalities we deduce (2.11). □

For other scalar Ostrowski's type inequalities, see [1]-[2] and [9].

3. Other Vector Inequalities

In [19], the authors have considered the following functional

$$D(f;u):=\int_a^b f(s)\,du(s)-[u(b)-u(a)]\cdot\frac{1}{b-a}\int_a^b f(t)\,dt, \tag{3.1}$$

provided that the Stieltjes integral $\int_a^b f(s)\,du(s)$ exists.

This functional plays an important role in approximating the Stieltjes integral $\int_a^b f(s)\,du(s)$ in terms of the Riemann integral $\int_a^b f(t)\,dt$ and the divided difference of the integrator u.

In [19], the following result in estimating the above functional $D(f;u)$ has been obtained:

$$|D(f;u)| \leq \frac{1}{2} L(M-m)(b-a), \tag{3.2}$$

provided u is $L-$*Lipschitzian* and f is *Riemann integrable* and with the property that there exists the constants $m, M \in \mathbb{R}$ such that

$$m \leq f(t) \leq M \quad \text{for any} \quad t \in [a,b]. \tag{3.3}$$

The constant $\frac{1}{2}$ is best possible in (3.2) in the sense that it cannot be replaced by a smaller quantity.

If one assumes that u is of *bounded variation* and f is $K-$*Lipschitzian*, then $D(f,u)$ satisfies the inequality [20]

$$|D(f;u)| \leq \frac{1}{2} K(b-a) \bigvee_a^b (u). \tag{3.4}$$

Here the constant $\frac{1}{2}$ is also best possible.

Now, for the function $u : [a,b] \to \mathbb{C}$, consider the following auxiliary mappings Φ, Γ and Δ [10]:

$$\Phi(t) := \frac{(t-a)u(b) + (b-t)u(a)}{b-a} - u(t), \qquad t \in [a,b],$$
$$\Gamma(t) := (t-a)[u(b)-u(t)] - (b-t)[u(t)-u(a)], \qquad t \in [a,b],$$
$$\Delta(t) := [u;b,t] - [u;t,a], \qquad t \in (a,b),$$

where $[u;\alpha,\beta]$ is the *divided difference* of u in α, β, i.e.,

$$[u;\alpha,\beta] := \frac{u(\alpha)-u(\beta)}{\alpha-\beta}.$$

The following representation of $D(f,u)$ may be stated, see [10] and [11]. Due to its importance in proving our new results we present here a short proof as well.

Lemma 4. *Let $f, u : [a,b] \to \mathbb{C}$ be such that the Stieltjes integral $\int_a^b f(t)\,du(t)$ and the Riemann integral $\int_a^b f(t)\,dt$ exist. Then*

$$\begin{aligned} D(f,u) &= \int_a^b \Phi(t)\,df(t) = \frac{1}{b-a}\int_a^b \Gamma(t)\,df(t) \\ &= \frac{1}{b-a}\int_a^b (t-a)(b-t)\Delta(t)\,df(t). \end{aligned} \tag{3.5}$$

Proof. Since $\int_a^b f(t)\,du(t)$ exists, hence $\int_a^b \Phi(t)\,df(t)$ also exists, and the integration by parts formula for Riemann-Stieltjes integrals gives that

$$\begin{aligned}\int_a^b \Phi(t)\,df(t) &= \int_a^b \left[\frac{(t-a)\,u(b)+(b-t)\,u(a)}{b-a} - u(t)\right] df(t) \\ &= \left[\frac{(t-a)\,u(b)+(b-t)\,u(a)}{b-a} - u(t)\right] f(t)\Big|_a^b \\ &\quad - \int_a^b f(t)\,d\left[\frac{(t-a)\,u(b)+(b-t)\,u(a)}{b-a} - u(t)\right] \\ &= -\int_a^b f(t)\left[\frac{u(b)-u(a)}{b-a}dt - du(t)\right] = D(f,u),\end{aligned}$$

proving the required identity. □

For recent inequalities related to $D(f;u)$ for various pairs of functions (f,u), see [12].

The following representation for a continuous function of selfadjoint operator may be stated:

Lemma 5. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m,M]$ for some real numbers $m < M$, $\{E_\lambda\}_\lambda$ be its spectral family and $f : [m,M] \to \mathbb{C}$ a continuous function on $[m,M]$. If $x, y \in H$, then we have the representation*

$$\langle f(A)\,x,y\rangle = \langle x,y\rangle \frac{1}{M-m}\int_m^M f(s)\,ds + \frac{1}{M-m}\int_{m-0}^M \langle [(t-m)(1_H - E_t) - (M-t)\,E_t]\,x,y\rangle\,df(t). \tag{3.6}$$

Proof. Utilising Lemma 4 we have

$$\int_m^M f(t)\,du(t) = [u(M)-u(m)]\cdot\frac{1}{M-m}\int_m^M f(s)\,ds + \int_m^M \left[\frac{(t-m)\,u(M)+(M-t)\,u(m)}{M-m} - u(t)\right] df(t), \tag{3.7}$$

for any continuous function $f : [m,M] \to \mathbb{C}$ and any function of bounded variation $u : [m,M] \to \mathbb{C}$.

Now, if we write the equality (3.7) for $u(t) = \langle E_t x, y\rangle$ with $x, y \in H$, then we get

$$\int_{m-0}^M f(t)\,d\langle E_t x,y\rangle = \langle x,y\rangle\cdot\frac{1}{M-m}\int_m^M f(s)\,ds + \int_{m-0}^M \left[\frac{(t-m)\langle x,y\rangle}{M-m} - \langle E_t x,y\rangle\right] df(t), \tag{3.8}$$

which, by the spectral representation (1.1), produces the desired result (3.6). □

The following result may be stated:

Theorem 4. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m,M]$ for some real numbers $m < M$ $\{E_\lambda\}_\lambda$ be its spectral family and $f : [m,M] \to \mathbb{C}$ a continuous function on $[m,M]$.*

1. If f is of bounded variation, then

$$\left|\langle f(A)x,y\rangle-\langle x,y\rangle\frac{1}{M-m}\int_m^M f(s)\,ds\right| \tag{3.9}$$
$$\leq \|y\|\bigvee_m^M(f)$$
$$\times\max_{t\in[m,M]}\left[\left(\frac{t-m}{M-m}\right)^2\|(1_H-E_t)x\|^2+\left(\frac{M-t}{M-m}\right)^2\|E_tx\|^2\right]^{1/2}$$
$$\leq\|x\|\,\|y\|\bigvee_m^M(f)$$

for any $x,y\in H$.

2. If f is Lipschitzian with the constant $L>0$, then

$$\left|\langle f(A)x,y\rangle-\langle x,y\rangle\frac{1}{M-m}\int_m^M f(s)\,ds\right| \tag{3.10}$$
$$\leq\frac{L\|y\|}{M-m}\int_{m-0}^M\left[(t-m)^2\|(1_H-E_t)x\|^2+(M-t)^2\|E_tx\|^2\right]^{1/2}dt$$
$$\leq\frac{1}{2}\left[1+\frac{\sqrt{2}}{2}\ln\left(\sqrt{2}+1\right)\right](M-m)L\|y\|\,\|x\|$$

for any $x,y\in H$.

3. If $f:[m,M]\to\mathbb{R}$ is monotonic nondecreasing, then

$$\left|\langle f(A)x,y\rangle-\langle x,y\rangle\frac{1}{M-m}\int_m^M f(s)\,ds\right| \tag{3.11}$$
$$\leq\frac{\|y\|}{M-m}\int_{m-0}^M\left[(t-m)^2\|(1_H-E_t)x\|^2+(M-t)^2\|E_tx\|^2\right]^{1/2}df(t)$$
$$\leq\|y\|\,\|x\|\int_m^M\left[\left(\frac{t-m}{M-m}\right)^2+\left(\frac{M-t}{M-m}\right)^2\right]^{1/2}df(t)$$
$$\leq\|y\|\,\|x\|\,[f(M)-f(m)]^{1/2}$$
$$\times\left[f(M)-f(m)-\frac{4}{M-m}\int_m^M\left(t-\frac{m+M}{2}\right)f(t)\,dt\right]^{1/2}$$
$$\leq\|y\|\,\|x\|\,[f(M)-f(m)]$$

for any $x,y\in H$.

Proof. If we assume that f is of bounded variation, then on applying the property (2.3) to the representation (3.6) we get

$$\left|\langle f(A)x,y\rangle - \langle x,y\rangle \frac{1}{M-m}\int_m^M f(s)\,ds\right| \le \frac{1}{M-m}\max_{t\in[m,M]}\left|\langle[(t-m)(1_H-E_t)-(M-t)E_t]x,y\rangle\right|\bigvee_m^M(f). \tag{3.12}$$

Now, on utilizing the Schwarz inequality and the fact that E_t is a projector for any $t\in[m,M]$, then we have

$$\begin{aligned}&\left|\langle[(t-m)(1_H-E_t)-(M-t)E_t]x,y\rangle\right| \\ &\le \|[(t-m)(1_H-E_t)-(M-t)E_t]x\|\,\|y\| \\ &= \left[(t-m)^2\|(1_H-E_t)x\|^2+(M-t)^2\|E_tx\|^2\right]^{1/2}\|y\| \\ &\le \left[(t-m)^2+(M-t)^2\right]^{1/2}\|x\|\,\|y\|\end{aligned} \tag{3.13}$$

for any $x,y\in H$ and for any $t\in[m,M]$.

Taking the maximum in (3.13) we deduce the desired inequality (3.9).

It is well known that if $p:[a,b]\to\mathbb{C}$ is a Riemann integrable function and $v:[a,b]\to\mathbb{C}$ is Lipschitzian with the constant $L>0$, i.e.,

$$|f(s)-f(t)|\le L|s-t| \text{ for any } t,s\in[a,b],$$

then the Riemann-Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and the following inequality holds

$$\left|\int_a^b p(t)\,dv(t)\right|\le L\int_a^b|p(t)|\,dt.$$

Now, on applying this property of the Riemann-Stieltjes integral to the representation (3.6), we get

$$\begin{aligned}&\left|\langle f(A)x,y\rangle - \langle x,y\rangle \frac{1}{M-m}\int_m^M f(s)\,ds\right| \\ &\le \frac{L}{M-m}\int_{m-0}^M\left|\langle[(t-m)(1_H-E_t)-(M-t)E_t]x,y\rangle\right|dt \\ &\le \frac{L\|y\|}{M-m}\int_{m-0}^M\left[(t-m)^2\|(1_H-E_t)x\|^2+(M-t)^2\|E_tx\|^2\right]^{1/2}dt \\ &\le L\|y\|\,\|x\|\int_m^M\left[\left(\frac{t-m}{M-m}\right)^2+\left(\frac{M-t}{M-m}\right)^2\right]^{1/2}dt,\end{aligned} \tag{3.14}$$

for any $x,y\in H$.

Now, if we change the variable in the integral by choosing $u=\frac{t-m}{M-m}$ then we get

$$\begin{aligned}
&\int_{m}^{M}\left[\left(\frac{t-m}{M-m}\right)^{2}+\left(\frac{M-t}{M-m}\right)^{2}\right]^{1/2}dt\\
&=(M-m)\int_{0}^{1}\left[u^{2}+(1-u)^{2}\right]^{1/2}du\\
&=\frac{1}{2}(M-m)\left[1+\frac{\sqrt{2}}{2}\ln\left(\sqrt{2}+1\right)\right],
\end{aligned}$$

which together with (3.14) produces the desired result (3.10).

From the theory of Riemann-Stieltjes integral is well known that if $p:[a,b]\to\mathbb{C}$ is of bounded variation and $v:[a,b]\to\mathbb{R}$ is continuous and monotonic nondecreasing, then the Riemann-Stieltjes integrals $\int_a^b p(t)\,dv(t)$ and $\int_a^b |p(t)|\,dv(t)$ exist and

$$\left|\int_{a}^{b}p(t)\,dv(t)\right|\leq\int_{a}^{b}|p(t)|\,dv(t).$$

Now, on applying this property of the Riemann-Stieltjes integral, we have from the representation (3.6)

$$\begin{aligned}
&\left|\langle f(A)x,y\rangle-\langle x,y\rangle\frac{1}{M-m}\int_{m}^{M}f(s)\,ds\right| \qquad (3.15)\\
&\leq\frac{1}{M-m}\int_{m-0}^{M}\left|\langle[(t-m)(1_H-E_t)-(M-t)E_t]x,y\rangle\right|df(t)\\
&\leq\frac{\|y\|}{M-m}\int_{m-0}^{M}\left[(t-m)^{2}\|(1_H-E_t)x\|^{2}+(M-t)^{2}\|E_tx\|^{2}\right]^{1/2}df(t)\\
&\leq\|y\|\|x\|\int_{m}^{M}\left[\left(\frac{t-m}{M-m}\right)^{2}+\left(\frac{M-t}{M-m}\right)^{2}\right]^{1/2}df(t),
\end{aligned}$$

for any $x,y\in H$ and the proof of the first and second inequality in (3.11) is completed.

For the last part we use the following Cauchy-Buniakowski-Schwarz integral inequality for the Riemann-Stieltjes integral with monotonic nondecreasing integrator v

$$\left|\int_{a}^{b}p(t)\,q(t)\,dv(t)\right|\leq\left[\int_{a}^{b}|p(t)|^{2}dv(t)\right]^{1/2}\left[\int_{a}^{b}|q(t)|^{2}dv(t)\right]^{1/2}$$

where $p,q:[a,b]\to\mathbb{C}$ are continuous on $[a,b]$.

By applying this inequality we conclude that

$$\begin{aligned}
&\int_{m}^{M}\left[\left(\frac{t-m}{M-m}\right)^{2}+\left(\frac{M-t}{M-m}\right)^{2}\right]^{1/2}df(t) \qquad (3.16)\\
&\leq\left[\int_{m}^{M}df(t)\right]^{1/2}\left[\int_{m}^{M}\left[\left(\frac{t-m}{M-m}\right)^{2}+\left(\frac{M-t}{M-m}\right)^{2}\right]df(t)\right]^{1/2}.
\end{aligned}$$

Further, integrating by parts in the Riemann-Stieltjes integral we also have that

$$\begin{aligned}(3.17)\qquad &\int_m^M \left[\left(\frac{t-m}{M-m}\right)^2+\left(\frac{M-t}{M-m}\right)^2\right] df(t)\\ &= f(M)-f(m)-\frac{4}{M-m}\int_m^M\left(t-\frac{m+M}{2}\right)f(t)\,dt\\ &\leq f(M)-f(m)\end{aligned}$$

where for the last part we used the fact that by the Čebyšev integral inequality for monotonic functions with the same monotonicity we have that

$$\begin{aligned}&\int_m^M\left(t-\frac{m+M}{2}\right)f(t)\,dt\\ &\geq \frac{1}{M-m}\int_m^M\left(t-\frac{m+M}{2}\right)dt\int_m^M f(t)\,dt=0.\end{aligned}$$

□

4. Some Applications for Particular Functions

1. Consider the function $f:(0,\infty)\to\mathbb{R}$ given by $f(t)=t^r$ with $r\in(0,1]$. This function is r-Hölder continuous with the constant $H>0$. Then, by applying Corollary 1 we can state the following result

Proposition 1. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A)\subseteq[m,M]$ for some real numbers $0<m<M$ and $\{E_\lambda\}_\lambda$ be its spectral family. Then for all r with $r\in(0,1]$ we have the inequality*

$$\begin{aligned}(4.1)\qquad &\left|\langle A^r x,y\rangle-\langle x,y\rangle\frac{M^{r+1}-m^{r+1}}{(r+1)(M-m)}\right|\\ &\leq\frac{1}{r+1}(M-m)^r\bigvee_m^M\left(\langle E_{(\cdot)}x,y\rangle\right)\leq\frac{1}{r+1}(M-m)^r\|x\|\|y\|\end{aligned}$$

for any $x,y\in H$.

The case of $p>1$ is incorporated in the following proposition:

Proposition 2. *With the same assumptions from Proposition 1 and if $p>1$, then we have*

$$\begin{aligned}(4.2)\qquad &\left|\langle A^p x,y\rangle-\frac{M^{p+1}-m^{p+1}}{(p+1)(M-m)}\langle x,y\rangle\right|\\ &\leq\frac{1}{2}pM^{p-1}(M-m)\bigvee_m^M\left(\langle E_{(\cdot)}x,y\rangle\right)\leq\frac{1}{2}pM^{p-1}(M-m)\|x\|\|y\|\end{aligned}$$

for any $x,y\in H$.

The case of negative powers except $p=-1$ goes likewise and we omit the details.

Now, if we apply Corollary 1 for the function $f(t)=-\frac{1}{t}$ with $t>0$, then we can state the following proposition:

Proposition 3. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $0 < m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. Then for any $x, y \in H$ we have the inequalities*

$$\left|\left\langle A^{-1}x, y\right\rangle - \frac{\ln M - \ln m}{M - m} \langle x, y\rangle\right| \leq \frac{1}{2}\frac{M - m}{m^2} \bigvee_m^M \left(\left\langle E_{(\cdot)}x, y\right\rangle\right) \leq \frac{1}{2}\frac{M - m}{m^2} \|x\| \|y\| . \tag{4.3}$$

2. Now, if we apply Corollary 1 to the function $f : (0, \infty) \to \mathbb{R}$, $f(t) = \ln t$, then we can state

Proposition 4. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $0 < m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. Then for any $x, y \in H$ we have the inequalities*

$$|\langle \ln Ax, y\rangle - \langle x, y\rangle \ln I(m, M)| \leq \frac{1}{2}\left(\frac{M}{m} - 1\right) \bigvee_m^M \left(\left\langle E_{(\cdot)}x, y\right\rangle\right) \leq \frac{1}{2}\left(\frac{M}{m} - 1\right) \|x\| \|y\| , \tag{4.4}$$

where $I(m, M)$ is the identric mean of m and M and is defined by

$$I(m, M) = \frac{1}{e}\left(\frac{M^M}{m^m}\right)^{1/(M-m)} .$$

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