Received 19/01/11

Power Series Inequalities Via a Refinement of Schwarz's Inequality

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Abstract. In this paper, we obtain some inequalities for functions defined by power series with nonnegative coefficients. In order to obtain these inequalities, a refinement of Schwarz inequality in inner product spaces is utilized. Natural applications for some elementary functions of interest are also provided.

1. Introduction

The famous Cauchy-Bunyakovsky-Schwarz (CBS)-inequality in the complex case [**9**] states that

$$\left|\sum_{j=1}^{n} x_j y_j\right|^2 \le \sum_{j=1}^{n} |x_j|^2 \sum_{j=1}^{n} |y_j|^2 \tag{1.1}$$

for $x_j, y_j \in \mathbb{C}$, $j \in \{1, 2, ..., n\}$ with equality holding in (1.1) if and only if the sequences $\{x_j\}, j \in \{1, 2, ..., n\}$ and $\{y_j\}, j \in \{1, 2, ..., n\}$ are proportional. This result is also called the Cauchy inequality, the Schwarz inequality or the Cauchy-Schwarz inequality.

If we consider an analytic function defined by a power series, $f(z) := \sum_{n=0}^{\infty} a_n z^n$ with complex coefficients a_n and convergent on the open disk $D(0, R) \subset \mathbb{C}, R > 0$ and apply the inequality (1.1), then we can deduce that

$$|f(z)|^2 \le \sum_{n=0}^{\infty} |a_n|^2 \sum_{n=0}^{\infty} |z|^{2n} = \frac{1}{1 - |z|^2} \sum_{n=0}^{\infty} |a_n|^2, \tag{1.2}$$

for any $z \in D(0, R) \cap D(0, 1)$. If we assume that the coefficients in the representation $f(z) := \sum_{n=0}^{\infty} a_n z^n$ are nonnegative, and on utilizing the weighted version of the (CBS)-inequality, namely

$$\left|\sum_{j=1}^{n} p_j x_j y_j\right|^2 \le \sum_{j=1}^{n} p_j |x_j|^2 \sum_{j=1}^{n} p_j |y_j|^2, \tag{1.3}$$

2000 *Mathematics Subject Classification.* 26D15; 33C05; 33C20.

Key words and phrases. Power series, Schwarz inequality, Complex functions, Polylogarithm.

This paper is in final form and no version of it will be submitted for publication elsewhere.

where $p_j \geq 0$, while $x_j, y_j \in \mathbb{C}$, $j \in \{1, 2, ..., n\}$, we can state that

$$|f(zw)|^2 \leq \sum_{n=0}^{\infty} a_n |z|^{2n} \sum_{n=0}^{\infty} a_n |w|^{2n} = f\left(|z|^2\right) f\left(|w|^2\right), \tag{1.4}$$

for any $z, w \in C$ with $zw, |z|^2, |w|^2 \in D(0, R)$.

Some refinements of the above inequality (1.4) concerning power series can be found in the literature, see [**1**], [**2**], [**5**], [**6**], [**7**], [**8**]. In particular, on utilizing the de Bruijn inequality [**10**] for power series, Cerone and Dragomir [**2**] obtained that

$$|f(az)|^2 \leq \frac{1}{2} f(a^2) \left[f\left(|z|^2\right) + \left|f(z^2)\right| \right], \tag{1.5}$$

where a is a real number and z a complex number such that $az, a^2, z^2, |z|^2 \in D(0, R)$. They also applied this inequality for various examples of fundamental complex functions including the exponential, trigonometric, hyperbolic, hypergeometric and polylogarithmic function.

Other refinement of (1.5) by using the Buzano inequality [**3**] in inner product spaces was provided by Ibrahim and Dragomir [**7**], namely

$$\left|f(\alpha\overline{x}) f(\overline{\beta}x)\right| \leq \frac{1}{2} \left(\left[f(|\alpha|^2) f\left(|\beta|^2\right) \right]^{1/2} + \left| f\left(\alpha\overline{\beta}\right) \right| \right) f\left(|x|^2\right), \tag{1.6}$$

for $\alpha, \beta, x \in \mathbb{C}$ with $\alpha\overline{x}, \overline{\beta}x, |\alpha|^2, |\beta|^2, \alpha\overline{\beta}, |x|^2 \in D(0, R)$.

In this paper, we present further generalizations of the above inequality (1.6) related to the Schwarz result in inner product spaces. Applications for some fundamental complex functions are also provided.

2. Some Results Via a Refinement of Schwarz's Inequality

In 1985, S.S. Dragomir [**4**] has obtained the following refinement of the Schwarz inequality in inner product space $(H; \langle \cdot, \cdot \rangle)$ over the real and complex number field $\mathbb{K}$:

$$\|x\| \|y\| \geq |\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle| + |\langle x, e\rangle \langle e, y\rangle| \geq |\langle x, y\rangle| \tag{2.1}$$

where $e \in H, \|e\| = 1$.

If in the first inequality (2.1), we choose $e = \frac{z}{\|z\|}$, $z \in H \setminus \{0\}$, then we get

$$\|x\| \|y\| \|z\|^2 - |\langle x, z\rangle \langle z, y\rangle| \geq \left| \langle x, y\rangle \|z\|^2 - \langle x, z\rangle \langle z, y\rangle \right| \tag{2.2}$$

for any $x, y, z \in H$.

If we write the inequality (2.2) for the particular inner product space $(\mathbb{K}^n; \langle \cdot, \cdot \rangle)$, where

$$\langle x, y\rangle_p = \sum_{j=1}^{n} p_j x_j \overline{y_j}$$

for $x=(x_1,x_2,...,x_n)$, $y=(y_1,y_2,...,y_n)\in\mathbb{K}^n$ and $p=(p_1,p_2,...,p_n)$ with $p_j\geq 0$, $j\in\{1,2,...,n\}$ then we get the discrete inequality

$$\left(\sum_{j=1}^{n}p_j|x_j|^2\right)^{1/2}\left(\sum_{j=1}^{n}p_j|y_j|^2\right)^{1/2}\sum_{j=1}^{n}p_j|z_j|^2-\left|\sum_{j=1}^{n}p_jx_j\overline{z_j}\sum_{j=1}^{n}p_jz_j\overline{y_j}\right| \geq\left|\sum_{j=1}^{n}p_jx_j\overline{y_j}\sum_{j=1}^{n}p_j|z_j|^2-\sum_{j=1}^{n}p_jx_j\overline{z_j}\sum_{j=1}^{n}p_jz_j\overline{y_j}\right|, \tag{2.3}$$

where $p_j\geq 0$, $x_j,y_j,z_j\in\mathbb{K}$, $j\in\{1,2,...,n\}$.

In particular, if we take in (2.3) $y_j=\overline{x_j}$ for $j\in\{1,2,...,n\}$, then we obtain

$$\sum_{j=1}^{n}p_j|x_j|^2\sum_{j=1}^{n}p_j|z_j|^2-\left|\sum_{j=1}^{n}p_jx_j\overline{z_j}\sum_{j=1}^{n}p_jx_jz_j\right| \geq\left|\sum_{j=1}^{n}p_jx_j^2\sum_{j=1}^{n}p_j|z_j|^2-\sum_{j=1}^{n}p_jx_j\overline{z_j}\sum_{j=1}^{n}p_jx_jz_j\right|, \tag{2.4}$$

for $p_j\geq 0$, $x_j,z_j\in\mathbb{K}$, $j\in\{1,2,...,n\}$.

On applying the inequality (2.3) for power series, we can deduce the following result.

THEOREM 1. *Let $f(z)=\sum_{n=0}^{\infty}a_nz^n$ be a power series with nonnegative coefficients a_n and convergent on the open disk $D(0,R)$. If $x,y,z\in\mathbb{C}$, so that $|x|^2,|y|^2,|z|^2$, $x\overline{z}$, $z\overline{y}$, $x\overline{y}\in D(0,R)$, then*

$$\left[f\left(|x|^2\right)f\left(|y|^2\right)\right]^{1/2}f\left(|z|^2\right)-|f(x\overline{z})f(z\overline{y})| \geq\left|f(x\overline{y})f\left(|z|^2\right)-f(x\overline{z})f(z\overline{y})\right|. \tag{2.5}$$

PROOF. If we choose $p_n=a_n$, $x_n=x^n$, $y_n=y^n$, $z_n=z^n$, $n\in\{0,1,2,...,m\}$ in (2.3), then we have

$$\left[\sum_{n=0}^{m}a_n\left(|x|^2\right)^n\right]^{1/2}\left[\sum_{n=0}^{m}a_n\left(|y|^2\right)^n\right]^{1/2}\sum_{n=0}^{m}a_n\left(|z|^2\right)^n -\left|\sum_{n=0}^{m}a_n(x\overline{z})^n\sum_{n=0}^{m}a_n(z\overline{y})^n\right| \geq\left|\sum_{n=0}^{m}a_n(x\overline{y})^n\sum_{n=0}^{m}a_n\left(|z|^2\right)^n-\sum_{n=0}^{m}a_n(x\overline{z})^n\sum_{n=0}^{m}a_n(x\overline{y})^n\right|. \tag{2.6}$$

Since $|x|^2$, $|y|^2$, $|z|^2$, $x\overline{z}$, $z\overline{y}$, $x\overline{y}$ belong to the convergence disk $D(0,R)$ and taking the limit as $m\to\infty$ in (2.6), we deduce the desired result (2.5). □

Some examples for particular functions that are generated by power series with nonnegative coefficients are as follows:

1. If we choose in the above inequality (2.5) $f(z)=\frac{1}{1-z}$, $z\in D(0,1)$, then we have

$$\frac{\left|(1-x\overline{z})(1-z\overline{y})\right|}{\left[\left(1-|x|^2\right)\left(1-|y|^2\right)\right]^{1/2}\left(1-|z|^2\right)}-1 \geq \left|\frac{(1-x\overline{z})(1-z\overline{y})}{(1-x\overline{y})\left(1-|z|^2\right)}-1\right|, \tag{2.7}$$

for any $x,y,z\in D(0,1)$. In particular for $y=\overline{x}$ in (2.7) we get

$$\frac{\left|(1-xz)(1-x\overline{z})\right|}{\left(1-|x|^2\right)\left(1-|z|^2\right)}-1 \geq \left|\frac{(1-xz)(1-x\overline{z})}{(1-x^2)\left(1-|z|^2\right)}-1\right| \tag{2.8}$$

for any $x,z\in D(0,1)$. Also for $z=\overline{x}$ in (2.7) we get

$$\frac{\left|\left(1-x^2\right)(1-\overline{x}\overline{y})\right|}{\left(1-|x|^2\right)^{3/2}\left(1-|y|^2\right)^{1/2}}-1 \geq \left|\frac{\left(1-x^2\right)(1-\overline{x}\overline{y})}{(1-x\overline{y})\left(1-|x|^2\right)}-1\right|, \tag{2.9}$$

for any $x,y\in D(0,1)$. If $z=a\in\mathbb{R}$ and $x,y\in\mathbb{C}$, then from (2.7),

$$\frac{\left|(1-ax)(1-a\overline{y})\right|}{\left[\left(1-|x|^2\right)\left(1-|y|^2\right)\right]^{1/2}(1-a^2)}-1 \geq \left|\frac{(1-ax)(1-a\overline{y})}{(1-x\overline{y})(1-a^2)}-1\right|, \tag{2.10}$$

for any $x,y\in D(0,1)$ and $a\in(-1,1)$.

2. If we apply (2.5) for $f(z)=\exp(z)$, $z\in\mathbb{C}$, then we get

$$\exp\left(\frac{|x|^2+|y|^2}{2}+|z|^2\right)-\left|\exp\left(x\overline{z}+z\overline{y}\right)\right| \geq \left|\exp\left(x\overline{y}+|z|^2\right)-\exp\left(x\overline{z}+z\overline{y}\right)\right|, \tag{2.11}$$

for any $x,y,z\in\mathbb{C}$. In particular for $y=\overline{x}$ in (2.11) we have

$$\exp\left(|x|^2+|z|^2\right)-\left|\exp\left[2x\operatorname{Re}(z)\right]\right| \geq \left|\exp\left(x^2+|z|^2\right)-\exp\left[2x\operatorname{Re}(z)\right]\right|,$$

for any $x,z\in\mathbb{C}$. Also for $z=\overline{x}$ in (2.11) we get

$$\exp\left(\frac{3|x|^2+|y|^2}{2}\right)-\left|\exp\left(x^2+\overline{x}\overline{y}\right)\right| \geq \left|\exp\left(x\overline{y}+|x|^2\right)-\exp\left(x^2+\overline{x}\overline{y}\right)\right|,$$

for any $x, y \in \mathbb{C}$. If $z = a \in \mathbb{R}$ and $x, y \in \mathbb{C}$, then from (2.11),

$$\exp\left(\frac{|x|^2 + |y|^2}{2} + a^2\right) - |\exp[a(x + \overline{y})]| \geq \left|\exp\left(x\overline{y} + a^2\right) - \exp[a(x + \overline{y})]\right|,$$

for any $x, y \in \mathbb{C}$ and $a \in \mathbb{R}$.

3. For the Koebe function $f(z) = \frac{z}{(1-z)^2}$, $z \in D(0,1)$, we get the following inequality,

$$\frac{1}{\left(1 - |x|^2\right)\left(1 - |y|^2\right)\left(1 - |z|^2\right)^2} - \frac{1}{\left|[(1 - x\overline{z})(1 - z\overline{y})]^2\right|} \geq \left|\frac{1}{\left[(1 - x\overline{y})\left(1 - |z|^2\right)\right]^2} - \frac{1}{[(1 - x\overline{z})(1 - z\overline{y})]^2}\right|, \tag{2.12}$$

for any $x, y, z \in D(0,1)$. In particular, for $y = \overline{x}$ in (2.12) we get

$$\frac{1}{\left[\left(1 - |x|^2\right)\left(1 - |z|^2\right)\right]^2} - \frac{1}{\left|[(1 - x\overline{z})(1 - xz)]^2\right|} \geq \left|\frac{1}{\left[(1 - x^2)\left(1 - |z|^2\right)\right]^2} - \frac{1}{[(1 - x\overline{z})(1 - xz)]^2}\right|, \tag{2.13}$$

for any $x, z \in D(0,1)$. Also for $z = \overline{x}$, we have

$$\frac{1}{\left(1 - |x|^2\right)^3\left(1 - |y|^2\right)} - \frac{1}{\left|[(1 - x^2)(1 - \overline{x}\overline{y})]^2\right|} \geq \left|\frac{1}{\left[(1 - x\overline{y})\left(1 - |x|^2\right)\right]^2} - \frac{1}{[(1 - x^2)(1 - \overline{x}\overline{y})]^2}\right|, \tag{2.14}$$

for any $x, y \in D(0,1)$. If $z = a \in \mathbb{R}$ and $x, y \in \mathbb{C}$ then

$$\frac{1}{\left(1 - |x|^2\right)\left(1 - |y|^2\right)(1 - a^2)^2} - \frac{1}{\left|[(1 - ax)(1 - a\overline{y})]^2\right|} \geq \left|\frac{1}{[(1 - x\overline{y})(1 - a^2)]^2} - \frac{1}{[(1 - ax)(1 - a\overline{y})]^2}\right|, \tag{2.15}$$

for any $x, y \in D(0,1)$ and $a \in (-1,1)$.

Remark 1. *If $z = 0$, then from (2.5) we obtain*

$$\left[f\left(|x|^2\right) f\left(|y|^2\right)\right]^{1/2} - |f(0)| \geq |f(x\overline{y}) - f(0)|, \tag{2.16}$$

where $f(0) = a_0 > 0$, $|x|^2$, $|y|^2$, $x\overline{y} \in D(0,R)$.

Some applications of the inequality (2.16) for particular functions of interest are as follows.

1. If we apply the inequality (2.16) for the function $f(z) = \exp(z)$, $z \in \mathbb{C}$, then we obtain the inequality

$$\exp\left(\frac{|x|^2 + |y|^2}{2}\right) - 1 \geq |\exp(x\overline{y}) - 1|, \tag{2.17}$$

for any $x, y \in \mathbb{C}$. Moreover, if $y = \overline{x}$, then from (2.17) we get

$$\exp\left(|x|^2\right) - 1 \geq \left|\exp\left(x^2\right) - 1\right|,$$

for any $x \in \mathbb{C}$.

2. If we apply the same inequality (2.16) for the function $f(z) = \cos(z)$, $z \in \mathbb{C}$, then we get the following inequality

$$\left[\cos\left(|x|^2\right)\cos\left(|y|^2\right)\right]^{1/2} - 1 \geq |\cos(x\overline{y}) - 1|, \tag{2.18}$$

for any $x, y \in \mathbb{C}$. Also, if $y = \overline{x}$, then from (2.18) we get

$$\cos\left(|x|^2\right) - 1 \geq \left|\cos\left(x^2\right) - 1\right|,$$

for any $x \in \mathbb{C}$.

3. For the function $f(z) = \frac{1}{1-z}$, $z \in D(0,1)$ and applying the inequality (2.16) we obtain

$$\frac{1}{\left[\left(1 - |x|^2\right)\left(1 - |y|^2\right)\right]^{1/2}} - 1 \geq \left|\frac{x\overline{y}}{1 - x\overline{y}}\right|,$$

for any $x, y \in \mathbb{C}$ with $|x|^2, |y|^2, x\overline{y} \in D(0,1)$.

Remark 2. *If $y = \overline{x}$ in (2.5), then we get*

$$f\left(|x|^2\right) f\left(|z|^2\right) - |f(xz) f(x\overline{z})| \geq \left|f\left(x^2\right) f\left(|z|^2\right) - f(xz) f(x\overline{z})\right|, \tag{2.19}$$

for $x, z \in \mathbb{C}$ with $|x|^2, |z|^2, x\overline{z}, zx \in D(0,R)$. Moreover, for $z = a \in \mathbb{R}$, from (2.19) we deduce

$$f\left(|x|^2\right) f\left(a^2\right) - |f(ax)|^2 \geq \left|f\left(x^2\right) f\left(a^2\right) - f^2(ax)\right|, \tag{2.20}$$

for any $x \in \mathbb{C}$, $a \in \mathbb{R}$. In (2.20), if we choose $a = 1$, then we have the inequality

$$f\left(|x|^2\right) f(1) - |f(x)|^2 \geq \left|f\left(x^2\right) f(1) - f^2(x)\right|, \tag{2.21}$$

for any $x \in \mathbb{C}$.

For some applications, we apply the inequality (2.21) for the function $f(z) = \exp(z)$, then we have

$$\exp\left(|x|^2 + 1\right) - |\exp(2x)| \geq \left|\exp\left(x^2 + 1\right) - \exp(2x)\right|, \tag{2.22}$$

for any $x \in \mathbb{C}$. Since $|\exp(2x)| \neq 0$, then (2.22) is equivalent with

$$\frac{\exp\left(|x|^2 + 1\right)}{|\exp(2x)|} - 1 \geq \left|\exp(x-1)^2 - 1\right|,$$

for any $x \in \mathbb{C}$.

REMARK 3. *If $z=\overline{x}$ in (2.5) then we get*

$$\left[f\left(|x|^{2}\right)f\left(|y|^{2}\right)\right]^{1/2}f\left(|x|^{2}\right)-\left|f\left(x^{2}\right)f\left(xy\right)\right| \geq f\left(x\overline{y}\right)f\left(|x|^{2}\right)-f\left(xy\right)\overline{f\left(xy\right)}, \tag{2.23}$$

for $x,y\in C$ with $x^{2},xy,|x|^{2},|y|^{2}\in D\left(0,R\right)$.

If we apply the inequality (2.23) for the function $f(z)=\exp(z)$, $z\in\mathbb{C}$, then we get

$$\exp\left(\frac{3|x|^{2}+|y|^{2}}{2}\right)-\left|\exp\left[x\left(x+y\right)\right]\right| \geq\left|\exp\left(x\overline{y}+|x|^{2}\right)-\exp\left(x^{2}+\overline{xy}\right)\right|, \tag{2.24}$$

for any $x,y\in\mathbb{C}$. Moreover if $x=a\in\mathbb{R}$, then from (2.24) we obtain

$$\exp\left(\frac{3a^{2}+|y|^{2}}{2}\right)\geq\left|\exp\left[a\left(a+y\right)\right]\right|, \tag{2.25}$$

for any $y\in\mathbb{C},a\in\mathbb{R}$.

In the following, we construct another power series which will have as coefficients the absolute values of the coefficient of the original series, namely $f_{A}\left(z\right):=\sum_{n=0}^{\infty}|a_{n}|z^{n}$. It obvious that this new power series will have the same radius of convergence as the original series.

THEOREM 2. *Let $f(z)=\sum_{n=0}^{\infty}a_{n}z^{n}$ be a power series with real coefficients a_{n} and convergent on $D\left(0,R\right)\subset\mathbb{C}$, $R>0$. If $x,y,z\in\mathbb{C}$, so that $|x|^{2},|y|^{2},|z|^{2}$, $x\overline{z}$, $z\overline{y}$, $x\overline{y}\in D\left(0,R\right)$, then*

$$\left[f_{A}\left(|x|^{2}\right)f_{A}\left(|y|^{2}\right)\right]^{1/2}f_{A}\left(|z|^{2}\right)-\left|f\left(x\overline{z}\right)f\left(z\overline{y}\right)\right| \geq\left|f_{A}\left(x\overline{y}\right)f_{A}\left(|z|^{2}\right)-f\left(x\overline{z}\right)f\left(z\overline{y}\right)\right|. \tag{2.26}$$

PROOF. Firstly, observe that for each $n\geq0$, $a_{n}=|a_{n}|\,sgn\left(a_{n}\right)$ where the sgn(x) is the sgn function defined to be 1 if $x>0$, -1 if $x<0$ and 0 if $x=0$. By choosing $p_{n}=|a_{n}|\geq0$, $x_{n}=x^{n}$, $y_{n}=y^{n}$, $z_{n}=sgn\left(a_{n}\right)z^{n}$, $n\geq0$ in (2.3) we have

$$\left|\sum_{n=0}^{m}a_{n}\left(x\overline{z}\right)^{n}\sum_{n=0}^{m}a_{n}\left(z\overline{y}\right)^{n}\right| =\left|\sum_{n=0}^{m}|a_{n}|\,sgn\left(a_{n}\right)x^{n}\left(\overline{z}\right)^{n}\sum_{n=0}^{m}|a_{n}|\,sgn\left(a_{n}\right)z^{n}\left(\overline{y}\right)^{n}\right|,$$

$$\leq \left(\sum_{n=0}^{m} |a_n| |x|^{2n}\right)^{1/2} \left(\sum_{n=0}^{m} |a_n| |y|^{2n}\right)^{1/2} \sum_{n=0}^{m} |a_n| |sgn(a_n) z^n|^2$$
$$- \left|\sum_{n=0}^{m} |a_n| x^n (\overline{y})^n \sum_{n=0}^{m} |a_n| |sgn(a_n) z^n|^2\right.$$
$$\left.- \sum_{n=0}^{m} |a_n| x^n [sgn(a_n) (\overline{z})^n] \times \sum_{n=0}^{m} |a_n| [sgn(a_n) z^n] (\overline{y})^n\right|$$
$$= \left(\sum_{n=0}^{m} |a_n| \left(|x|^2\right)^n\right)^{1/2} \left(\sum_{n=0}^{m} |a_n| \left(|y|^2\right)^n\right)^{1/2} \sum_{n=0}^{m} |a_n| \left(|z|^2\right)^n$$
$$- \left|\sum_{n=0}^{m} |a_n| (x\overline{y})^n \sum_{n=0}^{m} |a_n| \left(|z|^2\right)^n - \sum_{n=0}^{m} a_n (x\overline{z})^n \sum_{n=0}^{m} a_n (z\overline{y})^n\right|,$$

for any $x, y, z \in \mathbb{C}$ with $x\overline{y}, x\overline{z}, z\overline{y}, |x|^2, |y|^2, |z|^2 \in D(0, R)$. Taking the limit as $m \to \infty$, then we deduce the desired inequality (2.26). □

In what follows we provide some applications of the inequality (2.26) for particular functions of interest.

1. If we take the function

$$f(z) = \frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n, \; z \in D(0,1),$$

then

$$f_A(z) = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}, \; z \in D(0,1).$$

Applying the Theorem 2, we can state that

$$\left[\frac{1}{1-|x|^2} \cdot \frac{1}{1-|y|^2}\right]^{1/2} \left(\frac{1}{1-|z|^2}\right) - \left|\frac{1}{1+x\overline{z}} \cdot \frac{1}{1+z\overline{y}}\right|$$
$$= \frac{1}{\left[\left(1-|x|^2\right)\left(1-|y|^2\right)\right]^{1/2}\left(1-|z|^2\right)} - \frac{1}{|(1+x\overline{z})(1+z\overline{y})|},$$
$$= \frac{|(1+x\overline{z})(1+z\overline{y})| - \left[\left(1-|x|^2\right)\left(1-|y|^2\right)\right]^{1/2}\left(1-|z|^2\right)}{\left[\left(1-|x|^2\right)\left(1-|y|^2\right)\right]^{1/2}\left(1-|z|^2\right)|(1+x\overline{z})(1+z\overline{y})|},$$
$$\geq \left|\frac{1}{1-x\overline{y}} \cdot \frac{1}{1-|z|^2} - \frac{1}{1+x\overline{z}} \cdot \frac{1}{1+z\overline{y}}\right|,$$
$$= \frac{\left|(1+x\overline{z})(1+z\overline{y}) - (1-x\overline{y})\left(1-|z|^2\right)\right|}{\left|(1-x\overline{y})\left(1-|z|^2\right)\right| |(1+x\overline{z})(1+z\overline{y})|}.$$

Hence we have

$$\frac{|(1+x\overline{z})(1+z\overline{y})|}{\left[\left(1-|x|^2\right)\left(1-|y|^2\right)\right]^{1/2}\left(1-|z|^2\right)}-1\geq\left|\frac{(1+x\overline{z})(1+z\overline{y})}{(1-x\overline{y})\left(1-|z|^2\right)}-1\right|, \tag{2.27}$$

for any $x,y,z\in D(0,1)$. In particular if $y=\overline{x}$, $z=a\in\mathbb{R}$, then from (2.27) we get

$$\frac{|1+ax|^2}{\left(1-|x|^2\right)(1-a^2)}-1\geq\left|\frac{(1+ax)^2}{(1-x^2)(1-a^2)}-1\right|,$$

for any $x\in D(0,1)$, $a\in\mathbb{R}$.

2. For the function

$$f(z)=\exp(-z)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}z^n,\ \ z\in\mathbb{C},$$

we have the transform

$$f_A(z)=\sum_{n=0}^{\infty}\frac{1}{n!}z^n=\exp(z),\, z\in\mathbb{C}.$$

Utilising the inequality (2.26) we obtain

$$\begin{aligned}&\exp\left(\frac{|x|^2+|y|^2}{2}+|z|^2\right)|\exp(x\overline{z}+z\overline{y})|-1\\&\geq\left|\exp\left(x\overline{y}+|z|^2+x\overline{z}+z\overline{y}\right)-1\right|,\end{aligned} \tag{2.28}$$

for any $x,y,z\in\mathbb{C}$. In particular if $y=\overline{x}$, $z=a\in\mathbb{R}$, then from (2.28) we get

$$\exp\left(|x|^2+a^2\right)|\exp(2ax)|-1\geq\left|\exp\left(|x|^2+a+2ax\right)-1\right|,$$

for any $x\in\mathbb{C}$, $a\in\mathbb{R}$.

In the following result, we can state a connection between two power series, one having positive coefficients while the other has complex coefficients.

Theorem 3. *Let $g(z)=\sum_{n=0}^{\infty}g_nz^n$ and $f(z)=\sum_{n=0}^{\infty}a_nz^n$ be two power series with $g_n\in\mathbb{C}$ and $a_n>0$, $n\geq 0$. If f and g are convergent on $D(0,R_1)$ and $D(0,R_2)$ respectively, and the numerical series $\sum_{n=0}^{\infty}\frac{|g_n|^2}{a_n}$ is convergent, then we have the inequality*

$$\sum_{n=0}^{\infty}\frac{|g_n|^2}{a_n}f\left(|z|^2\right)-|g(z)g(\overline{z})|\geq\left|\sum_{n=0}^{\infty}\frac{|g_n|^2}{a_n}f\left(z^2\right)-g(z)\overline{g(\overline{z})}\right|, \tag{2.29}$$

for any $z\in\mathbb{C}$ with $z,z^2|z|^2\in D(0,R_1)\cap D(0,R_2)$.

Proof. On utilizing the inequality (2.4) for the choices $p_n = a_n$, $x_n = z^n$, $z_n = \frac{g_n}{a_n}$, $n \in \{0, 1, 2, ..., m\}$, we have

$$\begin{aligned}
(2.30)\qquad & \sum_{n=0}^{m} a_n |z|^{2n} \sum_{n=0}^{m} a_n \left|\frac{g_n}{a_n}\right|^2 - \left|\sum_{n=0}^{m} a_n z^n \overline{\left(\frac{g_n}{a_n}\right)} \sum_{n=0}^{n} a_n z^n \left(\frac{g_n}{a_n}\right)\right| \\
&= \sum_{n=0}^{m} a_n \left(|z|^2\right)^n \sum_{n=0}^{m} \frac{|g_n|^2}{a_n} - \left|\sum_{n=0}^{m} \overline{g_n} z^n \sum_{n=0}^{n} g_n z^n\right|, \\
&\geq \left|\sum_{n=0}^{m} a_n z^{2n} \sum_{n=0}^{m} a_n \left|\frac{g_n}{a_n}\right|^2 - \sum_{n=0}^{m} a_n z^n \overline{\left(\frac{g_n}{a_n}\right)} \sum_{n=0}^{n} a_n z^n \left(\frac{g_n}{a_n}\right)\right| \\
&= \left|\sum_{n=0}^{m} a_n \left(z^2\right)^n \sum_{n=0}^{m} \frac{|g_n|^2}{a_n} - \sum_{n=0}^{m} \overline{g_n} z^n \sum_{n=0}^{n} g_n z^n\right|,
\end{aligned}$$

for any $m \geq 0$.

Observe that $\sum_{n=0}^{m} \overline{g_n} z^n = \overline{\sum_{n=0}^{m} g_n \left(\overline{z}\right)^n}$ and then

$$\left|\sum_{n=0}^{m} \overline{g_n} z^n\right| = \left|\sum_{n=0}^{m} g_n \left(\overline{z}\right)^n\right|.$$

Replacing this in (2.30) we get

$$\begin{aligned}
(2.31)\qquad & \sum_{n=0}^{m} \frac{|g_n|^2}{a_n} \sum_{n=0}^{m} a_n \left(|z|^2\right)^n - \left|\sum_{n=0}^{m} g_n \left(\overline{z}\right)^n \sum_{n=0}^{n} g_n z^n\right| \\
&\geq \left|\sum_{n=0}^{m} a_n \left(z^2\right)^n \sum_{n=0}^{m} \frac{|g_n|^2}{a_n} - \sum_{n=0}^{m} \overline{g_n} z^n \sum_{n=0}^{n} g_n z^n\right|.
\end{aligned}$$

Since $z, z^2 |z|^2 \in D(0, R_1) \cap D(0, R_2)$, hence the series in (2.31) are convergent and letting $m \to \infty$, we deduce the desired inequality (2.29). □

Remark 4. *If the coefficients g_n, $n \geq 0$ are real, the we have the inequality*

$$\sum_{n=0}^{\infty} \frac{g_n^2}{a_n} f\left(|z|^2\right) - |g(z)g(\overline{z})| \geq \left|\sum_{n=0}^{\infty} \frac{g_n^2}{a_n} f\left(z^2\right) - g^2(z)\right|,$$

for any $z \in \mathbb{C}$ with $z, z^2, |z|^2 \in D(0, R_1) \cap D(0, R_2)$.

Corollary 1. *Let $g(z) = \sum_{n=0}^{\infty} g_n z^n$ be a power series with complex coefficients and convergent on the open disk $D(0, R)$. If the numerical series $\sum_{n=0}^{\infty} |g_n|^2$ is convergent, then*

$$(2.32)\qquad \left(\frac{1}{1 - |z|^2}\right) \sum_{n=0}^{\infty} |g_n|^2 - |g(z)g(\overline{z})| \geq \left|\left(\frac{1}{1 - z^2}\right) \sum_{n=0}^{\infty} |g_n|^2 - g(z)\overline{g\left(\overline{z}\right)}\right|,$$

for any $z \in D(0, 1) \cap D(0, R)$.

This follows from (2.29) for $f(z) = \frac{1}{1-z}$, $z \in D(0, 1)$.

If we consider the series expansion

$$\frac{1}{iz} \ln\left(\frac{1}{1 - iz}\right) = \sum_{n=0}^{\infty} \frac{i^n}{n+1} z^n, \ z \in D(0, 1) \setminus \{0\},$$

then on utilising the inequality (2.32) for the choice $g_n = \frac{i^n}{n+1}$ and taking into account that

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2) = \frac{\pi^2}{6},$$

where ζ is the Riemann zeta function, we can state the inequality

$$\frac{\pi^2}{6}\left(\frac{|z|^2}{1-|z|^2}\right) - \left|\ln\left(\frac{1}{1-iz}\right)\ln\left(\frac{1}{1-i\overline{z}}\right)\right| \geq \left|\frac{\pi^2}{6}\left(\frac{z^2}{1-z^2}\right) + \ln\left(\frac{1}{1-iz}\right)\ln\left(\frac{1}{1+iz}\right)\right|, \tag{2.33}$$

for any $z \in D(0,1)$.

Corollary 2. *Let $g(z) = \sum_{n=0}^{\infty} g_n z^n$ be a power series with complex coefficients and convergent on the open disk $D(0,R)$. If the numerical series $\sum_{n=0}^{\infty} n!\,|g_n|^2$ is convergent, then*

$$\sum_{n=0}^{\infty} n!\,|g_n|^2 \exp\left(|z|^2\right) - |g(z)g(\overline{z})| \geq \left|\sum_{n=0}^{\infty} n!\,|g_n|^2 \exp\left(z^2\right) - g(z)\overline{g(\overline{z})}\right|, \tag{2.34}$$

for any $z \in D(0,R)$.

This follows from Theorem 3 by choosing $f(z) = \exp(z)$.

If we apply the inequality (2.34) for the function $\sin(iz) = \sum_{n=0}^{\infty} \frac{i}{(2n+1)!} z^{2n+1}$, then we obtain the inequality

$$\sum_{n=0}^{\infty} \frac{n!}{(2n+1)!} \exp\left(|z|^2\right) - |\sin(iz)\sin(i\overline{z})| \geq \left|\sum_{n=0}^{\infty} \frac{n!}{(2n+1)!} \exp\left(z^2\right) - \sin^2(iz)\right|,$$

for any $z \in \mathbb{C}$.

3. Some Inequalities for the Polylogaritm

The polylogarithm $Li_n(z)$, also known as the *de Jonquières function* is the function defined by

$$Li_n(z) := \sum_{k=1}^{\infty} \frac{z^k}{k^n}, \tag{3.1}$$

defined in the complex plane over the unit disk $D(0,1)$.

Before we start our results for the polylogarithm that can be obtained on utilising the Schwarz inequality, we recall some concepts that will be used in the sequel.

The special case, $z = 1$ reduces to $Li_s(z) = \zeta(s)$, where ζ is the Riemann zeta function. The polylogarithm of nonnegative order arises in the sums of the form

$$Li_{-n}(r) = \sum_{k=1}^{\infty} k^n r^k = \frac{1}{(1-r)^{n+1}} \sum_{i=0}^{n} E_{n,i} r^{n-i},$$

where $E_{n,i}$ is an Eulerian number, namely, we recall that

$$E_{n,k} := \sum_{j=0}^{k+1} (-1)^j \binom{n+1}{j} (k-j+1)^n.$$

Polylogarithms also arise in sum of generalized harmonic numbers $H_{n,r}$ as

$$\sum_{n=1}^{\infty} H_{n,r} z^n = \frac{Li_r(z)}{1-z},$$

for $z \in D(0,1)$, where we recall that

$$H_{n,r} := \sum_{k=1}^{\infty} \frac{1}{k^r} \text{ and } H_{n,1} := H_n = \sum_{k=1}^{\infty} \frac{1}{k}.$$

Special forms of low-order polylogarithms include

$$Li_{-2}(z) = \frac{z(z+1)}{(1-z)^3},\ Li_{-1}(z) = \frac{z}{(1-z)^2},$$
$$Li_0(z) = \frac{z}{(1-z)},\ Li_1(z) = -\ln(1-z),\ z \in D(0,1).$$

At argument $z=-1$, the general polylogarithms becomes $Li_n(-1) = -\eta(x)$, where $\eta(x)$ is the Dirichlet eta function.

It is clear that Li_n being a power series with nonnegative coefficients and convergent on the open unit disk $D(0,1)$, so that all the above results hold true. Therefore we have, for instance the inequality:

$$\left[Li_n\left(|x|^2\right) Li_n\left(|y|^2\right)\right]^{1/2} Li_n\left(|z|^2\right) - |Li_n(x\overline{z}) Li_n(z\overline{y})| \geq \left|Li_n(x\overline{y}) Li_n\left(|z|^2\right) - Li_n(x\overline{z}) Li_n(z\overline{y})\right|, \tag{3.2}$$

for any $x, y, z \in \mathbb{C}$ with $|x|^2, |y|^2, |z|^2, x\overline{z}, z\overline{y}, x\overline{y} \in D(0,1)$ and n is a negative or positive integer.

In the following, we present some results that connect different order polylogarithms:

THEOREM 4. *Let $x, y, z \in \mathbb{C}$ with $|x|^2, |y|^2, |z|^2, x\overline{z}, z\overline{y}, x\overline{y} \in D(0,1)$ and p, q, r integers such that the following series exist. Then*

$$\begin{aligned} &\left[Li_{r+2q}\left(|x|^2\right) Li_{r+2q}\left(|y|^2\right)\right]^{1/2} Li_{r+2p}\left(|z|^2\right) \\ &- |Li_{r+p+q}(x\overline{z}) Li_{r+p+q}(z\overline{y})| \\ &\geq \left|Li_{r+2q}(x\overline{y}) Li_{r+2p}\left(|z|^2\right) - Li_{r+p+q}(x\overline{z}) Li_{r+p+q}(z\overline{y})\right|. \end{aligned} \tag{3.3}$$

PROOF. Utilising the discrete inequality (2.3) for $p_k = \frac{1}{k^r}$, $x_k = \frac{x^k}{k^q}$, $y_k = \frac{y^k}{k^q}$, $z_k = \frac{z^k}{k^p}$, $k \in \{1, 2, ..., m\}$, we have

$$\begin{aligned} &\left(\sum_{k=1}^{m} \frac{1}{k^r} \left|\frac{x^k}{k^q}\right|^2\right)^{1/2} \left(\sum_{k=1}^{m} \frac{1}{k^r} \left|\frac{y^k}{k^q}\right|^2\right)^{1/2} \sum_{k=1}^{m} \frac{1}{k^r} \left|\frac{z^k}{k^p}\right|^2 \\ &- \left|\sum_{k=1}^{n} \frac{1}{k^r} \frac{x^k}{k^q} \frac{(\overline{z})^k}{k^p} \sum_{k=1}^{n} \frac{1}{k^r} \frac{z^k}{k^p} \frac{(\overline{y})^k}{k^q}\right| \\ &\geq \left|\sum_{k=1}^{m} \frac{1}{k^r} \frac{x^k}{k^q} \frac{(\overline{y})^k}{k^q} \sum_{k=1}^{n} \frac{1}{k^r} \left|\frac{z^k}{k^p}\right|^2 - \sum_{k=1}^{m} \frac{1}{k^r} \frac{x^k}{k^q} \frac{(\overline{z})^k}{k^p} \sum_{k=1}^{m} \frac{1}{k^r} \frac{z^k}{k^p} \frac{(\overline{y})^k}{k^q}\right|, \end{aligned}$$

hence

$$\left(\sum_{k=1}^{m}\frac{1}{k^{r+2q}}\left(|x|^2\right)^k\right)^{1/2}\left(\sum_{k=1}^{m}\frac{1}{k^{r+2q}}\left(|y|^2\right)^k\right)^{1/2}\sum_{k=1}^{m}\frac{1}{k^{r+2p}}\left(|z|^2\right)^k - \left|\sum_{k=1}^{m}\frac{1}{k^{r+p+q}}(x\overline{z})^k\sum_{k=1}^{m}\frac{1}{k^{r+p+q}}(z\overline{y})^k\right| \geq \sum_{k=1}^{m}\frac{(x\overline{y})^k}{k^{r+2q}}\sum_{k=1}^{m}\frac{\left(|z|^2\right)^k}{k^{r+2p}} - \sum_{k=1}^{m}\frac{(x\overline{z})^k}{k^{r+p+q}}\sum_{k=1}^{m}\frac{(z\overline{y})^k}{k^{r+p+q}}, \tag{3.4}$$

for $m \geq 0$.

Taking the limit as $m \to \infty$ in (3.4), then we deduce the desired inequality (3.3). □

On making use of the above result (3.3), we can get some simpler inequalities as follows:

1. If $y = \overline{x}$, then from (3.3) we can state that

$$Li_{r+2q}\left(|x|^2\right)Li_{r+2p}\left(|z|^2\right) - |Li_{r+p+q}(xz)\,Li_{r+p+q}(x\overline{z})| \geq \left|Li_{r+2q}\left(x^2\right)L_{r+2p}\left(|z|^2\right) - Li_{r+p+q}(xz)\,Li_{r+p+q}(x\overline{z})\right|, \tag{3.5}$$

for $x, z \in \mathbb{C}$.

2. Moreover, if $z = a \in \mathbb{R}$, then from (3.5) we deduce the inequality

$$Li_{r+2q}\left(|x|^2\right)Li_{r+2p}\left(a^2\right) - |Li_{r+p+q}(ax)|^2 \geq \left|Li_{r+2q}\left(x^2\right)Li_{r+2p}\left(a^2\right) - Li^2_{r+p+q}(ax)\right|, \tag{3.6}$$

for any $x \in \mathbb{C}$, $a \in \mathbb{R}$.

3. If we choose $a = 1$ in (3.6), the we have

$$\zeta(r+2p)\,Li_{r+2q}\left(|x|^2\right) - |Li_{r+p+q}(x)|^2 \geq \left|\zeta(r+2p)\,Li_{r+2q}\left(x^2\right) - Li^2_{r+p+q}(x)\right| \tag{3.7}$$

where ζ is the Riemann zeta Function.

4. On utilising (3.7) and taking into account that some particular values of ζ are known, such as $\zeta(2) = \frac{\pi^2}{6}$, $\zeta(4) = \frac{\pi^4}{90}$, then we can state the following results:

$$\frac{\pi^2}{6}Li_{2q}\left(|x|^2\right) - |Li_{q+1}(x)|^2 \geq \left|\frac{\pi^2}{6}Li_{2q}\left(x^2\right) - Li^2_{q+1}(x)\right|;$$

$$\frac{\pi^4}{90}Li_{2q}\left(|x|^2\right) - |Li_{q+2}(x)|^2 \geq \left|\frac{\pi^4}{90}Li_{2q}\left(x^2\right) - Li^2_{q+2}(x)\right|;$$

and

$$\frac{\pi^4}{90}Li_{2(q+1)}\left(|x|^2\right) - |Li_{q+3}(x)|^2 \geq \left|\frac{\pi^4}{90}Li_{2(q+1)}\left(x^2\right) - Li^2_{q+3}(x)\right|$$

for any $x \in D(0,1)$ and q an integer.

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