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THE HADAMARD'S INEQUALITY FOR QUASI-CONVEX FUNCTIONS VIA FRACTIONAL INTEGRALS

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Abstract. In present note, firstly we give the Riemann-Liouville fractional integrals definitions. Secondly we use this Riemann-Liouville fractional integrals to establish some new integral inequalities for quasi-convex functions. Also, some applications for special means of real numbers are provided.

1. INTRODUCTION

Let real functions f be defined on some nonempty interval I of real line $\mathbb{R}$. The function f is said to be quasi-convex on I if inequality

$$f(tx+(1-t)y) \leq \max\{f(x), f(y)\}, \qquad (QC)$$

holds for all $x, y \in I$ and $t \in [0,1]$ (see [9]).

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function on the interval of I of real numbers and $a, b \in I$ with $a < b$. The following double inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

is well-known in the literature as Hadamard's inequality. For several recent results concerning the inequality (1.1) we refer the interested reader to ([1], [2], [8], [9], [11]-[15]). Clearly, any convex function is quasi-convex function. Furthermore, there exist quasi-convex functions which are not convex. For example, consider the following:

Let $f : \mathbb{R}^+ \to \mathbb{R}$,

$$f(x) = \ln x, \quad x \in \mathbb{R}^+.$$

This function is quasi-convex. However f is not convex functions.

In [9], Dragomir and Pearce proved the following results connected with the inequality (1.1):

Theorem 1. *Let $f : I \to \mathbb{R}$ be a Wright-quasi-convex map on I and suppose $a, b \in I \subseteq \mathbb{R}$ with $a < b$ and $f \in L_1[a,b]$. Then we have the inequality*

$$\frac{1}{b-a}\int_a^b f(t)dt \leq \max\{f(a), f(b)\}. \tag{1.2}$$

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Theorem 2. *Let $WQC(I)$ denote the class of Wright-quasi-convex functions on $I \subseteq \mathbb{R}$. Then*

$$QC(I) \subset WQC(I) \subset JQC(I).$$

In [11], Ion proved the following results connected with quasi-convex function:

Theorem 3. *Assume $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ is a differentiable function on (a, b). If $|f'|$ is quasi-convex on $[a, b]$ then the following inequality holds true*

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x)dx \right| \leq \frac{b - a}{4} \max \{ |f'(a)|, |f'(b)| \}. \tag{1.3}$$

Theorem 4. *Assume $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ is a differentiable function on (a, b). Assume $p \in \mathbb{R}$ with $p > 1$. If $|f'|^{p/(p-1)}$ is quasi-convex on $[a, b]$ then the following inequality holds true*

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x)dx \right| \leq \frac{b - a}{2(p + 1)^{1/p}} \left[\max \left\{ |f'(a)|^{\frac{p}{p-1}}, |f'(b)|^{\frac{p}{p-1}} \right\} \right]^{\frac{p-1}{p}}. \tag{1.4}$$

In [15], Alomari *et al.* proved the following theorem for quasi-convex function:

Theorem 5. *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$. If $|f'|^q$ is quasi-convex on $[a, b]$, $q \geq 1$, then the following inequality holds:*

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x)dx \right| \leq \frac{b - a}{4} \left(\max \left\{ |f'(a)|^q, |f'(b)|^q \right\} \right)^{\frac{1}{q}}. \tag{1.5}$$

In [12], Liu has generalized above results to the case that $|f'|^q$ is quasi-convex as:

Theorem 6. *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$ and $f' \in L[a, b]$. If $|f'|^q$ is quasi-convex on $[a, b]$, $q \geq 1$, then for any $\lambda \in [0, 1]$ and $x \in \left[a + \lambda \frac{b-a}{2}, b - \lambda \frac{b-a}{2}\right]$ we have*

$$\begin{aligned} &\left| (1 - \lambda) f(x) + \lambda \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(t)dt \right| \\ \leq &\left| \frac{2\lambda^2 - 2\lambda + 1}{4}(b - a) + \frac{1}{b - a} \left(x - \frac{a + b}{2} \right)^2 \right| \max \{ |f'(a)|, |f'(b)| \}. \end{aligned}$$

It should be noticed that this inequality has a uniform bound independent of q, and if we take $\lambda = 1$ then $x = \frac{a+b}{2}$ which implies that

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x)dx \right| \leq \frac{b - a}{4} \max \{ |f'(a)|, |f'(b)| \}. \tag{1.6}$$

Now we give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used throughout this paper.

Definition 1. *(see [10])Let $f \in L_1[a, b]$. The Riemann-Liouville integrals $J_{a^+}^\alpha f$ and $J_{b^-}^\alpha f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J_{a^+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - t)^{\alpha - 1} f(t)dt, \quad x > a$$

and

$$J_{b-}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int\limits_{x}^{b}(t-x)^{\alpha-1}f(t)dt,\quad x<b$$

respectively where $\Gamma(\alpha)=\int\limits_0^\infty e^{-u}u^{\alpha-1}du$. *Here is* $J_{a+}^{0}f(x)=J_{b-}^{0}f(x)=f(x)$.

In the case of $\alpha=1$, the fractional integral reduces to the classical integral.

For some recent results connected with fractional integral inequalities see ([3]-[7]).

In [13], Sarıkaya *et al.* proved the following Lemma and established some inequalities for fractional integrals

Lemma 1. *Let* $f:[a,b]\to\mathbb{R}$*, be a differentiable mapping on* (a,b) *with* $a<b$*. If* $f'\in L[a,b]$*, then the following equality for fractional integrals holds:*

$$\begin{aligned}&\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2\left(b-a\right)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]\\=&\frac{b-a}{2}\int\limits_0^1\left[(1-t)^{\alpha}-t^{\alpha}\right]f'(ta+(1-t)b)dt.\end{aligned}$$

The aim of this paper is to establish Hadamard type inequalities for quasi-convex functions via Riemann-Liouville fractional integral.

2. MAIN RESULTS

Theorem 7. *Let* $f:[a,b]\to\mathbb{R}$*, be positive function with* $0\le a<b$ *and* $f\in L_1[a,b]$*. If* f *is a quasi-convex function on* $[a,b]$*, then the following inequality for fractional integrals hold:*

$$\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]\le\max\left\{f(a),f(b)\right\}$$

with $\alpha>0$.

Proof. Since f is quasi-convex function on $[a,b]$, we have

$$f\left(ta+(1-t)b\right)\le\max\left\{f(a),f(b)\right\}$$

and

$$f\left((1-t)a+tb\right)\le\max\left\{f(a),f(b)\right\}.$$

By adding these inequalities we get

$$\frac{1}{2}\left[f\left(ta+(1-t)b\right)+f\left((1-t)a+tb\right)\right]\le\max\left\{f(a),f(b)\right\}\tag{2.1}$$

Then multiplying both sides of (2.1) by $t^{\alpha-1}$ and integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$\begin{aligned}
&\int_0^1 t^{\alpha-1} f\left(ta+(1-t)b\right) dt + \int_0^1 t^{\alpha-1} f\left((1-t)a+tb\right) dt \\
&= \int_b^a \left(\frac{b-u}{b-a}\right)^{\alpha-1} f(u)\frac{du}{a-b} + \int_a^b \left(\frac{v-a}{b-a}\right)^{\alpha-1} f(v)\frac{dv}{b-a} \\
&\leq \frac{2}{\alpha}\max\left\{f(a),f(b)\right\},
\end{aligned}$$

i.e.

$$\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)\right] \leq \max\left\{f(a),f(b)\right\}.$$

The proof is complete. □

Remark 1. *If we choose $\alpha=1$ in Theorem 7 with properties of gamma functions, we have the inequality (1.2).*

Theorem 8. *Let $f:[a,b]\to\mathbb{R}$, be a dfferentiable mapping on (a,b) with $a<b$. If $|f'|$ is quasi-convex on $[a,b], \alpha>0$, then the following inequality for fractional integrals hold:*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2\left(b-a\right)^{\alpha}}\left[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)\right]\right| \\
&\leq \frac{b-a}{\alpha+1}\left(1-\frac{1}{2^{\alpha}}\right)\max\left\{\left|f'(a)\right|,\left|f'(b)\right|\right\}.
\end{aligned} \tag{2.2}$$

Proof. Using Lemma 1 and the quasi-convex of $|f'|$ with properties of modulus, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2\left(b-a\right)^{\alpha}}\left[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)\right]\right| \\
&\leq \frac{b-a}{2}\int_0^1 \left|(1-t)^{\alpha}-t^{\alpha}\right|\left|f'(ta+(1-t)b)\right| dt \\
&\leq \frac{b-a}{2}\int_0^1 \left|(1-t)^{\alpha}-t^{\alpha}\right|\max\left\{\left|f'(a)\right|,\left|f'(b)\right|\right\} dt \\
&= \frac{b-a}{2}\max\left\{\left|f'(a)\right|,\left|f'(b)\right|\right\}\left\{\int_0^{\frac{1}{2}}\left[(1-t)^{\alpha}-t^{\alpha}\right] dt + \int_{\frac{1}{2}}^1 \left[t^{\alpha}-(1-t)^{\alpha}\right] dt\right\} \\
&= \frac{b-a}{\alpha+1}\left(1-\frac{1}{2^{\alpha}}\right)\max\left\{\left|f'(a)\right|,\left|f'(b)\right|\right\}.
\end{aligned}$$

where we use the fact that

$$\int_0^1 \left|(1-t)^{\alpha}-t^{\alpha}\right| dt = \int_0^{\frac{1}{2}}\left[(1-t)^{\alpha}-t^{\alpha}\right] dt + \int_{\frac{1}{2}}^1 \left[t^{\alpha}-(1-t)^{\alpha}\right] dt = \frac{2}{\alpha+1}\left(1-\frac{1}{2^{\alpha}}\right)$$

which completes the proof. □

Remark 2. *If we choose $\alpha = 1$ in (2.2), then the inequality (2.2) reduce to the inequality (1.3) of Theorem 3.*

Theorem 9. *Let $f : [a, b] \to \mathbb{R}$, be a dfferentiable mapping on (a, b) with $a < b$ such that $f' \in L_1[a, b]$. If $|f'|^q$ is quasi-convex on $[a, b]$, and $p > 1$, then the following inequality for fractional integrals hold:*

$$\begin{aligned} &\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha+1)}{2\,(b-a)^{\alpha}} \left[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a) \right] \right| \\ &\leq \frac{b-a}{2(\alpha p+1)^{\frac{1}{p}}} \left(\max\left\{ |f'(a)|^q, |f'(b)|^q \right\} \right)^{\frac{1}{q}} \end{aligned} \tag{2.3}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $\alpha \in [0, 1]$.

Proof. From Lemma 1 and using Hölder inequality with properties of modulus, we have

$$\begin{aligned} &\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha+1)}{2\,(b-a)^{\alpha}} \left[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a) \right] \right| \\ &\leq \frac{b-a}{2} \int_0^1 \left| (1-t)^{\alpha} - t^{\alpha} \right| \left| f'(ta + (1-t)b) \right| dt \\ &\leq \frac{b-a}{2} \left(\int_0^1 \left| (1-t)^{\alpha} - t^{\alpha} \right|^p dt \right)^{\frac{1}{p}} \left(\int_0^1 \left| f'(ta + (1-t)b) \right|^q dt \right)^{\frac{1}{q}}. \end{aligned}$$

We know that for $\alpha \in [0, 1]$ and $\forall t_1, t_2 \in [0, 1]$,

$$\left| t_1^{\alpha} - t_2^{\alpha} \right| \leq \left| t_1 - t_2 \right|^{\alpha},$$

hence

$$\begin{aligned} \int_0^1 \left| (1-t)^{\alpha} - t^{\alpha} \right|^p dt &\leq \int_0^1 |1-2t|^{\alpha p} dt \\ &= \int_0^{\frac{1}{2}} [1-2t]^{\alpha p} dt + \int_{\frac{1}{2}}^1 [2t-1]^{\alpha p} dt \\ &= \frac{1}{\alpha p+1}. \end{aligned}$$

Since $|f'|^q$ is quasi-convex on $[a, b]$,we get

$$\begin{aligned} &\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha+1)}{2\,(b-a)^{\alpha}} \left[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a) \right] \right| \\ &\leq \frac{b-a}{2(\alpha p+1)^{\frac{1}{p}}} \left(\max\left\{ |f'(a)|^q, |f'(b)|^q \right\} \right)^{\frac{1}{q}} \end{aligned}$$

which completes the proof. □

Remark 3. *If in Theorem 9, we choose $\alpha = 1$, then the inequality (2.3) become the inequality (1.4) of Theorem 4.*

Theorem 10. *Let $f : [a,b] \to \mathbb{R}$, be a differentiable mapping on (a,b) with $a < b$ such that $f' \in L_1[a,b]$. If $|f'|^q$ is quasi-convex on $[a,b]$ and $q \geq 1$, then the following inequality for fractional integrals hold:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)\right]\right| \leq \frac{b-a}{\alpha+1}\left(1-\frac{1}{2^{\alpha}}\right)\left(\max\left\{|f'(a)|^q, |f'(b)|^q\right\}\right)^{\frac{1}{q}} \tag{2.4}$$

with $\alpha > 0$.

Proof. From Lemma 1 and using power-mean inequality with properties of modulus, we can write

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)\right]\right| \\ \leq\ & \frac{b-a}{2}\int_0^1 |(1-t)^{\alpha} - t^{\alpha}|\, |f'(ta+(1-t)b)|\, dt \\ \leq\ & \frac{b-a}{2}\left(\int_0^1 |(1-t)^{\alpha} - t^{\alpha}|\, dt\right)^{1-\frac{1}{q}}\left(\int_0^1 |(1-t)^{\alpha} - t^{\alpha}|\, |f'(ta+(1-t)b)|^q\, dt\right)^{\frac{1}{q}}. \end{aligned}$$

Since $|f'|^q$ is quasi-convex on $[a,b]$, we have

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)\right]\right| \\ \leq\ & \frac{b-a}{2}\left(\max\left\{|f'(a)|^q, |f'(b)|^q\right\}\right)^{\frac{1}{q}}\left(\int_0^1 |(1-t)^{\alpha} - t^{\alpha}|\, dt\right) \\ =\ & \frac{b-a}{2}\left(\max\left\{|f'(a)|^q, |f'(b)|^q\right\}\right)^{\frac{1}{q}}\left\{\int_0^{\frac{1}{2}} [(1-t)^{\alpha} - t^{\alpha}]\, dt + \int_{\frac{1}{2}}^1 [t^{\alpha} - (1-t)^{\alpha}]\, dt\right\} \\ =\ & \frac{b-a}{\alpha+1}\left(1-\frac{1}{2^{\alpha}}\right)\left(\max\left\{|f'(a)|^q, |f'(b)|^q\right\}\right)^{\frac{1}{q}}, \end{aligned}$$

which completes the proof. □

Remark 4. *If in Theorem 10, we choose $\alpha = 1$, then the inequality (2.4) become the inequality (1.5) of Theorem 5.*

3. APPLICATIONS TO SPECIAL MEANS

We now consider the means for arbitrary real numbers α, β $(\alpha \neq \beta)$. We take

(1) *Arithmetic mean* :

$$A(\alpha,\beta) = \frac{\alpha+\beta}{2}, \quad \alpha, \beta \in \mathbb{R}^+.$$

(2) *Logarithmic mean*:

$$L(\alpha,\beta) = \frac{\alpha-\beta}{\ln|\alpha| - \ln|\beta|}, \quad |\alpha| \neq |\beta|,\ \alpha,\beta \neq 0,\ \alpha,\beta \in \mathbb{R}^+.$$

(3) *Generalized* $log - mean$:

$$L_n(\alpha,\beta)=\left[\frac{\beta^{n+1}-\alpha^{n+1}}{(n+1)(\beta-\alpha)}\right]^{\frac{1}{n}}, \qquad n\in\mathbb{Z}\backslash\{-1,0\},\ \alpha,\beta\in\mathbb{R}^+.$$

Now using the results of Section 2, we give some applications for special means of real numbers.

Proposition 1. *Let* $a,b\in\mathbb{R}^+$*,* $a<b$*, and* $n\in\mathbb{Z}$*. Then, we have*

$$|A(a^n,b^n)-L_n^n(a,b)|\leq\frac{b-a}{4}\max\left\{|a|^n,|b|^n\right\}.$$

Proof. The assertion follows from Theorem 8 applied to the quasi-convex mapping $f(x)=x^n$, $x\in\mathbb{R}$, and $\alpha=1$. □

Proposition 2. *Let* $a,b\in\mathbb{R}^+$*,* $a<b$*, and* $n\in\mathbb{Z}$*. Then, for all* $q\geq1$*, we have*

$$|A(a^n,b^n)-L_n^n(a,b)|\leq\frac{b-a}{4}\left(\max\left\{(|a|^n)^q,(|b|^n)^q\right\}\right)^{\frac{1}{q}}.$$

Proof. The assertion follows from Theorem 10 applied to the m-convex mapping $f(x)=x^n$, $x\in\mathbb{R}$, and $\alpha=1$. □

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