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Three error bounds for the Simpson quadrature rule

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Abstract. Three new error bounds are presented for the well-known Simpson quadrature rule in $L^1[a,b]$ and $L^\infty[a,b]$ spaces.

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1. Introduction

A general ($n+1$)-point weighted quadrature formula is denoted by

$$\int_a^b w(x)\, f(x)\, dx = \sum_{k=0}^{n} w_k\, f(x_k) + R_{n+1}[f], \tag{1}$$

where $w(x)$ is a positive weight function on $[a,b]$, $\{x_k\}_{k=0}^n$ and $\{w_k\}_{k=0}^n$ are respectively nodes and weight coefficients and $R_{n+1}[f]$ is the corresponding error [7].

Let $\mathbf{\Pi}_d$ be the set of algebraic polynomials of degree at most *d*. The quadrature formula (1) has degree of exactness *d* if for every $p \in \mathbf{\Pi}_d$ we have $R_{n+1}[p] = 0$. In addition, if $R_{n+1}[p] \neq 0$ for some $\mathbf{\Pi}_{d+1}$, formula (1) has precise degree of exactness *d*.

The convergence order of quadrature rule (1) depends on the smoothness of the function f as well as on its degree of exactness. It is well known that for given $n+1$ mutually different nodes $\{x_k\}_{k=0}^n$ we can always achieve a degree of exactness $d = n$ by interpolating at these nodes and integrating the interpolated polynomial instead of f. Namely, taking the node polynomial

$$\Psi_{n+1}(x)=\prod_{k=0}^{n}(x-x_k)\,,$$

by integrating the Lagrange interpolation formula

$$f(x)=\sum_{k=0}^{n} f(x_k)\,L(x;x_k)+r_{n+1}(f;x)\,,$$

where

$$L(x;x_k)=\frac{\Psi_{n+1}(x)}{\Psi'_{n+1}(x_k)(x-x_k)}\qquad (k=0,1,\ldots,n)\,,$$

we obtain (1), with

$$w_k=\frac{1}{\Psi'_{n+1}(x_k)}\int_a^b\frac{\Psi_{n+1}(x)\,w(x)}{x-x_k}\,dx\qquad (k=0,1,\ldots,n)\,,$$

and

$$R_{n+1}[f]=\int_a^b r_{n+1}(f;x)\,w(x)\,dx\,.$$

Note that for each $f\in\mathbf{\Pi}_n$ we have $r_{n+1}(f;x)=0$ and therefore $R_{n+1}[f]=0$.
Quadrature formulae obtained in this way are known as interpolatory. Usually the simplest interpolatory quadrature formula of type (1) with pre-determined nodes $\{x_k\}_{k=0}^{n}\in[a,b]$ is called a weighted Newton-Cotes formula. For $w(x)=1$ and the equidistant nodes $\{x_k\}_{k=0}^{n}=\{a+kh\}_{k=0}^{n}$ with $h=(b-a)/n$, the classical Newton-Cotes formulas are derived. One of the important cases of the classical Newton-Cotes formulas is the well-known Simpson's rule

$$\int_a^b f(t)\,dt=\frac{b-a}{6}\left(f(a)+4f(\frac{a+b}{2})+f(b)\right)+E(f)\,. \tag{2}$$

In this way, Simpson inequality [1-6] gives an error bound for the above quadrature rule. There are few known ways to estimate the residue value in (2). The main aim of this paper is to give three new estimations for $E(f)$ in $L^1[a,b]$ and $L^\infty[a,b]$ spaces. For this purpose, we should first consider some notations and inequalities.
Let $L^p[a,b]$ $(1\le p<\infty)$ denote the space of p-power integrable functions on the interval $[a,b]$ with the standard norm

$$\|f\|_p=\left(\int_a^b|f(t)|^p\,dt\right)^{1/p},$$

and $L^{\infty}[a,b]$ the space of all essentially bounded functions on $[a,b]$ with the norm

$$\| f\|_{\infty} = ess \sup_{x\in[a,b]} | f(x) | .$$

If $f \in L^1[a,b]$ and $g \in L^{\infty}[a,b]$, the following inequality is well known

$$\left| \int_a^b f(x)\, g(x)\, dx \right| \le \| f \|_1 \| g \|_{\infty} .$$

Recently in [8], a main inequality has been introduced that can estimate the error of Simpson quadrature rule. In other words we have:

Theorem A. *Let* $f : \mathbf{I} \to \mathbf{R}$, *where* $\mathbf{I}$ *is an interval, be a differentiable function in the interior* $\mathbf{I}^0$ *of* $\mathbf{I}$, *and let* $[a,b] \subset \mathbf{I}^0$. *If* α_0, β_0 *are two real constants such that* $\alpha_0 \le f'(t) \le \beta_0$ *for all* $t \in [a,b]$, *then for any* $\lambda \in [1/2,1]$ *and all* $x \in [\frac{a+(2\lambda-1)b}{2\lambda}, \frac{b+(2\lambda-1)a}{2\lambda}] \subseteq [a,b]$ *we have*

$$\begin{aligned} &\left| f(x) - \frac{1}{\lambda(b-a)} \int_a^b f(t)\, dt - \frac{f(b)-f(a)}{b-a} x + \frac{(2\lambda-1)a+b}{2\lambda(b-a)} f(b) - \frac{a+(2\lambda-1)b}{2\lambda(b-a)} f(a) \right| \\ &\le \frac{\beta_0-\alpha_0}{4(b-a)} \frac{\lambda^2+(1-\lambda)^2}{\lambda} \left((x-a)^2 + (b-x)^2 \right). \end{aligned} \tag{3}$$

As we observe, replacing $x = (a+b)/2$ and $\lambda = 2/3$ in (3) gives an error bound for the Simpson rule as

$$\left| \int_a^b f(t)\, dt - \frac{b-a}{6} \left(f(a) + 4 f(\frac{a+b}{2}) + f(b) \right) \right| \le \frac{5}{72} (b-a)^2 (\beta_0 - \alpha_0).$$

To introduce three new error bounds for the Simpson quadrature rule in $L^1[a,b]$ and $L^{\infty}[a,b]$ spaces we first consider the following kernel on $[a,b]$:

$$K(t) = \begin{cases} t - \frac{5a+b}{6} & t \in [a, \frac{a+b}{2}], \\ t - \frac{a+5b}{6} & t \in (\frac{a+b}{2}, b]. \end{cases} \tag{4}$$

After some calculations, it can be directly concluded that

$$\int_a^b f'(t)\,K(t)\,dt=\frac{b-a}{6}\left(f(a)+4f(\frac{a+b}{2})+f(b)\right)-\int_a^b f(t)\,dt, \tag{5}$$

and

$$\max_{t\in[a,b]}\left|K(t)\right|=\frac{1}{3}(b-a).$$

2. Main Results

Theorem 1. *Let* $f:\mathbf{I}\to\mathbf{R}$ *, where* $\mathbf{I}$ *is an interval, be a function differentiable in the interior* $\mathbf{I}^0$ *of* $\mathbf{I}$ *, and let* $[a,b]\subset\mathbf{I}^0$ *. If* $\alpha(x)\le f'(x)\le\beta(x)$ *for any* $\alpha,\beta\in C[a,b]$ *and* $x\in[a,b]$ *then the following inequality holds*

$$\begin{aligned}
m_1&=\int_a^{\frac{5a+b}{6}}(t-\frac{5a+b}{6})\,\beta(t)\,dt+\int_{\frac{5a+b}{6}}^{\frac{a+b}{2}}(t-\frac{5a+b}{6})\,\alpha(t)\,dt\\
&\quad+\int_{\frac{a+b}{2}}^{\frac{a+5b}{6}}(t-\frac{a+5b}{6})\,\beta(t)\,dt+\int_{\frac{a+5b}{6}}^{b}(t-\frac{a+5b}{6})\,\alpha(t)\,dt\\
&\le\frac{b-a}{6}\left(f(a)+4f(\frac{a+b}{2})+f(b)\right)-\int_a^b f(t)\,dt\le \qquad (6)\\
M_1&=\int_a^{\frac{5a+b}{6}}(t-\frac{5a+b}{6})\,\alpha(t)\,dt+\int_{\frac{5a+b}{6}}^{\frac{a+b}{2}}(t-\frac{5a+b}{6})\,\beta(t)\,dt\\
&\quad+\int_{\frac{a+b}{2}}^{\frac{a+5b}{6}}(t-\frac{a+5b}{6})\,\alpha(t)\,dt+\int_{\frac{a+5b}{6}}^{b}(t-\frac{a+5b}{6})\,\beta(t)\,dt.
\end{aligned}$$

Proof. By referring to the kernel (4) and identity (5) we first have

$$\begin{aligned}
&\int_a^b K(t)\left(f'(t)-\frac{\alpha(t)+\beta(t)}{2}\right)dt\\
&=\frac{b-a}{6}\left(f(a)+4f(\frac{a+b}{2})+f(b)\right)-\int_a^b f(t)\,dt-\frac{1}{2}\left(\int_a^b K(t)(\alpha(t)+\beta(t))\,dt\right)\\
&=\frac{b-a}{6}\left(f(a)+4f(\frac{a+b}{2})+f(b)\right)-\int_a^b f(t)\,dt \qquad (7)\\
&-\frac{1}{2}\left(\int_a^{\frac{a+b}{2}}(t-\frac{5a+b}{6})(\alpha(t)+\beta(t))\,dt+\int_{\frac{a+b}{2}}^{b}(t-\frac{a+5b}{6})(\alpha(t)+\beta(t))\,dt\right).
\end{aligned}$$

On the other hand, the given assumption $\alpha(t)\le f'(t)\le\beta(t)$ results in

$$\left| f'(t)-\frac{\alpha(t)+\beta(t)}{2}\right| \le \frac{\beta(t)-\alpha(t)}{2}. \tag{8}$$

Therefore, one can conclude from (7) and (8) that

$$\begin{aligned}
&\Bigg| \frac{b-a}{6}\left(f(a)+4f(\frac{a+b}{2})+f(b)\right)-\int_a^b f(t)\,dt \\
&-\frac{1}{2}\left(\int_a^{\frac{a+b}{2}}(t-\frac{5a+b}{6})(\alpha(t)+\beta(t))\,dt+\int_{\frac{a+b}{2}}^b (t-\frac{a+5b}{6})(\alpha(t)+\beta(t))\,dt\right)\Bigg| \\
&=\left|\int_a^b K(t)\left(f'(t)-\frac{\alpha(t)+\beta(t)}{2}\right)dt\right| \le \int_a^b |K(t)|\frac{\beta(t)-\alpha(t)}{2}\,dt \\
&=\frac{1}{2}\left(\int_a^{\frac{a+b}{2}}\left|t-\frac{5a+b}{6}\right|(\beta(t)-\alpha(t))\,dt+\int_{\frac{a+b}{2}}^b\left|t-\frac{a+5b}{6}\right|(\beta(t)-\alpha(t))\,dt\right).
\end{aligned} \tag{9}$$

After re-arranging (9) we obtain

$$\begin{aligned}
m_1 &= \int_a^{\frac{a+b}{2}}\left(\left(t-\frac{5a+b}{6}-\left|t-\frac{5a+b}{6}\right|\right)\frac{\beta(t)}{2}+\left(t-\frac{5a+b}{6}+\left|t-\frac{5a+b}{6}\right|\right)\frac{\alpha(t)}{2}\right)dt \\
&+\int_{\frac{a+b}{2}}^b\left(\left(t-\frac{a+5b}{6}-\left|t-\frac{a+5b}{6}\right|\right)\frac{\beta(t)}{2}+\left(t-\frac{a+5b}{6}+\left|t-\frac{a+5b}{6}\right|\right)\frac{\alpha(t)}{2}\right)dt \\
&=\int_a^{\frac{5a+b}{6}}(x-\frac{5a+b}{6})\,\beta(x)\,dx+\int_{\frac{5a+b}{6}}^{\frac{a+b}{2}}(x-\frac{5a+b}{6})\,\alpha(x)\,dx \\
&+\int_{\frac{a+b}{2}}^{\frac{a+5b}{6}}(x-\frac{a+5b}{6})\,\beta(x)\,dx+\int_{\frac{a+5b}{6}}^b(x-\frac{a+5b}{6})\,\alpha(x)\,dx,
\end{aligned}$$

and

$$\begin{aligned}
M_1 &= \int_a^{\frac{a+b}{2}}\left(\left(t-\frac{5a+b}{6}-\left|t-\frac{5a+b}{6}\right|\right)\frac{\alpha(t)}{2}+\left(t-\frac{5a+b}{6}+\left|t-\frac{5a+b}{6}\right|\right)\frac{\beta(t)}{2}\right)dt \\
&+\int_{\frac{a+b}{2}}^b\left(\left(t-\frac{a+5b}{6}-\left|t-\frac{a+5b}{6}\right|\right)\frac{\alpha(t)}{2}+\left(t-\frac{a+5b}{6}+\left|t-\frac{a+5b}{6}\right|\right)\frac{\beta(t)}{2}\right)dt \\
&=\int_a^{\frac{5a+b}{6}}(x-\frac{5a+b}{6})\,\alpha(x)\,dx+\int_{\frac{5a+b}{6}}^{\frac{a+b}{2}}(x-\frac{5a+b}{6})\,\beta(x)\,dx \\
&+\int_{\frac{a+b}{2}}^{\frac{a+5b}{6}}(x-\frac{a+5b}{6})\,\alpha(x)\,dx+\int_{\frac{a+5b}{6}}^b(x-\frac{a+5b}{6})\,\beta(x)\,dx.
\end{aligned}$$

■

The advantage of theorem 1 is that necessary computations in bounds m_1 and M_1 are just in terms of the pre-assigned functions $\alpha(t),\beta(t)$ (not f').

Special case 1. Substituting $\alpha(x)=\alpha_1 x+\alpha_0 \neq 0$ and $\beta(x)=\beta_1 x+\beta_0 \neq 0$ in (6) gives

$$\left|\int_a^b f(t)\,dt-\frac{b-a}{6}\left(f(a)+4f(\frac{a+b}{2})+f(b)\right)\right| \leq \frac{5(b-a)^2}{144}\left((\beta_1-\alpha_1)(a+b)+2(\beta_0-\alpha_0)\right).$$

In particular, replacing $\alpha_1=\beta_1=0$ in above inequality leads to one of the results of [9] as

$$\left|\int_a^b f(t)\,dt-\frac{b-a}{6}\left(f(a)+4f(\frac{a+b}{2})+f(b)\right)\right| \leq \frac{5}{72}(b-a)^2(\beta_0-\alpha_0).$$

Remark 1. Although $\alpha(x)\leq f'(x)\leq \beta(x)$ is a straightforward condition in theorem 1, however sometimes one might not be able to easily obtain both bounds of $\alpha(x)$ and $\beta(x)$ for f'. In this case, we can make use of two analogue theorems. The first one would be helpful when f' is unbounded from above and the second one would be helpful when f' is unbounded from below.

Theorem 2. *Let* $f:\mathbf{I}\to\mathbf{R}$*, where* $\mathbf{I}$ *is an interval, be a function differentiable in the interior* $\mathbf{I}^0$ *of* $\mathbf{I}$*, and let* $[a,b]\subset\mathbf{I}^0$*. If* $\alpha(x)\leq f'(x)$ *for any* $\alpha\in C[a,b]$ *and* $x\in[a,b]$ *then*

$$\begin{aligned}
&\int_a^{\frac{a+b}{2}}(t-\frac{5a+b}{6})\alpha(t)\,dt+\int_{\frac{a+b}{2}}^b(t-\frac{a+5b}{6})\alpha(t)\,dt-\frac{b-a}{3}\left(f(b)-f(a)-\int_a^b\alpha(t)\,dt\right)\\
&\qquad\leq \frac{b-a}{6}\left(f(a)+4f(\frac{a+b}{2})+f(b)\right)-\int_a^b f(t)\,dt\leq \qquad (10)\\
&\int_a^{\frac{a+b}{2}}(t-\frac{5a+b}{6})\alpha(t)\,dt+\int_{\frac{a+b}{2}}^b(t-\frac{a+5b}{6})\alpha(t)\,dt+\frac{b-a}{3}\left(f(b)-f(a)-\int_a^b\alpha(t)\,dt\right).
\end{aligned}$$

Proof. Since

$$\begin{aligned}
\int_a^b K(t)\left(f'(t)-\alpha(t)\right)dt &= \frac{b-a}{6}\left(f(a)+4f(\frac{a+b}{2})+f(b)\right)-\int_a^b f(t)\,dt-\left(\int_a^b K(t)\alpha(t)\,dt\right)\\
&= \frac{b-a}{6}\left(f(a)+4f(\frac{a+b}{2})+f(b)\right)-\int_a^b f(t)\,dt\\
&\qquad -\left(\int_a^{\frac{a+b}{2}}(t-\frac{5a+b}{6})\alpha(t)\,dt+\int_{\frac{a+b}{2}}^b(t-\frac{a+5b}{6})\alpha(t)\,dt\right),
\end{aligned}$$

so we have

$$\left|\frac{b-a}{6}\left(f(a)+4f(\frac{a+b}{2})+f(b)\right)-\int_a^b f(t)\,dt\right.$$
$$\left.-\left(\int_a^{\frac{a+b}{2}}(t-\frac{5a+b}{6})\alpha(t)\,dt+\int_{\frac{a+b}{2}}^b(t-\frac{a+5b}{6})\alpha(t)\,dt\right)\right| \tag{11}$$
$$=\left|\int_a^b K(t)\left(f'(t)-\alpha(t)\right)dt\right|\le\int_a^b\left|K(t)\right|\left(f'(t)-\alpha(t)\right)dt$$
$$\le\max_{t\in[a,b]}\left|K(t)\right|\int_a^b\left(f'(t)-\alpha(t)\right)dt=\frac{b-a}{3}\left(f(b)-f(a)-\int_a^b\alpha(t)\,dt\right).$$

After re-arranging (11), the main inequality (10) will be derived. ■

Special case 2. If $\alpha(x)=\alpha_1 x+\alpha_0\neq 0$ then (10) becomes

$$\left|\int_a^b f(t)\,dt-\frac{b-a}{6}\left(f(a)+4f(\frac{a+b}{2})+f(b)\right)\right|\le\frac{(b-a)^2}{3}\left(\frac{f(b)-f(a)}{b-a}-\left(\alpha_0+\frac{a+b}{2}\alpha_1\right)\right),$$

if and only if $\alpha_1 x+\alpha_0\le f'(x)\quad\forall x\in[a,b]$. In particular, replacing $\alpha_1=0$ in above inequality leads to theorem 1, relation (4) of [10] as

$$\left|\int_a^b f(t)\,dt-\frac{b-a}{6}\left(f(a)+4f(\frac{a+b}{2})+f(b)\right)\right|\le\frac{(b-a)^2}{3}\left(\frac{f(b)-f(a)}{b-a}-\alpha_0\right).$$

1.3. Theorem 3. *Let* $f:\mathbf{I}\to\mathbf{R}$, *where* $\mathbf{I}$ *is an interval, be a function differentiable in the interior* $\mathbf{I}^0$ *of* $\mathbf{I}$, *and let* $[a,b]\subset\mathbf{I}^0$. *If* $f'(x)\le\beta(x)$ *for any* $\beta\in C[a,b]$ *and* $x\in[a,b]$ *then*

$$\int_a^{\frac{a+b}{2}}(t-\frac{5a+b}{6})\beta(t)\,dt+\int_{\frac{a+b}{2}}^b(t-\frac{a+5b}{6})\beta(t)\,dt-\frac{b-a}{3}\left(\int_a^b\beta(t)\,dt-f(b)+f(a)\right)$$
$$\le\frac{b-a}{6}\left(f(a)+4f(\frac{a+b}{2})+f(b)\right)-\int_a^b f(t)\,dt\le \tag{12}$$
$$\int_a^{\frac{a+b}{2}}(t-\frac{5a+b}{6})\beta(t)\,dt+\int_{\frac{a+b}{2}}^b(t-\frac{a+5b}{6})\beta(t)\,dt+\frac{b-a}{3}\left(\int_a^b\beta(t)\,dt-f(b)+f(a)\right).$$

Proof. Since

$$\int_a^b K(t)\left(f'(t)-\beta(t)\right)dt$$

$$=\frac{b-a}{6}\left(f(a)+4f(\frac{a+b}{2})+f(b)\right)-\int_a^b f(t)\,dt-\left(\int_a^b K(t)\,\beta(t)\,dt\right)$$

$$=\frac{b-a}{6}\left(f(a)+4f(\frac{a+b}{2})+f(b)\right)-\int_a^b f(t)\,dt$$

$$-\left(\int_a^{\frac{a+b}{2}}(t-\frac{5a+b}{6})\,\beta(t)\,dt+\int_{\frac{a+b}{2}}^b(t-\frac{a+5b}{6})\,\beta(t)\,dt\right),$$

so we have

$$\left|\frac{b-a}{6}\left(f(a)+4f(\frac{a+b}{2})+f(b)\right)-\int_a^b f(t)\,dt\right.$$

$$\left.-\left(\int_a^{\frac{a+b}{2}}(t-\frac{5a+b}{6})\,\beta(t)\,dt+\int_{\frac{a+b}{2}}^b(t-\frac{a+5b}{6})\,\beta(t)\,dt\right)\right| \tag{13}$$

$$=\left|\int_a^b K(t)\left(f'(t)-\beta(t)\right)dt\right|\le\int_a^b\left|K(t)\right|\left(\beta(t)-f'(t)\right)dt$$

$$\le\max_{t\in[a,b]}\left|K(t)\right|\int_a^b\left(\beta(t)-f'(t)\right)dt=\frac{b-a}{3}\left(\int_a^b\beta(t)\,dt-f(b)+f(a)\right).$$

After re-arranging (13), the main inequality (12) will be derived. ■

Special case 3. If $\beta(x)=\beta_1 x+\beta_0\neq 0$ in (12), then

$$\left|\int_a^b f(t)\,dt-\frac{b-a}{6}\left(f(a)+4f(\frac{a+b}{2})+f(b)\right)\right|\le\frac{(b-a)^2}{3}\left(\beta_0+\frac{a+b}{2}\beta_1-\frac{f(b)-f(a)}{b-a}\right).$$

if and only if $f'(x)\le\beta_1 x+\beta_0\quad\forall x\in[a,b]$. In particular, replacing $\beta_1=0$ in above inequality leads to theorem 1, relation (5) of [10] as

$$\left|\int_a^b f(t)\,dt-\frac{b-a}{6}\left(f(a)+4f(\frac{a+b}{2})+f(b)\right)\right|\le\frac{(b-a)^2}{3}\left(\beta_0-\frac{f(b)-f(a)}{b-a}\right).$$

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