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An Analogue of the Ostrowski-Grüss Inequality and Applications

Mohammad Masjed-Jamei [a] **Sever S. Dragomir** [b,*]

[a] *Department of Mathematics, K.N.Toosi University of Technology, P.O.Box 16315-1618, Tehran, Iran,*
E-mail: mmjamei@kntu.ac.ir , mmjamei@yahoo.com

[b] *Research Group in Mathematical Inequalities & Applications, School of Engineering and Science, Victoria University, P. O. Box 14428, Melbourne City, MC Victoria 8001, Australia,*
E-mail: sever.dragomir@vu.edu.au , [*] *Corresponding author*

Abstract. A new analogue of the Ostrowski-Grüss inequality is introduced in three different cases for functions in $L^1[a,b]$ and $L^\infty[a,b]$ spaces and its application is given for deriving error bounds of some quadrature rules.

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1. Introduction

Let $L^p[a,b]$ $(1 \le p < \infty)$ denote the space of *p*-power integrable functions on the interval $[a,b]$ with the standard norm

$$\|f\|_p = \left(\int_a^b |f(t)|^p \, dt \right)^{1/p},$$

and $L^\infty[a,b]$ the space of all essentially bounded functions on $[a,b]$ with the norm

$$\|f\|_\infty = ess \sup_{x\in[a,b]} |f(x)|.$$

If $f \in L^1[a,b]$ and $g \in L^\infty[a,b]$, the following inequality is well known

$$\left| \int_a^b f(x)\, g(x)\, dx \right| \le \| f \|_1 \| g \|_\infty .$$

For two absolutely continuous functions $f, g : [a,b] \to \mathbf{R}$ such that $f, g, fg \in L^1[a,b]$, the Chebyshev functional [1] is defined by

$$\mathbf{T}(f,g) = \frac{1}{b-a}\int_a^b \left(f(x) - \frac{1}{b-a}\int_a^b f(x)\,dx \right)\left(g(x) - \frac{1}{b-a}\int_a^b g(x)\,dx \right) dx$$
$$= \frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \frac{1}{(b-a)^2}\left(\int_a^b f(x)\,dx \right)\left(\int_a^b g(x)\,dx \right).$$

In 1934 Grüss [6] showed that

$$|\mathbf{T}(f,g)| \le \frac{1}{4}(M_1 - m_1)(M_2 - m_2), \tag{1}$$

where m_1, m_2, M_1 and M_2 are real numbers satisfying the conditions

$$m_1 \le f(x) \le M_1 \quad \text{and} \quad m_2 \le g(x) \le M_2 \quad \text{for all } x \in [a,b].$$

The constant $1/4$ is the best possible number in (1) in the sense that it cannot be replaced by a smaller quantity.

An inequality related to the Chebyshev functional is due to Ostrowski [13] in 1938. If $f : [a,b] \to \mathbf{R}$ is a differentiable function with bounded derivative, then

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \left(\frac{(x-a)^2 + (b-x)^2}{2(b-a)} \right) \| f' \|_\infty \quad \text{for all } x \in [a,b]. \tag{2}$$

The Ostrowski inequality also plays an important role in numerical quadrature rules, see e.g. [4,8].

In 1997, Dragomir and Wang [3] introduced a mixture type of the two mentioned inequalities (1) and (2), called the Ostrowski-Grüss inequality, and proved the following theorem.

Theorem A. *If* $f : [a,b] \to \mathbf{R}$ *is a differentiable function with bounded derivative and* $\alpha_0 \le f'(x) \le \beta_0$ *for all* $x \in [a,b]$, *then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(b) - f(a)}{b-a}\left(x - \frac{a+b}{2} \right) \right| \le \frac{1}{4}(b-a)(\beta_0 - \alpha_0). \tag{3}$$

Due to the importance of inequality (3), many refinements and generalizations have been so far presented in the literature. For instance, Cheng in [2] gave a sharp version of Ostrowski-Grüss inequality and proved that instead of $1/4$ in the right hand side of (3), the constant $1/8$ should be replaced. See also [9,10]. In 2007, Liu [7] applied inequality (3) for (l, L)-Lipschitzian functions as follows:

Theorem B. *Let* $f:[a,b]\to\mathbf{R}$ *be a* (l,L)*-Lipschitzian function on* $[a,b]$, *i.e.*

$$l(x_2-x_1)\le f(x_2)-f(x_1)\le L(x_2-x_1)\quad \text{for}\quad a\le x_1\le x_2\le b,$$

where $l,L\in\mathbf{R}$ *with* $l<L$. *Then for all* $x\in[a,b]$ *we have*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{f(b)-f(a)}{b-a}\left(x-\frac{a+b}{2}\right)\right|\le\frac{b-a}{8}(L-l).$$

The following theorem, due to Niezgoda [12], is probably the most recent work about Ostrowski-Grüss inequality in $L^p[a,b]$ spaces.

Theorem C. *Let* $f:\mathbf{I}\to\mathbf{R}$, *where* $\mathbf{I}$ *is an interval, be a function differentiable in the interior* $\mathbf{I}^0$ *of* $\mathbf{I}$, *and let* $[a,b]\subset\mathbf{I}^0$. *Suppose that* $f',\alpha,\beta\in L^p[a,b]$ $(1\le p<\infty)$ *are functions such that* $\alpha(t)+\beta(t)$ *is a constant function and* $\alpha(t)\le f'(t)\le\beta(t)$ *for all* $t\in[a,b]$. *Then for any* $x\in[a,b]$ *we have the following inequality*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{f(b)-f(a)}{b-a}\left(x-\frac{a+b}{2}\right)\right|\le\begin{cases}\dfrac{1}{4}\|\beta-\alpha\|_p\dfrac{(b-a)^{1/q}}{(q+1)^{1/q}} & (1\le q<\infty),\\[2mm] \dfrac{1}{4}\|\beta-\alpha\|_1 & (q=\infty),\end{cases}$$

where $1/p+1/q=1$.

In this paper we introduce a new analogue of the Ostrowski-Grüss inequality in three different cases and apply them for some quadrature rules. For this purpose, let us first consider the following kernel on $[a,b]$:

$$K(x;t)=\begin{cases}t-x+\dfrac{b-a}{2} & t\in[a,x],\\[2mm] t-x-\dfrac{b-a}{2} & t\in(x,b].\end{cases}\tag{4}$$

Note in this kernel that $t-x+(b-a)/2$ is always positive for $t\in[a,x]$ and $x\in[a,(a+b)/2]$ while $t-x-(b-a)/2$ is negative for $t\in(x,b]$ and $x\in[(a+b)/2,b]$.

After some computations, one can directly conclude that

$$\frac{1}{b-a}\int_a^b f'(t)\,K(x;t)\,dt=f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{f(b)-f(a)}{b-a}\left(x-\frac{a+b}{2}\right).\tag{5}$$

2. Main Results

Theorem 1. *Let* $f:\mathbf{I}\to\mathbf{R}$, *where* $\mathbf{I}$ *is an interval, be a function differentiable in the interior* $\mathbf{I}^0$ *of* $\mathbf{I}$, *and let* $[a,b]\subset\mathbf{I}^0$. *If* $\alpha(x)\le f'(x)\le\beta(x)$ *for any* $\alpha,\beta\in C[a,b]$ *and* $x\in[a,b]$ *then the following inequality holds*

$$\begin{aligned} m(x)&=\frac{1}{b-a}\left(\int_{-\frac{b-a}{2}}^{\frac{a+b}{2}-x}\left(\frac{z+|z|}{2}\alpha(z+x+\frac{b-a}{2})+\frac{z-|z|}{2}\beta(z+x+\frac{b-a}{2})\right)dz\right.\\ &\qquad\left.+\int_{\frac{a+b}{2}-x}^{\frac{b-a}{2}}\left(\frac{z+|z|}{2}\alpha(z+x-\frac{b-a}{2})+\frac{z-|z|}{2}\beta(z+x-\frac{b-a}{2})\right)dz\right)\\ &\le f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{f(b)-f(a)}{b-a}\left(x-\frac{a+b}{2}\right)\le\\ M(x)&=\frac{1}{b-a}\left(\int_{-\frac{b-a}{2}}^{\frac{a+b}{2}-x}\left(\frac{z-|z|}{2}\alpha(z+x+\frac{b-a}{2})+\frac{z+|z|}{2}\beta(z+x+\frac{b-a}{2})\right)dz\right.\\ &\qquad\left.+\int_{\frac{a+b}{2}-x}^{\frac{b-a}{2}}\left(\frac{z-|z|}{2}\alpha(z+x-\frac{b-a}{2})+\frac{z+|z|}{2}\beta(z+x-\frac{b-a}{2})\right)dz\right). \end{aligned} \tag{6}$$

Proof. By referring to the kernel (4) and identity (5) we first have

$$\begin{aligned} &\int_a^b K(x;t)\left(f'(t)-\frac{\alpha(t)+\beta(t)}{2}\right)dt\\ &=(b-a)f(x)-\int_a^b f(t)\,dt-(f(b)-f(a))\left(x-\frac{a+b}{2}\right)-\frac{1}{2}\left(\int_a^b K(x;t)(\alpha(t)+\beta(t))\,dt\right)\\ &=(b-a)f(x)-\int_a^b f(t)\,dt-(f(b)-f(a))\left(x-\frac{a+b}{2}\right)\\ &-\frac{1}{2}\left(\int_a^x(t-x+\frac{b-a}{2})(\alpha(t)+\beta(t))\,dt+\int_x^b(t-x-\frac{b-a}{2})(\alpha(t)+\beta(t))\,dt\right). \end{aligned} \tag{7}$$

On the other hand, the given assumption $\alpha(t)\le f'(t)\le\beta(t)$ results in

$$\left|f'(t)-\frac{\alpha(t)+\beta(t)}{2}\right|\le\frac{\beta(t)-\alpha(t)}{2}. \tag{8}$$

Therefore, one can conclude from (7) and (8) that

$$\left|(b-a)f(x)-\int_a^b f(t)\,dt-(f(b)-f(a))\left(x-\frac{a+b}{2}\right)\right.$$
$$\left.-\frac{1}{2}\left(\int_a^x (t-x+\frac{b-a}{2})(\alpha(t)+\beta(t))\,dt+\int_x^b (t-x-\frac{b-a}{2})(\alpha(t)+\beta(t))\,dt\right)\right|$$
$$=\left|\int_a^b K(x;t)\left(f'(t)-\frac{\alpha(t)+\beta(t)}{2}\right)dt\right|\le\int_a^b |K(x;t)|\frac{\beta(t)-\alpha(t)}{2}\,dt \tag{9}$$
$$=\frac{1}{2}\left(\int_a^x\left|t-x+\frac{b-a}{2}\right|(\beta(t)-\alpha(t))\,dt+\int_x^b\left|t-x-\frac{b-a}{2}\right|(\beta(t)-\alpha(t))\,dt\right).$$

After re-arranging (9) we obtain

$$m(x)=\frac{1}{b-a}\left(\int_a^x\left(\left(t-x+\frac{b-a}{2}-\left|t-x+\frac{b-a}{2}\right|\right)\frac{\beta(t)}{2}+\left(t-x+\frac{b-a}{2}+\left|t-x+\frac{b-a}{2}\right|\right)\frac{\alpha(t)}{2}\right)dt\right.$$
$$\left.+\int_x^b\left(\left(t-x-\frac{b-a}{2}-\left|t-x-\frac{b-a}{2}\right|\right)\frac{\beta(t)}{2}+\left(t-x-\frac{b-a}{2}+\left|t-x-\frac{b-a}{2}\right|\right)\frac{\alpha(t)}{2}\right)dt\right)$$
$$=\frac{1}{b-a}\left(\int_{-\frac{b-a}{2}}^{\frac{a+b}{2}-x}\left(\frac{z+|z|}{2}\alpha(z+x+\frac{b-a}{2})+\frac{z-|z|}{2}\beta(z+x+\frac{b-a}{2})\right)dz\right.$$
$$\left.+\int_{\frac{a+b}{2}-x}^{\frac{b-a}{2}}\left(\frac{z+|z|}{2}\alpha(z+x-\frac{b-a}{2})+\frac{z-|z|}{2}\beta(z+x-\frac{b-a}{2})\right)dz\right),$$

and

$$M(x)=\frac{1}{b-a}\left(\int_a^x\left(\left(t-x+\frac{b-a}{2}+\left|t-x+\frac{b-a}{2}\right|\right)\frac{\beta(t)}{2}+\left(t-x+\frac{b-a}{2}-\left|t-x+\frac{b-a}{2}\right|\right)\frac{\alpha(t)}{2}\right)dt\right.$$
$$\left.+\int_x^b\left(\left(t-x-\frac{b-a}{2}+\left|t-x-\frac{b-a}{2}\right|\right)\frac{\beta(t)}{2}+\left(t-x-\frac{b-a}{2}-\left|t-x-\frac{b-a}{2}\right|\right)\frac{\alpha(t)}{2}\right)dt\right)$$
$$=\frac{1}{b-a}\left(\int_{-\frac{b-a}{2}}^{\frac{a+b}{2}-x}\left(\frac{z-|z|}{2}\alpha(z+x+\frac{b-a}{2})+\frac{z+|z|}{2}\beta(z+x+\frac{b-a}{2})\right)dz\right.$$
$$\left.+\int_{\frac{a+b}{2}-x}^{\frac{b-a}{2}}\left(\frac{z-|z|}{2}\alpha(z+x-\frac{b-a}{2})+\frac{z+|z|}{2}\beta(z+x-\frac{b-a}{2})\right)dz\right).$$

■

The advantage of theorem 1 is that necessary computations in bounds (6) are just in terms of the pre-assigned functions $\alpha(t),\beta(t)$ (not f'). Moreover, note that $m(x)$ and $M(x)$ in (6) can be simplified if $x\in[a,(a+b)/2]$ or $x\in[(a+b)/2,b]$. For instance, if $x\in[a,(a+b)/2]$ then

$$m(x)=\frac{1}{b-a}\left(\int_{-\frac{b-a}{2}}^{0} z\,\beta(z+x+\frac{b-a}{2})\,dz+\int_{0}^{\frac{a+b}{2}-x} z\,\alpha(z+x+\frac{b-a}{2})\,dz+\int_{\frac{a+b}{2}-x}^{\frac{b-a}{2}} z\,\alpha(z+x-\frac{b-a}{2})\,dz\right),$$

and

$$M(x)=\frac{1}{b-a}\left(\int_{-\frac{b-a}{2}}^{0} z\,\alpha(z+x+\frac{b-a}{2})\,dz+\int_{0}^{\frac{a+b}{2}-x} z\,\beta(z+x+\frac{b-a}{2})\,dz+\int_{\frac{a+b}{2}-x}^{\frac{b-a}{2}} z\,\beta(z+x-\frac{b-a}{2})\,dz\right).$$

Special case 1. Substituting $\alpha(x)=\alpha_0\neq 0$ and $\beta(x)=\beta_0\neq 0$ in (6) gives the sharp version of Ostrowski-Grüss inequality as

$$\left|\, f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{f(b)-f(a)}{b-a}\left(x-\frac{a+b}{2}\right)\right|\leq\frac{1}{8}(b-a)(\beta_0-\alpha_0)\,.$$

Remark 1. Although $\alpha(x)\leq f'(x)\leq\beta(x)$ is a straightforward condition in theorem 1, however sometimes one might not be able to easily obtain both bounds of $\alpha(x)$ and $\beta(x)$ for f'. In this case, we can make use of two analogue theorems. The first one would be helpful when f' is unbounded from above and the second one would be helpful when f' is unbounded from below.

Theorem 2. *Let* $f:\mathbf{I}\to\mathbf{R}$, *where* $\mathbf{I}$ *is an interval, be a function differentiable in the interior* $\mathbf{I}^0$ *of* $\mathbf{I}$, *and let* $[a,b]\subset\mathbf{I}^0$. *If* $\alpha(x)\leq f'(x)$ *for any* $\alpha\in C[a,b]$ *and* $x\in[a,b]$ *then*

$$\int_a^b(t-x)\,\alpha(t)\,dt+\frac{b-a}{2}\left(\int_a^x\alpha(t)\,dt-\int_x^b\alpha(t)\,dt\right)-\max\left\{\left|\frac{a+b}{2}-x\right|,\frac{b-a}{2}\right\}\left(f(b)-f(a)-\int_a^b\alpha(t)\,dt\right)$$
$$\leq(b-a)\,f(x)-\int_a^b f(t)\,dt-(f(b)-f(a))\left(x-\frac{a+b}{2}\right)\leq$$
$$\int_a^b(t-x)\,\alpha(t)\,dt+\frac{b-a}{2}\left(\int_a^x\alpha(t)\,dt-\int_x^b\alpha(t)\,dt\right)+\max\left\{\left|\frac{a+b}{2}-x\right|,\frac{b-a}{2}\right\}\left(f(b)-f(a)-\int_a^b\alpha(t)\,dt\right). \tag{10}$$

Proof. Since

$$\int_a^b K(x;t)\left(f'(t)-\alpha(t)\right)dt$$
$$=(b-a)\,f(x)-\int_a^b f(t)\,dt-(f(b)-f(a))\left(x-\frac{a+b}{2}\right)-\left(\int_a^b K(x;t)\,\alpha(t)\,dt\right)$$
$$=(b-a)\,f(x)-\int_a^b f(t)\,dt-(f(b)-f(a))\left(x-\frac{a+b}{2}\right)$$
$$-\left(\int_a^x(t-x+\frac{b-a}{2})\,\alpha(t)\,dt+\int_x^b(t-x-\frac{b-a}{2})\,\alpha(t)\,dt\right),$$

so we have

$$\left|(b-a)f(x)-\int_a^b f(t)\,dt-(f(b)-f(a))\left(x-\frac{a+b}{2}\right)\right.$$
$$\left.-\left(\int_a^x (t-x+\frac{b-a}{2})\,\alpha(t)\,dt+\int_x^b (t-x-\frac{b-a}{2})\,\alpha(t)\,dt\right)\right|$$
$$=\left|\int_a^b K(x;t)\left(f'(t)-\alpha(t)\right)dt\right|\le\int_a^b \left|K(x;t)\right|\left(f'(t)-\alpha(t)\right)dt \tag{11}$$
$$\le\max_{t\in[a,b]}\left|K(x;t)\right|\int_a^b \left(f'(t)-\alpha(t)\right)dt=\max\left\{\left|\frac{a+b}{2}-x\right|,\frac{b-a}{2}\right\}\left(f(b)-f(a)-\int_a^b \alpha(t)\,dt\right).$$

After re-arranging (11), the main inequality (10) will be derived. ■

Special case 2. If $\alpha(x)=\alpha_1 x+\alpha_0\neq 0$ then (10) becomes

$$\frac{\alpha_1}{12}\left((b-a)^2+6(x-a)(x-b)\right)-\max\left\{\left|\frac{a+b}{2}-x\right|,\frac{b-a}{2}\right\}\left(\frac{f(b)-f(a)}{b-a}-\alpha_0-\frac{a+b}{2}\alpha_1\right)$$
$$\le f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{f(b)-f(a)}{b-a}\left(x-\frac{a+b}{2}\right)\le$$
$$\frac{\alpha_1}{12}\left((b-a)^2+6(x-a)(x-b)\right)+\max\left\{\left|\frac{a+b}{2}-x\right|,\frac{b-a}{2}\right\}\left(\frac{f(b)-f(a)}{b-a}-\alpha_0-\frac{a+b}{2}\alpha_1\right),$$

if and only if $\alpha_1 x+\alpha_0\le f'(x)\quad\forall x\in[a,b]$.

1.3. Theorem 3. *Let* $f:\mathbf{I}\to\mathbf{R}$, *where* $\mathbf{I}$ *is an interval, be a function differentiable in the interior* $\mathbf{I}^0$ *of* $\mathbf{I}$, *and let* $[a,b]\subset\mathbf{I}^0$. *If* $f'(x)\le\beta(x)$ *for any* $\beta\in C[a,b]$ *and* $x\in[a,b]$ *then*

$$\int_a^b (t-x)\,\beta(t)\,dt+\frac{b-a}{2}\left(\int_a^x \beta(t)\,dt-\int_x^b \beta(t)\,dt\right)-\max\left\{\left|\frac{a+b}{2}-x\right|,\frac{b-a}{2}\right\}\left(\int_a^b \beta(t)\,dt-f(b)+f(a)\right)$$
$$\le(b-a)f(x)-\int_a^b f(t)\,dt-(f(b)-f(a))\left(x-\frac{a+b}{2}\right)\le$$
$$\int_a^b (t-x)\,\beta(t)\,dt+\frac{b-a}{2}\left(\int_a^x \beta(t)\,dt-\int_x^b \beta(t)\,dt\right)+\max\left\{\left|\frac{a+b}{2}-x\right|,\frac{b-a}{2}\right\}\left(\int_a^b \beta(t)\,dt-f(b)+f(a)\right). \tag{12}$$

Proof. Since

$$\int_a^b K(x;t)\left(f'(t)-\beta(t)\right)dt$$

$$=(b-a)f(x)-\int_a^b f(t)\,dt-(f(b)-f(a))\left(x-\frac{a+b}{2}\right)-\left(\int_a^b K(x;t)\,\beta(t)\,dt\right)$$

$$=(b-a)f(x)-\int_a^b f(t)\,dt-(f(b)-f(a))\left(x-\frac{a+b}{2}\right)$$

$$-\left(\int_a^x (t-x+\frac{b-a}{2})\,\beta(t)\,dt+\int_x^b (t-x-\frac{b-a}{2})\,\beta(t)\,dt\right),$$

so we have

$$\left|(b-a)f(x)-\int_a^b f(t)\,dt-(f(b)-f(a))\left(x-\frac{a+b}{2}\right)\right.$$

$$\left.-\left(\int_a^x (t-x+\frac{b-a}{2})\,\beta(t)\,dt+\int_x^b (t-x-\frac{b-a}{2})\,\beta(t)\,dt\right)\right|$$

$$=\left|\int_a^b K(x;t)\left(f'(t)-\beta(t)\right)dt\right|\le\int_a^b \left|K(x;t)\right|\left(\beta(t)-f'(t)\right)dt \tag{13}$$

$$\le\max_{t\in[a,b]}\left|K(x;t)\right|\int_a^b\left(\beta(t)-f'(t)\right)dt=\max\left\{\left|\frac{a+b}{2}-x\right|,\frac{b-a}{2}\right\}\left(\int_a^b\beta(t)\,dt-f(b)+f(a)\right).$$

After re-arranging (13), the main inequality (12) will be derived. ■

Special case 3. If $\beta(x)=\beta_1 x+\beta_0\neq 0$ in (12), then

$$\frac{\beta_1}{12}\left((b-a)^2+6(x-a)(x-b)\right)-\max\left\{\left|\frac{a+b}{2}-x\right|,\frac{b-a}{2}\right\}\left(\beta_0+\frac{a+b}{2}\beta_1-\frac{f(b)-f(a)}{b-a}\right)$$

$$\le f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{f(b)-f(a)}{b-a}\left(x-\frac{a+b}{2}\right)\le$$

$$\frac{\beta_1}{12}\left((b-a)^2+6(x-a)(x-b)\right)+\max\left\{\left|\frac{a+b}{2}-x\right|,\frac{b-a}{2}\right\}\left(\beta_0+\frac{a+b}{2}\beta_1-\frac{f(b)-f(a)}{b-a}\right),$$

if and only if $\alpha_1 x+\alpha_0\le f'(x)\quad\forall x\in[a,b]$.

3. Applications in numerical integration

A general ($n+1$)-point weighted quadrature formula is denoted by

$$\int_a^b w(x)\,f(x)\,dx=\sum_{k=0}^{n} w_k\ f(x_k)+R_{n+1}[f], \tag{14}$$

where $w(x)$ is a positive weight function on $[a,b]$, $\{x_k\}_{k=0}^n$ and $\{w_k\}_{k=0}^n$ are respectively nodes and weight coefficients and $R_{n+1}[f]$ is the corresponding error [5].

Let $\mathbf{\Pi}_d$ be the set of algebraic polynomials of degree at most *d*. The quadrature formula (14) has degree of exactness *d* if for every $p \in \mathbf{\Pi}_d$ we have $R_{n+1}[p] = 0$. In addition, if $R_{n+1}[p] \neq 0$ for some $\mathbf{\Pi}_{d+1}$, formula (14) has precise degree of exactness *d*.

The convergence order of quadrature rule (14) depends on the smoothness of the function f as well as on its degree of exactness. It is well known that for given $n+1$ mutually different nodes $\{x_k\}_{k=0}^n$ we can always achieve a degree of exactness $d = n$ by interpolating at these nodes and integrating the interpolated polynomial instead of f. Namely, taking the node polynomial

$$\Psi_{n+1}(x) = \prod_{k=0}^{n} (x - x_k),$$

by integrating the Lagrange interpolation formula

$$f(x) = \sum_{k=0}^{n} f(x_k)\, L(x; x_k) + r_{n+1}(f; x),$$

where

$$L(x; x_k) = \frac{\Psi_{n+1}(x)}{\Psi'_{n+1}(x_k)(x - x_k)} \qquad (k = 0,1,\ldots,n),$$

we obtain (14), with

$$w_k = \frac{1}{\Psi'_{n+1}(x_k)} \int_a^b \frac{\Psi_{n+1}(x)\, w(x)}{x - x_k}\, dx \qquad (k = 0,1,\ldots,n),$$

and

$$R_{n+1}[f] = \int_a^b r_{n+1}(f; x)\, w(x)\, dx.$$

We should note that for each $f \in \mathbf{\Pi}_n$ we have $r_{n+1}(f; x) = 0$ and therefore $R_{n+1}[f] = 0$.

Quadrature formulae obtained in this way are known as interpolatory. If a quadrature is not of the interpolatory type, i.e. if it does not follow the concept of the degree of exactness, then it would be a nonstandard quadrature rule, see e.g. [11].

Usually the simplest interpolatory quadrature formula of type (14) with pre-determined nodes $\{x_k\}_{k=0}^n \in [a,b]$ is called a weighted Newton-Cotes formula. For $w(x) = 1$ and the equidistant nodes $\{x_k\}_{k=0}^n = \{a + kh\}_{k=0}^n$ with $h = (b-a)/n$, the classical Newton-Cotes formulas including the midpoint rule for $n=0$ and $w(x)=1$, the trapezoidal rule for $n=1$ and $w(x)=1$ and so on are derived.

In this section we use theorems 1, 2 and 3 to obtain error bounds for midpoint and trapezoidal quadrature rules and also four further nonstandard quadratures as follows:

$$I_1(f): \quad \int_a^b f(x)\,dx \cong (b-a)\, f(\frac{a+b}{2}),$$

$$I_2(f): \quad \int_a^b f(x)\,dx \cong \frac{b-a}{2}\big(f(a)+f(b)\big),$$

$$I_3(f): \quad \int_a^b f(t)\,dt \cong \frac{b-a}{2}\left(-f(a)+2f(\frac{a+b}{2})+f(b)\right),$$

$$I_4(f): \quad \int_a^b f(t)\,dt \cong \frac{b-a}{2}\left(f(a)+2f(\frac{a+b}{2})-f(b)\right),$$

$$I_5(f): \quad \int_a^b f(x)\,dx \cong (b-a)\, f(a),$$

$$I_6(f): \quad \int_a^b f(x)\,dx \cong (b-a)\, f(b).$$

Corollary 1. If $\alpha(x) \le f'(x) \le \beta(x)$ for any $x \in [a,b]$ and $\alpha, \beta \in C[a,b]$ then by replacing $x = \frac{a+b}{2} \in [a,b]$ in (6), the error of midpoint rule $I_1(f)$ can be bounded as

$$\int_{-\frac{b-a}{2}}^{0}\left(\frac{z+|z|}{2}\alpha(z+b)+\frac{z-|z|}{2}\beta(z+b)\right)dz+\int_0^{\frac{b-a}{2}}\left(\frac{z+|z|}{2}\alpha(z+a)+\frac{z-|z|}{2}\beta(z+a)\right)dz$$
$$\le (b-a)f(\frac{a+b}{2})-\int_a^b f(t)\,dt \le \tag{15}$$
$$\int_{-\frac{b-a}{2}}^{0}\left(\frac{z-|z|}{2}\alpha(z+b)+\frac{z+|z|}{2}\beta(z+b)\right)dz+\int_0^{\frac{b-a}{2}}\left(\frac{z-|z|}{2}\alpha(z+a)+\frac{z+|z|}{2}\beta(z+a)\right)dz.$$

For instance, if $\alpha(x)=\alpha_1 x+\alpha_0 \ne 0$ and $\beta(x)=\beta_1 x+\beta_0 \ne 0$ in (15) then we get

$$\frac{(b-a)^2}{4}\left(\frac{b-a}{6}(\alpha_1+\beta_1)+\frac{\alpha_0+a\,\alpha_1-(\beta_0+b\,\beta_1)}{2}\right) \le (b-a)\, f(\frac{a+b}{2})-\int_a^b f(t)\,dt$$
$$\le \frac{(b-a)^2}{4}\left(\frac{b-a}{6}(\alpha_1+\beta_1)+\frac{\beta_0+a\,\beta_1-(\alpha_0+b\,\alpha_1)}{2}\right).$$

Corollary 2. If $\alpha(x) \le f'(x) \le \beta(x)$ for any $x \in [a,b]$ and $\alpha, \beta \in C[a,b]$ then by replacing $x = a$ or $x = b$ in (6), the error of trapezoidal rule $I_2(f)$ can be bounded as

$$\begin{aligned}
&\int_{-\frac{b-a}{2}}^{\frac{b-a}{2}} \left(\frac{z+|z|}{2} \alpha(z + \frac{a+b}{2}) + \frac{z-|z|}{2} \beta(z + \frac{a+b}{2}) \right) dz \\
&\le \frac{b-a}{2} \big(f(a) + f(b) \big) - \int_a^b f(t)\,dt \le \qquad (16) \\
&\int_{-\frac{b-a}{2}}^{\frac{b-a}{2}} \left(\frac{z-|z|}{2} \alpha(z + \frac{a+b}{2}) + \frac{z+|z|}{2} \beta(z + \frac{a+b}{2}) \right) dz.
\end{aligned}$$

Corollary 3. If $\alpha(x) \le f'(x) \le \beta(x)$ for any $x \in [a,b]$ and $\alpha, \beta \in C[a,b]$ then the error of nonstandard quadrature $I_3(f)$ can be bounded as

$$\begin{aligned}
&\int_a^{\frac{a+b}{2}} (t-a)\,\alpha(t)\,dt + \int_{\frac{a+b}{2}}^b (t-b)\,\alpha(t)\,dt + \frac{b-a}{2} \int_a^b \alpha(t)\,dt \\
&\le \frac{b-a}{2} \left(-f(a) + 2f(\frac{a+b}{2}) + f(b) \right) - \int_a^b f(t)\,dt \le \qquad (17) \\
&\int_a^{\frac{a+b}{2}} (t-a)\,\beta(t)\,dt + \int_{\frac{a+b}{2}}^b (t-b)\,\beta(t)\,dt + \frac{b-a}{2} \int_a^b \beta(t)\,dt.
\end{aligned}$$

Proof. To prove (17) we need to use the results of both theorems 2 and 3 simultaneously such that by replacing $x = (a+b)/2$ in (10) we first obtain

$$\begin{aligned}
&\int_a^b (t-a)\,\alpha(t)\,dt + \frac{b-a}{2} \left(\int_a^{\frac{a+b}{2}} \alpha(t)\,dt - \int_{\frac{a+b}{2}}^b \alpha(t)\,dt \right) \\
&\le \frac{b-a}{2} \left(-f(a) + 2f(\frac{a+b}{2}) + f(b) \right) - \int_a^b f(t)\,dt, \qquad (18)
\end{aligned}$$

provided that $\alpha(t) \le f'(t) \quad \forall t \in [a,b]$. On the other hand, replacing $x = (a+b)/2$ in (12) gives

$$\begin{aligned}
&\frac{b-a}{2} \left(-f(a) + 2f(\frac{a+b}{2}) + f(b) \right) - \int_a^b f(t)\,dt \le \qquad (19) \\
&\int_a^b (t-a)\,\beta(t)\,dt + \frac{b-a}{2} \left(\int_a^{\frac{a+b}{2}} \beta(t)\,dt - \int_{\frac{a+b}{2}}^b \beta(t)\,dt \right),
\end{aligned}$$

provided that $f'(t) \le \beta(t) \quad \forall t \in [a,b]$. Now by combining two latter results (18) and (19) the inequality (20) is derived.

Corollary 4. If $\alpha(x) \le f'(x) \le \beta(x)$ for any $x \in [a,b]$ and $\alpha, \beta \in C[a,b]$ then the error of nonstandard quadrature $I_4(f)$ can be bounded as

$$\begin{aligned}
&\int_a^{\frac{a+b}{2}} (t-a)\,\beta(t)\,dt + \int_{\frac{a+b}{2}}^b (t-b)\,\beta(t)\,dt - \frac{b-a}{2}\int_a^b \beta(t)\,dt \\
&\quad \le \frac{b-a}{2}\left(f(a) + 2f(\frac{a+b}{2}) - f(b) \right) - \int_a^b f(t)\,dt \le \\
&\int_a^{\frac{a+b}{2}} (t-a)\,\alpha(t)\,dt + \int_{\frac{a+b}{2}}^b (t-b)\,\alpha(t)\,dt - \frac{b-a}{2}\int_a^b \alpha(t)\,dt.
\end{aligned} \tag{20}$$

Proof. The proof of (20) is similar to that of corollary 3 if one replaces $x = (a+b)/2$ in respectively (10) and (12) and then combines them together.

Corollary 5. If $\alpha(x) \le f'(x) \le \beta(x)$ for any $x \in [a,b]$ and $\alpha, \beta \in C[a,b]$ then by replacing $x = b$ in respectively (10) and (12), the error of nonstandard quadrature $I_5(f)$ can be bounded as

$$\int_a^b (t-b)\,\beta(t)\,dt \le (b-a)\,f(a) - \int_a^b f(t)\,dt \le \int_a^b (t-b)\,\alpha(t)\,dt. \tag{21}$$

Corollary 6. If $\alpha(x) \le f'(x) \le \beta(x)$ for any $x \in [a,b]$ and $\alpha, \beta \in C[a,b]$ then by replacing $x = a$ in respectively (10) and (12), the error of nonstandard quadrature $I_6(f)$ can be bounded as

$$\int_a^b (t-a)\,\alpha(t)\,dt \le (b-a)\,f(b) - \int_a^b f(t)\,dt \le \int_a^b (t-a)\,\beta(t)\,dt. \tag{22}$$

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