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A SURVEY ON JESSEN'S TYPE INEQUALITIES FOR POSITIVE FUNCTIONALS

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Abstract. Some recent inequalities related to the celebrated Jessen's result for positive linear or sublinear functionals and convex functions are surveyed.

1. Introduction

Let L be a linear class of real-valued functions $g : E \to \mathbb{R}$ having the properties

(L1) $f, g \in L$ imply $(\alpha f + \beta g) \in L$ for all $\alpha, \beta \in \mathbb{R}$;
(L2) $\mathbf{1} \in L$, i.e., if $f_0(t) = 1$, $t \in E$ then $f_0 \in L$.

An *isotonic linear functional* $A : L \to \mathbb{R}$ is a functional satisfying

(A1) $A(\alpha f + \beta g) = \alpha A(f) + \beta A(g)$ for all $f, g \in L$ and $\alpha, \beta \in \mathbb{R}$.
(A2) If $f \in L$ and $f \geq 0$, then $A(f) \geq 0$.
The mapping A is said to be *normalised if*
(A3) $A(\mathbf{1}) = 1$.

Isotonic, that is, order-preserving, linear functionals are natural objects in analysis which enjoy a number of convenient properties. Thus, they provide, for example, Jessen's inequality, which is a functional form of Jensen's inequality (see [2] and [14]).

We note that common examples of such isotonic linear functionals A are given by

$$A(g) = \int_E g d\mu \text{ or } A(g) = \sum_{k \in E} p_k g_k,$$

where μ is a positive measure on E in the first case and E is a subset of the natural numbers $\mathbb{N}$, in the second ($p_k \geq 0, k \in E$).

We recall Jessen's inequality (see also [12]).

Theorem 1 (Jessen Inequality)**.** *Let $\phi : I \subseteq \mathbb{R} \to \mathbb{R}$ (I is an interval), be a convex function and $f : E \to I$ such that $\phi \circ f$, $f \in L$. If $A : L \to \mathbb{R}$ is an isotonic linear and normalised functional, then*

$$\phi(A(f)) \leq A(\phi \circ f). \tag{1.1}$$

A counterpart of this result was proved by Beesack and Pečarić in [2] for compact intervals $I = [\alpha, \beta]$.

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Theorem 2 (Beesack & Pečarić, 1985, [2])**.** *Let $\phi : [\alpha, \beta] \subset \mathbb{R} \to \mathbb{R}$ be a convex function and $f : E \to [\alpha, \beta]$ such that $\phi \circ f$, $f \in L$. If $A : L \to \mathbb{R}$ is an isotonic linear and normalised functional, then*

$$A(\phi \circ f) \leq \frac{\beta - A(f)}{\beta - \alpha}\phi(\alpha) + \frac{A(f) - \alpha}{\beta - \alpha}\phi(\beta). \tag{1.2}$$

Remark 1. *Note that (1.2) is a generalisation of the inequality*

$$A(\phi) \leq \frac{b - A(e_1)}{b - a}\phi(a) + \frac{A(e_1) - a}{b - a}\phi(b) \tag{1.3}$$

due to Lupaş [13] *(see for example* [2, Theorem A]*), which assumed $E = [a, b]$, L satisfies (L1), (L2), $A : L \to \mathbb{R}$ satisfies (A1), (A2), $A(\mathbf{1}) = 1$, ϕ is convex on E and $\phi \in L$, $e_1 \in L$, where $e_1(x) = x$, $x \in [a, b]$.*

The following inequality is well known in the literature as the Hermite-Hadamard inequality

$$\varphi\left(\frac{a + b}{2}\right) \leq \frac{1}{b - a}\int_a^b \varphi(t)\,dt \leq \frac{\varphi(a) + \varphi(b)}{2}, \tag{1.4}$$

provided that $\varphi : [a, b] \to \mathbb{R}$ is a convex function.

Using Theorem 1 and Theorem 2, we may state the following generalisation of the Hermite-Hadamard inequality for isotonic linear functionals ([15] and [16]).

Theorem 3 (Pečarić & Beesack, 1991, [15])**.** *Let $\phi : [a, b] \subset \mathbb{R} \to \mathbb{R}$ be a convex function and $e : E \to [a, b]$ with e, $\phi \circ e \in L$. If $A \to \mathbb{R}$ is an isotonic linear and normalised functional, with $A(e) = \frac{a+b}{2}$, then*

$$\varphi\left(\frac{a + b}{2}\right) \leq A(\phi \circ e) \leq \frac{\varphi(a) + \varphi(b)}{2}. \tag{1.5}$$

For other results concerning convex functions and isotonic linear functionals, see [15], [16], [5], [17], [18] and [12] and the recent monograph [7].

2. Generalizations of Hermite-Hadamard's Inequalities for Isotonic Linear Functionals

2.1. **Some generalizations.** The following lemma holds [16]:

Lemma 1. *Let X be a real linear space and C its convex subset. Then the following statements are equivalent for a mapping $F : X \to \mathbb{R}$:*

- *(i) f is convex on C;*
- *(ii) for all $x, y \in C$ the mapping $g_{x,y} : [0, 1] \to \mathbb{R}$, $g_{x,y}(t) := f(tx + (1 - t)y)$ is convex on $[0, 1]$.*

Proof. "$(i) \Rightarrow (ii)$". Suppose $x, y \in C$ and let $t_1, t_2 \in [0, 1]$, $\lambda_1, \lambda_2 \geq 0$ with $\lambda_1 + \lambda_2 = 1$. Then

$$\begin{aligned} g_{x,y}(\lambda_1 t_1 + \lambda_2 t_2) &= f[(\lambda_1 t_1 + \lambda_2 t_2)x + (1 - \lambda_1 t_1 - \lambda_2 t_2)y] \\ &= f[(\lambda_1 t_1 + \lambda_2 t_2)x + [\lambda_1(1 - t_1) + \lambda_2(1 - t_2)]y] \\ &\leq \lambda_1 f(t_1 x + (1 - t_1)y) + \lambda_2 f(t_2 x + (1 - t_2)y). \end{aligned}$$

That is, $g_{x,y}$ is convex on $[0,1]$.
"$(ii) \Rightarrow (i)$". Now, let $x, y \in C$ and $\lambda_1, \lambda_2 \geq 0$ with $\lambda_1 + \lambda_2 = 1$. Then we have:

$$\begin{aligned} f(\lambda_1 x + \lambda_2 y) &= f(\lambda_1 x + (1-\lambda_1) y) = g_{x,y}(\lambda_1 \cdot 1 + \lambda_2 \cdot 0) \\ &\leq \lambda_1 g_{x,y}(1) + \lambda_2 g_{x,y}(0) = \lambda_1 f(x) + \lambda_2 f(y). \end{aligned}$$

That is, f is convex on C and the statement is proved. □

The following generalization of Hermite-Hadamard's inequality for isotonic linear functionals holds [16]:

Theorem 4 (Pečarić & Dragomir, 1991, [16])**.** *Let $f : C \subseteq X \to \mathbb{R}$ be a convex function on C, L and A satisfy conditions L1, L2 and A1, A2, and $h : E \to \mathbb{R}$, $0 \leq h(t) \leq 1$, $h \in L$ is such that $g_{x,y} \circ h \in L$ for x, y given in C. If $A(\mathbb{I}) = 1$, then we have the inequality*

$$\begin{aligned} f(A(h)x + (1-A(h))y) &\leq A[f(hx + (\mathbb{I}-h)y)] \\ &\leq A(h) f(x) + (1-A(h)) f(y). \end{aligned} \tag{2.1}$$

Proof. Consider the mapping $g_{x,y} : [0,1] \to \mathbb{R}$, $g_{x,y}(s) := f(sx + (1-s)y)$. Then, by the above lemma, we have that $g_{x,y}$ is convex on $[0,1]$. For all $t \in E$ we have:

$$g_{x,y}(h(t) \cdot 1 + (1-h(t)) \cdot 0) \leq h(t) g_{x,y}(1) + (1-h(t)) g_{x,y}(0),$$

which implies that

$$A(g_{x,y}(h)) \leq A(h) g_{x,y}(1) + (1-A(h)) g_{x,y}(0).$$

That is,

$$A[f(hx + (\mathbb{I}-h)y)] \leq A(h) f(x) + (1-A(h)) f(y).$$

On the other hand, by Jessen's inequality, applied for $g_{x,y}$ we have:

$$g_{x,y}(A(h)) \leq A(g_{x,y}(h)),$$

which gives:

$$f(A(h)x + (1-A(h))y) \leq A[f(hx + (\mathbb{I}-h)y)]$$

and the proof is completed. □

Remark 2. *If $h : E \to [0,1]$ is such that $A(h) = \frac{1}{2}$, we get from the inequality* (2.1) *that*

$$f\left(\frac{x+y}{2}\right) \leq A[f(hx + (\mathbb{I}-h)y)] \leq \frac{f(x) + f(y)}{2}, \tag{2.2}$$

for all x, y in C.

Consequences

a) If $A = \int_0^1$, $E = [0,1]$, $h(t) = t$, $C = [x,y] \subset \mathbb{R}$, then we recapture from (2.1) the classical inequality of Hermite and Hadamard, because

$$\int_0^1 f(tx + (1-t)y)\,dt = \frac{1}{y-x} \int_x^y f(t)\,dt.$$

b) If $A = \frac{2}{\pi}\int_0^{\frac{\pi}{2}}, E = \left[0, \frac{\pi}{2}\right], h(t) = \sin^2 t, C \subseteq \mathbb{R}$, then, from (2.2) we get

$$f\left(\frac{x+y}{2}\right) \leq \frac{2}{\pi}\int_0^{\frac{\pi}{2}} f\left(x\sin^2 t + y\cos^2 t\right) dt \leq \frac{f(x)+f(y)}{2},$$

$x, y \in C$, which is a new inequality of Hadamard's type. This is as $\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \sin^2 t dt = \frac{1}{2}$.

c) If $A = \int_0^1, E = [0,1], h(t) = t$ and X is a normed linear space, then (2.2) implies that for $f(x) = \|x\|^p$, $x \in X, p \geq 1$:

$$\left\|\frac{x+y}{2}\right\|^p \leq \int_0^1 \|tx + (1-t)y\|^p dt \leq \frac{\|x\|^p + \|y\|^p}{2}$$

for all $x, y \in X$.

d) If $A = \frac{1}{n}\sum_{i=1}^n, E = \{1, ..., n\}, \sum_{i=1}^n t_i = \frac{n}{2}, C \subseteq \mathbb{R}, n \geq 1$, then from (2.2) we also have

$$f\left(\frac{x+y}{2}\right) \leq \frac{1}{n}\sum_{i=1}^n f(t_i x + (1-t_i)y) \leq \frac{f(x)+f(y)}{2}$$

for all $x, y \in C$, which is a discrete variant of the Hermite-Hadamard inequality.

To give a symmetric generalization of the Hermite-Hadamard inequality, we present the following lemma which is interesting in itself [5].

Lemma 2. *Let X be a real linear space and C be its convex subset. If $f : C \to \mathbb{R}$ is convex on C, then for all x, y in C the mapping $g_{x,y} : [0,1] \to \mathbb{R}$ given by*

$$g_{x,y}(t) := \frac{1}{2}\left[f(tx + (1-t)y) + f((1-t)x + ty)\right]$$

is also convex on $[0,1]$. In addition, we have the inequality

$$f\left(\frac{x+y}{2}\right) \leq g_{x,y}(t) \leq \frac{f(x)+f(y)}{2} \tag{2.3}$$

for all $x, y \in C$ and $t \in [0,1]$.

Proof. Suppose $x, y \in C$ and let $t_1, t_2 \in [0,1]$, $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$. Then

$$\begin{aligned}
& g_{x,y}(\alpha t_1 + \beta t_2) \\
&= \frac{1}{2}\left[f((\alpha t_1 + \beta t_2)x + (1 - \alpha t_1 - \beta t_2)y)\right. \\
&+ \left. f((1 - \alpha t_1 - \beta t_2)x + (\alpha t_1 + \beta t_2)y)\right] \\
&= \frac{1}{2}\left(f[\alpha(t_1 x + (1-t_1)y) + \beta(t_2 x + (1-t_2)y)]\right. \\
&+ \left. f[\alpha((1-t_1) + t_1 xy) + \beta((1-t_2)x + t_2 y)]\right) \\
&\leq \frac{1}{2}\left(\alpha f[t_1 x + (1-t_1)y] + \beta f[t_2 x + (1-t_2)y]\right. \\
&+ \left. \alpha f[(1-t_1) + t_1 xy] + \beta f[(1-t_2)x + t_2 y]\right) \\
&= \alpha g_{x,y}(t_1) + \beta g_{x,y}(t_2),
\end{aligned}$$

which shows that $g_{x,y}$ is convex on $[0,1]$.
By the convexity of f we can state that

$$g_{x,y}(t) \geq f\left[\frac{1}{2}(tx + (1-t)y + (1-t)x + ty)\right] = f\left(\frac{x+y}{2}\right).$$

In addition,

$$g_{x,y}(t) \leq \frac{1}{2}[tf(x) + (1-t)f(y) + (1-t)f(x) + tf(y)] \leq \frac{f(x)+f(y)}{2}$$

for all t in $[0,1]$, which completes the proof. □

Remark 3. *By the inequality* (2.3) *we deduce the bounds*

$$\sup_{t\in[0,1]} g_{x,y}(t) = \frac{f(x)+f(y)}{2} \quad \text{and} \quad \inf_{t\in[0,1]} g_{x,y}(t) = f\left(\frac{x+y}{2}\right)$$

for all x, y in C.

The following symmetric generalization of the Hermite-Hadamard inequality holds [5]:

Theorem 5 (Dragomir, 1992, [5])**.** *Let $f : C \subseteq X \to \mathbb{R}$ be a convex function on the convex set C, where L and A satisfy the conditions L1, L2 and A1,A2. Also, $h : E \to \mathbb{R}$, $0 \leq h(t) \leq 1$ $(t \in E)$, and $h \in L$ is such that $f(hx + (1-h)y)$, $f((1-h)x + hy)$ belong to L for x, y fixed in C. If $A(\mathbb{I}) = 1$, then we have the inequality:*

$$\begin{aligned} f\left(\frac{x+y}{2}\right) &\leq \frac{1}{2}[f(A(h)x + (1-A(h))y) + f((1-A(h))x + A(h)y)] \\ &\leq \frac{1}{2}(A[f(hx + (\mathbb{I}-h)y)] + A[f((\mathbb{I}-h)x + hy)]) \\ &\leq \frac{f(x)+f(y)}{2} \end{aligned} \tag{2.4}$$

Proof. Let us consider the mapping $g_{x,y} : [0,1] \to \mathbb{R}$ given above. Then, by the above Lemma we know that $g_{x,y}$ is convex on $[0,1]$.
Applying Jensen's inequality to the mapping $g_{x,y}$ we get:

$$g_{x,y}(A(h)) \leq A(g_{x,y}(h)).$$

However,

$$g_{x,y}(A(h)) = \frac{1}{2}[f(A(h)x + (1-A(h))y) + f((1-A(h))x + A(h)y)]$$

and

$$A(g_{x,y}(h)) = \frac{1}{2}(A[f(hx + (\mathbb{I}-h)y)] + A[f((\mathbb{I}-h)x + hy)])$$

and the second inequality in (2.4) is proved.
To prove the first inequality in (2.4) we observe, by (2.3), that

$$f\left(\frac{x+y}{2}\right) \leq g_{x,y}(A(h)) \text{ as } 0 \leq A(h) \leq 1,$$

which is exactly the desired outcome.
Finally, by the convexity of f, we observe that

$$\frac{1}{2}\left[f\left(hx+\left(\mathbb{I}-h\right)y\right)+f\left(\left(\mathbb{I}-h\right)x+hy\right)\right]\leq\frac{f\left(x\right)+f\left(y\right)}{2}$$

on E.
By applying the functional A, since $A\left(\mathbb{I}\right)=1$, we obtain the last part of (2.4). □

Remark 4. *The above theorem can also be proved by the use of Theorem 4 and by Lemma 2. We shall omit the details.*

Note that, if we choose $A=\int_0^1, E=[0,1], h(t)=t, C=[x,y]\subset\mathbb{R}$, we recapture, by (2.4), the Hermite-Hadamard inequality for integrals. This is because

$$\int_0^1 f\left(tx+\left(1-t\right)y\right)dt=\int_0^1 f\left(\left(1-t\right)x+ty\right)dt=\frac{1}{y-x}\int_x^y f\left(t\right)dt.$$

Consequences

a) Let $h:[0,1]\to[0,1]$ be a Riemann integrable function on $[0,1]$ and $p\geq 1$. Then, for all x,y vectors in the normed space $(X;\|\cdot\|)$ we have the inequality:

$$\begin{aligned}&\left\|\frac{x+y}{2}\right\|^p\\&\leq\frac{1}{2}\left[\left\|\left(1-\int_0^1 h\left(t\right)dt\right)x+\left(\int_0^1 h\left(t\right)dt\right)y\right\|^p\right.\\&\left.+\left\|\left(\int_0^1 h\left(t\right)dt\right)x+\left(1-\int_0^1 h\left(t\right)dt\right)y\right\|^p\right]\\&\leq\frac{1}{2}\left[\int_0^1\left\|\left(h\left(t\right)\right)x+\left(1-h\left(t\right)\right)y\right\|^p dt+\int_0^1\left\|\left(1-h\left(t\right)\right)x+\left(h\left(t\right)\right)y\right\|^p dt\right]\\&\leq\frac{\|x\|^p+\|y\|^p}{2}.\end{aligned}$$

If we choose $h(t)=t$, we get the inequality obtained above

$$\left\|\frac{x+y}{2}\right\|^p\leq\int_0^1\left\|tx+\left(1-t\right)y\right\|^p dt\leq\frac{\|x\|^p+\|y\|^p}{2}$$

for all $x,y\in X$.

b) Let $f:C\subseteq X\to\mathbb{R}$ be a convex function on the convex set C of a linear space X, $t_i\in[0,1]$ $\left(i=\overline{1,n}\right)$. Then we have the inequality:

$$\begin{aligned}&f\left(\frac{x+y}{2}\right)\\&\leq\frac{1}{2}\left[f\left(\frac{1}{n}\sum_{i=1}^n t_ix+\frac{1}{n}\sum_{i=1}^n\left(1-t_i\right)y\right)+f\left(\frac{1}{n}\sum_{i=1}^n\left(1-t_i\right)x+\frac{1}{n}\sum_{i=1}^n t_iy\right)\right]\\&\leq\frac{1}{2n}\left[\sum_{i=1}^n f\left(t_ix+\left(1-t_i\right)y\right)+\sum_{i=1}^n f\left(\left(1-t_i\right)x+t_iy\right)\right]\\&\leq\frac{f\left(x\right)+f\left(y\right)}{2}.\end{aligned}$$

If we put in the above inequality $t_i = \sin^2 \alpha_i, \alpha_i \in \mathbb{R}$ $\left(i = \overline{1, n}\right)$, then we have:

$$\begin{aligned} & f\left(\frac{x+y}{2}\right) \\ & \leq \frac{1}{2}\left(f\left[\left(\frac{1}{n}\sum_{i=1}^{n}\sin^2\alpha_i\right)x + \left(\frac{1}{n}\sum_{i=1}^{n}\cos^2\alpha_i\right)y\right]\right. \\ & \left. + f\left[\left(\frac{1}{n}\sum_{i=1}^{n}\cos^2\alpha_i\right)x + \left(\frac{1}{n}\sum_{i=1}^{n}\sin^2\alpha_i\right)y\right]\right) \\ & \leq \frac{1}{2n}\sum_{i=1}^{n}\left(f\left[\left(\sin^2\alpha_i\right)x + \left(\cos^2\alpha_i\right)y\right]\right. \\ & \left. + f\left[\left(\cos^2\alpha_i\right)x + \left(\sin^2\alpha_i\right)y\right]\right) \\ & \leq \frac{f(x)+f(y)}{2}. \end{aligned}$$

2.2. **Applications for Special Means.**

(1) For $x, y \geq 0$, let us consider the weighted means:

$$A_\alpha(x, y) := \alpha x + (1-\alpha) y$$

and

$$G_\alpha(x, y) := x^\alpha y^{1-\alpha}$$

where $\alpha \in [0, 1]$.

If $h : [0, 1] \to [0, 1]$ is an integrable mapping on $[0, 1]$, then, by Theorem 4 we have the inequality:

$$A_{\int_0^1 h(t)dt}(x, y) \geq \exp\left[\int_0^1 \ln\left[A_{h(t)}(x, y)\right] dt\right] \geq G_{\int_0^1 h(t)dt}(x, y). \tag{2.5}$$

If $\int_0^1 h(t)\,dt = \frac{1}{2}$, we get

$$A(x, y) \geq \exp\left[\int_0^1 \ln\left[A_{h(t)}(x, y)\right] dt\right] \geq G(x, y) \tag{2.6}$$

which is a refinement of the classic $A. - G.$ inequality.

In particular, if in this inequality we choose $h(t) = t,\ t \in [0, 1]$, we recapture the well-known result for the identric mean:

$$A(x, y) \geq I(x, y) \geq G(x, y).$$

Now, if we use Theorem 5, we can state the following weighted refinement of the classical $A. - G.$ inequality:

$$\begin{aligned} A(x, y) & \geq G\left(A_{\int_0^1 h(t)dt}(x, y), A_{\int_0^1 h(t)dt}(x, y)\right) \\ & \geq \exp\left[\int_0^1 \ln\left[G\left(A_{h(t)}(x, y), A_{h(t)}(y, x)\right)\right] dt\right] \geq G(x, y). \end{aligned} \tag{2.7}$$

If $\int_0^1 h(t)\,dt = \frac{1}{2}$, then, by (2.7) we get the following refinement of the $A. - G.$ inequality:

$$A(x, y) \geq \exp\left[\int_0^1 \ln\left[G\left(A_{h(t)}(x, y), A_{h(t)}(y, x)\right)\right] dt\right] \geq G(x, y). \tag{2.8}$$

If, in the above inequality we choose $h(t)=t, t\in[0,1]$, then we get the inequality

$$A(x,y)\geq\exp\left[\int_0^1\ln\left[G\left(A_t(x,y),A_t(y,x)\right)\right]dt\right]\geq G(x,y). \tag{2.9}$$

(2) Some discrete refinements of $A.-G.$ means inequality can also be done. If $\bar{x}=(x_1,...,x_n)\in\mathbb{R}_+^n$, we can denote by $G_n(\bar{x})$ the geometric mean of $\bar{x}$, i.e., $G_n(\bar{x}):=\left(\prod_{i=1}^n x_i\right)^{\frac{1}{n}}$.
If $\bar{t}=(t_1,...,t_n)\in[0,1]^n$, we can define the vector in $\mathbb{R}_+^n$ given by

$$\bar{A}_{\bar{t}}(x,y):=\left(A_{t_1}(x,y),...,A_{t_n}(x,y)\right)$$

where $x,y\geq0$.
Applying, now, Theorem 4 for the convex mapping $f(x)=-\ln x$ and the linear functional $A:=\frac{1}{n}\sum_{i=1}^n t_i$, we get the inequality

$$A_{\tilde{t}}(x,y)\geq G_n\left(\bar{A}_{\bar{t}}(x,y)\right)\geq G_{\tilde{t}}(x,y) \tag{2.10}$$

where $\tilde{t}:=\frac{1}{n}\sum_{i=1}^n t_i\in[0,1]$ and $x,y\geq0$.
If we choose t_i so that $\tilde{t}=\frac{1}{2}$, we get

$$A(x,y)\geq G_n\left(\bar{A}_{\bar{t}}(x,y)\right)\geq G(x,y) \tag{2.11}$$

which is a discrete refinement of the classical $A.-G.$ inequality.
In addition, if we use Theorem 5, we can state that

$$\begin{aligned}A(x,y)&\geq G_n\left(A_{\bar{t}}(x,y),A_{\bar{t}}(y,x)\right)\\&\geq G\left(G_n\left(\bar{A}_{\bar{t}}(x,y)\right),G_n\left(\bar{A}_{\bar{t}}(y,x)\right)\right)\geq G(x,y),\end{aligned} \tag{2.12}$$

which is another refinement of the $A.-G.$ inequality.

3. The Concepts of $m-\Psi-$Convex and $M-\Psi-$Convex Functions

3.1. **Some Preliminary Results.** Assume that the mapping $\Psi:I\subseteq\mathbb{R}\to\mathbb{R}$ (I is an interval) is convex on I and $m\in\mathbb{R}$. We shall say that the mapping $\phi:I\to\mathbb{R}$ is $m-\Psi-$ *lower convex* if $\phi-m\Psi$ is a convex mapping on I. We may introduce the class of functions [6]

$$\mathcal{L}(I,m,\Psi):=\{\phi:I\to\mathbb{R}|\phi-m\Psi\ \text{ is convex on } I\}. \tag{3.1}$$

Similarly, for $M\in\mathbb{R}$ and Ψ as above, we can introduce the class of $M-\Psi-$*upper convex* functions by [6]

$$\mathcal{U}(I,M,\Psi):=\{\phi:I\to\mathbb{R}|M\Psi-\phi\ \text{ is convex on } I\}. \tag{3.2}$$

The intersection of these two classes will be called the class of $(m,M)-\Psi-$*convex functions* and will be denoted by [6]

$$\mathcal{B}(I,m,M,\Psi):=\mathcal{L}(I,m,\Psi)\cap\mathcal{U}(I,M,\Psi). \tag{3.3}$$

Remark 5. *If $\Psi\in\mathcal{B}(I,m,M,\Psi)$, then $\phi-m\Psi$ and $M\Psi-\phi$ are convex and then $(\phi-m\Psi)+(M\Psi-\phi)$ is also convex which shows that $(M-m)\Psi$ is convex, implying that $M\geq m$ (as Ψ is assumed not to be the trivial convex function $\Psi(t)=0$, $t\in I$).*

The above concepts may be introduced in the general case of a convex subset in a real linear space, but we do not consider this extension here.

In [10], S.S. Dragomir and N.M. Ionescu introduced the concept of $g-$*convex dominated mappings,* for a mapping $f : I \to \mathbb{R}$. We recall this, by saying, for a given convex function $g : I \to \mathbb{R}$, the function $f : I \to \mathbb{R}$ is $g-$*convex dominated* iff $g + f$ and $g - f$ are convex mappings on I. In [10], the authors pointed out a number of inequalities for convex dominated functions related to Jensen's, Fuchs', Pečarić's, Barlow-Proschan and Vasić-Mijalković results, etc.

We observe that the concept of $g-$convex dominated functions can be obtained as a particular case from $(m, M)-\Psi-$convex functions by choosing $m = -1$, $M = 1$ and $\Psi = g$.

The following lemma holds [6].

Lemma 3. *Let $\Psi, \phi : I \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable functions on $\mathring{I}$ and Ψ is a convex function on $\mathring{I}$.*

(i) *For $m \in \mathbb{R}$, the function $\phi \in \mathcal{L}\left(\mathring{I}, m, \Psi\right)$ iff*

$$m\left[\Psi(x) - \Psi(y) - \Psi'(y)(x - y)\right] \le \phi(x) - \phi(y) - \phi'(y)(x - y) \tag{3.4}$$

for all $x, y \in \mathring{I}$.

(ii) *For $M \in \mathbb{R}$, the function $\phi \in \mathcal{U}\left(\mathring{I}, M, \Psi\right)$ iff*

$$\phi(x) - \phi(y) - \phi'(y)(x - y) \le M\left[\Psi(x) - \Psi(y) - \Psi'(y)(x - y)\right] \tag{3.5}$$

for all $x, y \in \mathring{I}$.

(iii) *For $M, m \in \mathbb{R}$ with $M \ge m$, the function $\phi \in \mathcal{B}\left(\mathring{I}, m, M, \Psi\right)$ iff both (3.4) and (3.5) hold.*

Proof. Follows by the fact that a differentiable mapping $h : I \to \mathbb{R}$ is convex on $\mathring{I}$ iff $h(x) - h(y) \ge h'(y)(x - y)$ for all $x, y \in \mathring{I}$. □

Another elementary fact for twice differentiable functions also holds [6].

Lemma 4. *Let $\Psi, \phi : I \subseteq \mathbb{R} \to \mathbb{R}$ be twice differentiable on $\mathring{I}$ and Ψ is convex on $\mathring{I}$.*

(i) *For $m \in \mathbb{R}$, the function $\phi \in \mathcal{L}\left(\mathring{I}, m, \Psi\right)$ iff*

$$m\Psi''(t) \le \phi''(t) \quad \text{for all } t \in \mathring{I}. \tag{3.6}$$

(ii) *For $M \in \mathbb{R}$, the function $\phi \in \mathcal{U}\left(\mathring{I}, M, \Psi\right)$ iff*

$$\phi''(t) \le M\Psi''(t) \quad \text{for all } t \in \mathring{I}. \tag{3.7}$$

(iii) *For $M, m \in \mathbb{R}$ with $M \ge m$, the function $\phi \in \mathcal{B}\left(\mathring{I}, m, M, \Psi\right)$ iff both (3.6) and (3.7) hold.*

Proof. Follows by the fact that a twice differentiable function $h : I \to \mathbb{R}$ is convex on $\mathring{I}$ iff $h''(t) \ge 0$ for all $t \in \mathring{I}$. □

We consider the $p-$logarithmic mean of two positive numbers given by

$$L_p(a, b) := \begin{cases} a & \text{if } \quad b = a, \\ \\ \left[\dfrac{b^{p+1} - a^{p+1}}{(p+1)(b - a)}\right]^{\frac{1}{p}} & \text{if } \quad a \ne b \end{cases}$$

and $p\in\mathbb{R}\diagdown\{-1,0\}$.

The following proposition holds [6].

Proposition 1. *Let $\phi:(0,\infty)\to\mathbb{R}$ be a differentiable mapping.*

(i) *For $m\in\mathbb{R}$, the function $\phi\in\mathcal{L}\left((0,\infty),m,(\cdot)^p\right)$ with $p\in(-\infty,0)\cup(1,\infty)$ iff*

$$mp\left(x-y\right)\left[L_{p-1}^{p-1}\left(x,y\right)-y^{p-1}\right]\leq\phi\left(x\right)-\phi\left(y\right)-\phi'\left(y\right)\left(x-y\right) \tag{3.8}$$

for all $x,y\in(0,\infty)$.

(ii) *For $M\in\mathbb{R}$, the function $\phi\in\mathcal{U}\left((0,\infty),M,(\cdot)^p\right)$ with $p\in(-\infty,0)\cup(1,\infty)$ iff*

$$\phi\left(x\right)-\phi\left(y\right)-\phi'\left(y\right)\left(x-y\right)\leq Mp\left(x-y\right)\left[L_{p-1}^{p-1}\left(x,y\right)-y^{p-1}\right] \tag{3.9}$$

for all $x,y\in(0,\infty)$.

(iii) *For $M,m\in\mathbb{R}$ with $M\geq m$, the function $\phi\in\mathcal{B}\left((0,\infty),M,(\cdot)^p\right)$ with $p\in(-\infty,0)\cup(1,\infty)$ iff both (3.8) and (3.9) hold.*

The proof follows by Lemma 3 applied for the convex mapping $\Psi(t)=t^p$, $p\in(-\infty,0)\cup(1,\infty)$ and via some elementary computation. We omit the details.

The following corollary is useful in practice [6].

Corollary 1. *Let $\phi:(0,\infty)\to\mathbb{R}$ be a differentiable function.*

(i) *For $m\in\mathbb{R}$, the function ϕ is $m-$quadratic-lower convex (i.e., for $p=2$) iff*

$$m\left(x-y\right)^2\leq\phi\left(x\right)-\phi\left(y\right)-\phi'\left(y\right)\left(x-y\right) \tag{3.10}$$

for all $x,y\in(0,\infty)$.

(ii) *For $M\in\mathbb{R}$, the function ϕ is $M-$quadratic-upper convex iff*

$$\phi\left(x\right)-\phi\left(y\right)-\phi'\left(y\right)\left(x-y\right)\leq M\left(x-y\right)^2 \tag{3.11}$$

for all $x,y\in(0,\infty)$.

(iii) *For $m,M\in\mathbb{R}$ with $M\geq m$, the function ϕ is $(m,M)-$quadratic convex if both (3.10) and (3.11) hold.*

The following proposition holds [6].

Proposition 2. *Let $\phi:(0,\infty)\to\mathbb{R}$ be a twice differentiable function.*

(i) *For $m\in\mathbb{R}$, the function $\phi\in\mathcal{L}\left((0,\infty),m,(\cdot)^p\right)$ with $p\in(-\infty,0)\cup(1,\infty)$ iff*

$$p\left(p-1\right)mt^{p-2}\leq\phi''\left(t\right)\quad\text{for all } t\in(0,\infty). \tag{3.12}$$

(ii) *For $M\in\mathbb{R}$, the function $\phi\in\mathcal{U}\left((0,\infty),M,(\cdot)^p\right)$ with $p\in(-\infty,0)\cup(1,\infty)$ iff*

$$\phi''\left(t\right)\leq p\left(p-1\right)Mt^{p-2}\quad\text{for all } t\in(0,\infty). \tag{3.13}$$

(iii) *For $m,M\in\mathbb{R}$ with $M\geq m$, the function $\phi\in\mathcal{B}\left((0,\infty),m,M,(\cdot)^p\right)$ with $p\in(-\infty,0)\cup(1,\infty)$ iff both (3.12) and (3.13) hold.*

As can be easily seen, the above proposition offers the practical criterion of deciding when a twice differentiable mapping is $(\cdot)^p-$lower or $(\cdot)^p-$upper convex and which weights the constant m and M are.

The following corollary is useful in practice [6].

Corollary 2. *Assume that the mapping $\phi : (a,b) \subseteq \mathbb{R} \to \mathbb{R}$ is twice differentiable.*

(i) If $\inf_{t \in (a,b)} \phi''(t) = k > -\infty$, *then* ϕ *is* $\frac{k}{2}-$*quadratic lower convex on* (a,b);

(ii) If $\sup_{t \in (a,b)} \phi''(t) = K < \infty$, *then* ϕ *is* $\frac{K}{2}-$*quadratic upper convex on* (a,b).

4. Jessen's Inequality for $m-\Psi-$Convex and $M-\Psi-$Convex Functions

4.1. **Some Jessen Type Inequality.** We start with the following result [7].

Theorem 6 (Dragomir, 2002, [7])**.** *Let $\Psi : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function and $f : E \to I$ such that $\Psi \circ f$, $f \in L$ and $A : L \to \mathbb{R}$ be an isotonic linear and normalised functional.*

(i) If $\phi \in \mathcal{L}(I,m,\Psi)$ and $\phi \circ f \in L$, then we have the inequality

$$m\left[A(\Psi \circ f) - \Psi(A(f))\right] \le A(\phi \circ f) - \phi(A(f)). \tag{4.1}$$

(ii) If $\phi \in \mathcal{U}(I,M,\Psi)$ and $\phi \circ f \in L$, then we have the inequality

$$A(\phi \circ f) - \phi(A(f)) \le M\left[A(\Psi \circ f) - \Psi(A(f))\right]. \tag{4.2}$$

(iii) If $\phi \in \mathcal{B}(I,m,M,\Psi)$ and $\phi \circ f \in L$, then both (4.1) and (4.2) hold.

Proof. The proof is as follows.

(i) As $\phi \in \mathcal{L}(I,m,\Psi)$ and $\phi \circ f \in L$, it follows that $\phi - m\Psi$ is convex and $(\phi - m\Psi) \circ f \in L$.

Applying Jessen's inequality for the convex mapping $\phi - m\Psi$, we get

$$(\phi - m\Psi)(A(f)) \le A\left[(\phi - m\Psi) \circ f\right]. \tag{4.3}$$

However,

$$(\phi - m\Psi)(A(f)) = \phi(A(f)) - m\Psi(A(f))$$

and

$$A\left[(\phi - m\Psi) \circ f\right] = A(\phi \circ f) - mA(\phi \circ f)$$

and then, by (3.3), we deduce the desired result (4.1).

(ii) Follows in a similar manner by taking into account that $\phi \circ f \in L$ and $\phi \in \mathcal{U}(I,M,\Psi)$ imply $M\Psi - \phi$ is convex and $(M\Psi - \phi) \circ f \in L$.

(iii) Follows by (i) and (ii).

□

The following corollary is useful in practice [7].

Corollary 3. *Let $\Psi : I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable convex function on $\mathring{I}$, $f : E \to I$ such that $\Psi \circ f$, $f \in L$ and $A : L \to \mathbb{R}$ be an isotonic linear and normalised functional.*

(i) If $\phi : I \to \mathbb{R}$ is twice differentiable and $\phi''(t) \ge m\Psi''(t)$, $t \in \mathring{I}$ (where m is a given real number), then (4.1) holds, provided that $\phi \circ f \in L$.

(ii) If $\phi : I \to \mathbb{R}$ is twice differentiable and $\phi''(t) \le M\Psi''(t)$, $t \in \mathring{I}$ (where M is a given real number), then (4.2) holds, provided that $\phi \circ f \in L$.

(iii) If $\phi : I \to \mathbb{R}$ is twice differentiable and $m\Psi''(t) \le \phi''(t) \le M\Psi''(t)$, $t \in \mathring{I}$, then both (4.1) and (4.2) hold, provided $\phi \circ f \in L$.

Some particular important cases of the above corollary are embodied in the following propositions [7].

Proposition 3. *Assume that the mapping $\phi : I \subseteq \mathbb{R} \to \mathbb{R}$ is twice differentiable on $\mathring{I}$.*

(i) *If $\inf_{t\in\mathring{I}} \phi''(t) = k > -\infty$, then we have the inequality*

$$\frac{k}{2}\left[A\left(f^2\right) - [A(f)]^2\right] \le A(\phi\circ f) - \phi(A(f)) \tag{4.4}$$

provided that $\phi\circ f, f^2, f \in L$.

(ii) *If $\sup_{t\in\mathring{I}} \phi''(t) = K < \infty$, then we have the inequality*

$$A(\phi\circ f) - \phi(A(f)) \le \frac{K}{2}\left[A\left(f^2\right) - [A(f)]^2\right] \tag{4.5}$$

provided that $\phi\circ f, f^2, f \in L$.

(iii) *If $-\infty < k \le \phi''(t) \le K < \infty$, $t \in \mathring{I}$, then both (4.4) and (4.5) hold, provided that $\phi\circ f, f^2, f \in L$.*

The proof follows by Corollary 3 applied for $\Psi(t) = \frac{1}{2}t^2$ and $m = k$, $M = K$.

Another result is the following one [7].

Proposition 4. *Assume that the mapping $\phi : I \subseteq (0,\infty) \to \mathbb{R}$ is twice differentiable on $\mathring{I}$. Let $p \in (-\infty, 0) \cup (1,\infty)$ and define $g_p : I \to \mathbb{R}$, $g_p(t) = \phi''(t)\,t^{2-p}$.*

(i) *If $\inf_{t\in\mathring{I}} g_p(t) = \gamma > -\infty$, then we have the inequality*

$$\frac{\gamma}{p(p-1)}\left[A(f^p) - [A(f)]^p\right] \le A(\phi\circ f) - \phi(A(f)) \tag{4.6}$$

provided that $\phi\circ f, f^p, f \in L$.

(ii) *If $\sup_{t\in\mathring{I}} g_p(t) = \Gamma < \infty$, then we have the inequality*

$$A(\phi\circ f) - \phi(A(f)) \le \frac{\Gamma}{p(p-1)}\left[A(f^p) - [A(f)]^p\right] \tag{4.7}$$

provided that $\phi\circ f, f^p, f \in L$.

(iii) *If $-\infty < \gamma \le g_p(t) \le \Gamma < \infty$, $t \in \mathring{I}$, then both (4.6) and (4.7) hold, provided that $\phi\circ f, f^p, f \in L$.*

Proof. The proof is as follows.

(i) We have for the auxiliary mapping $h_p(t) = \phi(t) - \frac{\gamma}{p(p-1)}t^p$ that

$$\begin{aligned} h_p''(t) &= \phi''(t) - \gamma t^{p-2} = t^{p-2}\left(t^{2-p}\phi''(t) - \gamma\right) \\ &= t^{p-2}\left(g_p(t) - \gamma\right) \ge 0. \end{aligned}$$

That is, h_p is convex or, equivalently, $\phi \in \mathcal{L}\left(I, \frac{\gamma}{p(p-1)}, (\cdot)^p\right)$. Applying Corollary 3, we deduce (4.6).

(ii) Goes similarly.

(iii) Follows by (i) and (ii).

□

The following proposition also holds [7].

Proposition 5. *Assume that the mapping $\phi : I \subseteq (0,\infty) \to \mathbb{R}$ is twice differentiable on $\mathring{I}$. Define $l(t) = t^2\phi''(t)$, $t \in I$.*

(i) *If* $\inf_{t\in\mathring{I}} l(t) = s > -\infty$, *then we have the inequality*

$$s\left[\ln\left[A(f)\right] - A(\ln f)\right] \leq A(\phi\circ f) - \phi(A(f)), \tag{4.8}$$

provided that $\phi\circ f, \ln f, f \in L$ *and* $A(f) > 0$.

(ii) *If* $\sup_{t\in\mathring{I}} l(t) = S < \infty$, *then we have the inequality*

$$A(\phi\circ f) - \phi(A(f)) \leq S\left[\ln\left[A(f)\right] - A(\ln f)\right], \tag{4.9}$$

provided that $\phi\circ f, \ln f, f \in L$ *and* $A(f) > 0$.

(iii) *If* $-\infty < s \leq l(t) \leq S < \infty$ *for* $t \in\mathring{I}$, *then both (4.8) and (4.9) hold, provided that* $\phi\circ f, \ln f, f \in L$ *and* $A(f) > 0$.

Proof. The proof is as follows.

(i) Define the auxiliary mapping $h(t) = \phi(t) + s\ln t$. Then

$$h''(t) = \phi''(t) - \frac{s}{t^2} = \frac{1}{t^2}\left(\phi''(t)\,t^2 - s\right) \geq 0$$

which shows that h is convex or, equivalently, $\phi \in \mathcal{L}(I, s, -\ln(\cdot))$. Applying Corollary 3, we deduce (4.8).

(ii) Goes similarly.

(iii) Follows by (i) and (ii).

□

Finally, the following result also holds [7].

Proposition 6. *Assume that the mapping* $\phi : I \subseteq (0,\infty) \to \mathbb{R}$ *is twice differentiable on* $\mathring{I}$. *Define* $\tilde{I}(t) = t\phi''(t)$, $t \in I$.

(i) *If* $\inf_{t\in\mathring{I}} \tilde{I}(t) = \delta > -\infty$, *then we have the inequality*

$$\delta\left[A\left[f\ln f\right] - A(f)\ln A(f)\right] \leq A(\phi\circ f) - \phi(A(f)), \tag{4.10}$$

provided that $\phi\circ f, f\ln f, f \in L$ *and* $A(f) > 0$.

(ii) *If* $\sup_{t\in\mathring{I}} \tilde{I}(t) = \Delta < \infty$, *then we have the inequality*

$$A(\phi\circ f) - \phi(A(f)) \leq \Delta\left[A\left[f\ln f\right] - A(f)\ln A(f)\right], \tag{4.11}$$

provided that $\phi\circ f, f\ln f, f \in L$ *and* $A(f) > 0$.

(iii) *If* $-\infty < \delta \leq \tilde{I}(t) \leq \Delta < \infty$ *for* $t \in\mathring{I}$, *then both (4.10) and (4.11) hold, provided that* $\phi\circ f, f\ln f, f \in L$ *and* $A(f) > 0$.

Proof. The proof is as follows.

(i) Define the auxiliary mapping $h(t) = \phi(t) + \delta t\ln t$, $t \in I$. Then

$$h''(t) = \phi''(t) - \frac{\delta}{t} = \frac{1}{t^2}\left[\phi''(t)\,t - \delta\right] = \frac{1}{t}\left[\tilde{I}(t) - \delta\right] \geq 0$$

which shows that h is convex or, equivalently, $\phi \in \mathcal{L}(I, \delta, (\cdot)\ln(\cdot))$. Applying Corollary 3, we deduce (4.10).

(ii) Goes similarly.

(iii) Follows by (i) and (ii).

□

4.2. **Some Particular Inequalities.** We know, by Proposition 3, that

$$\frac{1}{2}k\left[A\left(f^{2}\right)-[A(f)]^{2}\right]\leq A(\phi\circ f)-\phi(A(f))\leq\frac{1}{2}K\left[A\left(f^{2}\right)-[A(f)]^{2}\right],\tag{4.12}$$

provided that $\phi : I \subseteq \mathbb{R} \to \mathbb{R}$ is twice differentiable on $\mathring{I}$, $-\infty < k \leq \phi''(t) \leq K < \infty$, $t \in \mathring{I}$, $f : E \to I$, $\phi \circ f$, f^2, $f \in L$ and $A : L \to \mathbb{R}$ is an isotonic linear and normalised functional.

The following inequalities have been established in [7].

(1) We assume that $0 < m \leq f \leq M < \infty$, where m, M are real numbers. Then, by (4.12) applied for $\phi : [m, M] \to \mathbb{R}$, $\phi(t) = -\ln t$, we have the inequality

$$\frac{1}{2M^{2}}\left[A\left(f^{2}\right)-[A(f)]^{2}\right]\leq\ln[A(f)]-A[\ln(f)]\leq\frac{1}{2m^{2}}\left[A\left(f^{2}\right)-[A(f)]^{2}\right],\tag{4.13}$$

provided that $\ln f, f^2, f \in L$ and $A(f) > 0$.

Note that (4.13) is equivalent to

$$\exp\left[\frac{1}{2M^{2}}\left[A\left(f^{2}\right)-[A(f)]^{2}\right]\right]\leq\frac{[A(f)]}{\exp[A[\ln(f)]]}\leq\exp\left[\frac{1}{2m^{2}}\left[A\left(f^{2}\right)-[A(f)]^{2}\right]\right].\tag{4.14}$$

(2) If we apply (4.12) for $\phi : [m, M] \to \mathbb{R}$, $\phi(t) = t^p$, $p \in (-\infty, 0) \cup (1, \infty)$, then we have the inequality:

$$\frac{p(p-1)}{2}m^{p-2}\left[A\left(f^{2}\right)-[A(f)]^{2}\right]\leq A(f^{p})-[A(f)]^{p}\leq\frac{p(p-1)}{2}M^{p-2}\left[A\left(f^{2}\right)-[A(f)]^{2}\right]\tag{4.15}$$

if $p > 2$, and

$$\frac{p(p-1)}{2}M^{p-2}\left[A\left(f^{2}\right)-[A(f)]^{2}\right]\leq A(f^{p})-[A(f)]^{p}\leq\frac{p(p-1)}{2}m^{p-2}\left[A\left(f^{2}\right)-[A(f)]^{2}\right]\tag{4.16}$$

if $p \in (-\infty, 0) \cup (1, 2)$, provided that f^2, f^p, $f \in L$.

(3) If we apply (4.13) for $\phi : [m, M] \to \mathbb{R}$, $\phi(t) = t \ln t$, then we have the inequality

$$\frac{1}{2M}\left[A\left(f^{2}\right)-[A(f)]^{2}\right]\leq A[f\ln f]-A(f)\ln A(f)\leq\frac{1}{2m}\left[A\left(f^{2}\right)-[A(f)]^{2}\right],\tag{4.17}$$

provided that $f \ln f, f^2, f \in L$ and $A(f) > 0$.

Note that the inequality (4.17) is equivalent with

$$\exp\left\{\frac{1}{2M}\left[A\left(f^2\right)-\left[A\left(f\right)\right]^2\right]\right\}\leq\frac{\exp\left[A\left(f\ln f\right)\right]}{\left[A\left(f\right)\right]^{A(f)}}\leq\exp\left\{\frac{1}{2m}\left[A\left(f^2\right)-\left[A\left(f\right)\right]^2\right]\right\}. \tag{4.18}$$

(4) If we assume that $-\infty<m\leq f\leq M<\infty$, and apply the inequality (4.12) for $\phi\left(t\right)=e^t$, $t\in\mathbb{R}$, we obtain

$$\frac{1}{2}\exp\left(m\right)\left[A\left(f^2\right)-\left[A\left(f\right)\right]^2\right]\leq A\left(\exp\left(f\right)\right)-\exp\left(A\left(f\right)\right)\leq\frac{1}{2}\exp\left(M\right)\left[A\left(f^2\right)-\left[A\left(f\right)\right]^2\right], \tag{4.19}$$

provided that $\exp\left(f\right)$, $f^2, f\in L$.

Using Proposition 4, we may state that

$$\frac{\gamma}{p\left(p-1\right)}\left[A\left(f^p\right)-\left[A\left(f\right)\right]^p\right]\leq A\left(\phi\circ f\right)-\phi\left(A\left(f\right)\right)\leq\frac{\Gamma}{p\left(p-1\right)}\left[A\left(f^p\right)-\left[A\left(f\right)\right]^p\right], \tag{4.20}$$

provided that $\phi:I\subseteq\mathbb{R}\to\mathbb{R}$ is twice differentiable on $\mathring{I}$, $\gamma\leq\phi''\left(t\right)t^{2-p}\leq\Gamma$, $t\in\mathring{I}$, $f:E\to I$, $\phi\circ f$, f^p, $f\in L$ and $A:L\to\mathbb{R}$ is an isotonic linear and normalised functional.

5. If we consider $\phi\left(t\right)=-\ln t$ and assume that $0<m\leq f\leq M<\infty$, then

$$\frac{m^{-p}}{p\left(p-1\right)}\left[A\left(f^p\right)-\left[A\left(f\right)\right]^p\right]\leq\ln\left(A\left(f\right)\right)-A\left(\ln\left(f\right)\right)\leq\frac{M^{-p}}{p\left(p-1\right)}\left[A\left(f^p\right)-\left[A\left(f\right)\right]^p\right] \tag{4.21}$$

if $p\in\left(-\infty,0\right)$ and

$$\frac{M^{-p}}{p\left(p-1\right)}\left[A\left(f^p\right)-\left[A\left(f\right)\right]^p\right]\leq\ln\left(A\left(f\right)\right)-A\left(\ln\left(f\right)\right)\leq\frac{m^{-p}}{p\left(p-1\right)}\left[A\left(f^p\right)-\left[A\left(f\right)\right]^p\right] \tag{4.22}$$

if $p\in\left(1,\infty\right)$, provided that $f^p,\ln f,f\in L$ and $A\left(f\right)>0$.

6. If we consider $\phi\left(t\right)=t\ln t$ and assume that $0<m\leq f\leq M<\infty$, then

$$\frac{m^{1-p}}{p\left(p-1\right)}\left[A\left(f^p\right)-\left[A\left(f\right)\right]^p\right]\leq A\left(f\ln f\right)-A\left(f\right)\ln A\left(f\right)\leq\frac{M^{1-p}}{p\left(p-1\right)}\left[A\left(f^p\right)-\left[A\left(f\right)\right]^p\right] \tag{4.23}$$

if $p\in\left(-\infty,0\right)$ and

$$\frac{M^{1-p}}{p\left(p-1\right)}\left[A\left(f^p\right)-\left[A\left(f\right)\right]^p\right]\leq A\left(f\ln f\right)-A\left(f\right)\ln A\left(f\right)\leq\frac{m^{1-p}}{p\left(p-1\right)}\left[A\left(f^p\right)-\left[A\left(f\right)\right]^p\right] \tag{4.24}$$

if $p\in(1,\infty)$, $f\ln f$, f^p, $f\in L$ and $A(f)>0$.

Finally, by Proposition 5, we may state the inequality

$$s\left[\ln\left[A(f)\right]-A(\ln f)\right]\le A(\phi\circ f)-\phi(A(f))\le S\left[\ln\left[A(f)\right]-A(\ln f)\right], \tag{4.25}$$

provided that $\phi:I\subseteq(0,\infty)$ is twice differentiable on $\mathring{I}$, $-\infty<s\le t^2\phi''(t)\le S<\infty$, $\phi\circ f$, $\ln f$, $f\in L$ and $A(f)>0$.

7. If we assume that $0<m\le f\le M<\infty$ and apply (4.25) for $\phi(t)=t\ln t$, we have the inequality

$$\begin{aligned} m\left[\ln\left[A(f)\right]-A(\ln f)\right] &\le A(f\ln f)-A(f)\ln A(f) \\ &\le M\left[\ln\left[A(f)\right]-A(\ln f)\right] \end{aligned} \tag{4.26}$$

provided that $\ln f$, $f\ln f$, $f\in L$ and $A(f)>0$.

Note that (4.26) is equivalent to

$$\left(\frac{A(f)}{\exp\left[A(\ln f)\right]}\right)^m\le\frac{\exp\left[A(f\ln f)\right]}{\left[A(f)\right]^{A(f)}}\le\left(\frac{A(f)}{\exp\left[A(\ln f)\right]}\right)^M. \tag{4.27}$$

4.3. **Applications for Hermite-Hadamard Inequalities.** The following integral inequalities were obtained in [7].

a) Suppose that $\phi:[a,b]\subset\mathbb{R}\to\mathbb{R}$ is twice differentiable and $-\infty<k\le\phi''(t)\le K<\infty$ for $t\in(a,b)$. If in (4.12) we choose $f=e$, i.e., $f(x)=x$, $x\in[a,b]$ and $A(f):=\frac{1}{b-a}\int_a^b f(t)\,dt$ and take into account that

$$\begin{aligned} A\left(f^2\right)-\left(A(f)\right)^2 &= \frac{1}{b-a}\int_a^b x^2dx-\left(\frac{1}{b-a}\int_a^b xdx\right)^2 \\ &= \frac{(b-a)^2}{12}, \end{aligned}$$

then we get the inequality (see also [7, p. 40])

$$\frac{(b-a)^2}{24}\cdot k\le\frac{1}{b-a}\int_a^b\phi(x)\,dx-\phi\left(\frac{a+b}{2}\right)\le\frac{(b-a)^2}{24}\cdot K. \tag{4.28}$$

b) Now, if we assume that $\phi:[a,b]\subset(0,\infty)\to\mathbb{R}$ is twice differentiable over $\gamma\le t^{2-p}\phi''(t)\le\Gamma$, $t\in(a,b)$, $p\in(-\infty,0)\cup(1,\infty)$, then by (4.20), in which we choose $f=e$, $A(f):=\frac{1}{b-a}\int_a^b f(t)\,dt$ and taking into account that

$$\begin{aligned} A(f^p)-\left(A(f)\right)^p &= \frac{1}{b-a}\int_a^b x^pdx-\left(\frac{1}{b-a}\int_a^b xdx\right)^p \\ &= L_p^p(a,b)-A^p(a,b), \end{aligned}$$

we get

$$\begin{aligned} \frac{\gamma}{p(p-1)}\left[L_p^p(a,b)-A^p(a,b)\right] &\le \frac{1}{b-a}\int_a^b\phi(x)\,dx-\phi\left(\frac{a+b}{2}\right) \\ &\le \frac{\Gamma}{p(p-1)}\left[L_p^p(a,b)-A^p(a,b)\right]. \end{aligned} \tag{4.29}$$

c) If $\phi : [a,b] \subset (0,\infty) \to \mathbb{R}$ is twice differentiable and satisfies the condition $s \le t^2\phi''(t) \le S$, $t \in (a,b)$, then by Proposition 5 applied for $f = e$, $A(f) := \frac{1}{b-a}\int_a^b f(t)\,dt$ and taking into account that

$$\ln[A(f)] - A(\ln f) = \ln\left(\frac{1}{b-a}\int_a^b x dx\right) - \frac{1}{b-a}\int_a^b \ln x dx$$
$$= \ln[A(a,b)] - \ln I(a,b) = \ln\left[\frac{A(a,b)}{I(a,b)}\right],$$

we get the inequality

$$s\ln\left[\frac{A(a,b)}{I(a,b)}\right] \le \frac{1}{b-a}\int_a^b \phi(x)\,dx - \phi\left(\frac{a+b}{2}\right) \le S\ln\left[\frac{A(a,b)}{I(a,b)}\right] \tag{4.30}$$

or, equivalently,

$$\left[\frac{A(a,b)}{I(a,b)}\right]^s \le \frac{\exp\left[\frac{1}{b-a}\int_a^b \phi(x)\,dx\right]}{\exp\left[\phi\left(\frac{a+b}{2}\right)\right]} \le \left[\frac{A(a,b)}{I(a,b)}\right]^S. \tag{4.31}$$

d) Finally, if we assume that the twice differentiable function $\phi : [a,b] \subset (0,\infty) \to \mathbb{R}$ satisfies the condition $\delta \le t\phi''(t) \le \Delta$, $t \in (a,b)$, then by Proposition 6 and by the same selection of f and A and taking into account that

$$\begin{aligned}
&A(f\ln f) - A(f)\ln A(f) \\
&= \frac{1}{b-a}\int_a^b x\ln x dx - \frac{1}{b-a}\int_a^b x dx \cdot \frac{1}{b-a}\int_a^b \ln x dx \\
&= \frac{1}{4(b-a)}\left[b^2\ln b^2 - a^2\ln a^2 - \left(b^2-a^2\right)\right] - A(a,b)\ln I(a,b) \\
&= \frac{A(a,b)}{2(b^2-a^2)}\left[b^2\ln b^2 - a^2\ln a^2 - \left(b^2-a^2\right)\right] - A(a,b)\ln I(a,b) \\
&= \frac{A(a,b)}{2}\ln I\left(a^2,b^2\right) - A(a,b)\ln I(a,b) \\
&= \ln\left[\left(\frac{\sqrt{I(a^2,b^2)}}{I(a,b)}\right)^{A(a,b)}\right],
\end{aligned}$$

we may state the inequality

$$\begin{aligned}
\delta\ln\left[\left(\frac{\sqrt{I(a^2,b^2)}}{I(a,b)}\right)^{A(a,b)}\right] &\le \frac{1}{b-a}\int_a^b \phi(x)\,dx - \phi\left(\frac{a+b}{2}\right) \\
&\le \Delta\ln\left[\left(\frac{\sqrt{I(a^2,b^2)}}{I(a,b)}\right)^{A(a,b)}\right],
\end{aligned} \tag{4.32}$$

or, equivalently,

$$\left(\frac{\sqrt{I(a^2,b^2)}}{I(a,b)}\right)^{\delta A(a,b)} \le \frac{\exp\left[\frac{1}{b-a}\int_a^b \phi(x)\,dx\right]}{\exp\left[\phi\left(\frac{a+b}{2}\right)\right]} \le \left(\frac{\sqrt{I(a^2,b^2)}}{I(a,b)}\right)^{\Delta A(a,b)}. \tag{4.33}$$

5. Lupaş-Beesack-Pečarić Inequality for $m-\Psi-$Convex and $M-\Psi-$Convex Functions

5.1. **Some Lupaş-Beesack-Pečarić Type Inequalities.** We now prove the following result [8].

Theorem 7 (Dragomir, 2002, [8])**.** *Let $\Psi : [\alpha,\beta] \subset \mathbb{R} \to \mathbb{R}$ be a convex function and $f : I \to [\alpha,\beta]$ such that $\Psi \circ f$, $f \in L$ and $A : L \to \mathbb{R}$ is an isotonic linear and normalised functional.*

(*i*) *If $\phi \in \mathcal{L}(I,m,\Psi)$ and $\phi \circ f \in L$, then we have the inequality*

$$m\left[\frac{\beta - A(f)}{\beta-\alpha}\Psi(\alpha) + \frac{A(f)-\alpha}{\beta-\alpha}\Psi(\beta) - A(\Psi\circ f)\right] \leq \frac{\beta - A(f)}{\beta-\alpha}\phi(\alpha) + \frac{A(f)-\alpha}{\beta-\alpha}\phi(\beta) - A(\phi\circ f). \tag{5.1}$$

(*ii*) *If $\phi \in \mathcal{U}(I,M,\Psi)$ and $\phi \circ f \in L$, then*

$$\frac{\beta - A(f)}{\beta-\alpha}\phi(\alpha) + \frac{A(f)-\alpha}{\beta-\alpha}\phi(\beta) - A(\phi\circ f) \leq M\left[\frac{\beta - A(f)}{\beta-\alpha}\Psi(\alpha) + \frac{A(f)-\alpha}{\beta-\alpha}\Psi(\beta) - A(\Psi\circ f)\right]. \tag{5.2}$$

(*iii*) *If $\phi \in \mathcal{B}(I,m,M,\Psi)$ and $\phi \circ f \in L$, then both (5.1) and (5.2) hold.*

Proof. The proof is as follows.

(*i*) As $\phi \in \mathcal{L}(I,m,\Psi)$ and $\phi \circ f \in L$, it follows that $\phi - m\Psi$ is convex and $(\phi - m\Psi)\circ f \in L$.
Applying Lupaş-Beesack-Pečarić's inequality for the convex function $\phi - m\Psi$, we get

$$A((\phi - m\Psi)\circ f) \leq \frac{\beta - A(f)}{\beta-\alpha}(\phi - m\Psi)(\alpha) + \frac{A(f)-\alpha}{\beta-\alpha}(\phi - m\Psi)(\beta). \tag{5.3}$$

However,

$$A((\phi - m\Psi)\circ f) = A(\phi\circ f) - mA(\Psi\circ f)$$

and then, after some simple computation, (5.3) is equivalent to (5.1).

(*ii*) Goes likewise and we omit the details.

(*iii*) Follows by (*i*) and (*ii*).

□

The following corollary is useful in practice [8].

Corollary 4. *Let $\Psi : I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable convex function on $\mathring{I}$, $f : E \to I$ such that $\Psi \circ f$, $f \in L$ and $A : L \to \mathbb{R}$ is an isotonic linear and normalised functional.*

(*i*) *If $\phi : I \to \mathbb{R}$ is twice differentiable, $\phi \circ f \in L$ and $\phi''(t) \geq m\Psi''(t)$, $t \in \mathring{I}$ (where m is a given real number), then (5.1) holds.*

(*ii*) *If $\phi : I \to \mathbb{R}$ is twice differentiable, $\phi \circ f \in L$ and $\phi''(t) \leq M\Psi''(t)$, $t \in \mathring{I}$ (where m is a given real number), then (5.2) holds.*

(*iii*) *If $m\Psi''(t) \leq \phi''(t) \leq M\Psi''(t)$, $t \in \mathring{I}$, then both (5.1) and (5.2) hold.*

Some particular important cases of the above corollary are embodied in the following propositions [8].

Proposition 7. *Assume that the function $\phi : I \subseteq \mathbb{R} \to \mathbb{R}$ is twice differentiable on $\mathring{I}$.*

(*i*) *If $\inf_{t\in\mathring{I}} \phi''(t) = k > -\infty$, then we have the inequality:*

$$\frac{k}{2}\left[(\alpha+\beta) A(f) - \alpha\beta - A\left(f^2\right)\right] \leq \frac{\beta - A(f)}{\beta-\alpha}\phi(\alpha) + \frac{A(f)-\alpha}{\beta-\alpha}\phi(\beta) - A(\phi\circ f), \tag{5.4}$$

provided that $\phi\circ f, f^2, f \in L$.

(*ii*) *If $\sup_{t\in\mathring{I}} \phi''(t) = K < \infty$, then we have the inequality*

$$\frac{\beta - A(f)}{\beta-\alpha}\phi(\alpha) + \frac{A(f)-\alpha}{\beta-\alpha}\phi(\beta) - A(\phi\circ f) \leq \frac{K}{2}\left[(\alpha+\beta) A(f) - \alpha\beta - A\left(f^2\right)\right]. \tag{5.5}$$

provided that $\phi\circ f, f^2, f \in L$.

(*iii*) *If $-\infty < k \leq \phi''(t) \leq K < \infty$, $t \in \mathring{I}$, then both (5.4) and (5.5) hold, provided that $\phi\circ f, f^2, f \in L$.*

Proof. The proof is as follows.

(*i*) Consider the auxiliary mapping $h(t) := \phi(t) - \frac{1}{2}kt^2$. Then $h''(t) = \phi''(t) - k \geq 0$ i.e., h is convex, or, equivalently, $\phi \in \mathcal{L}\left(I, \frac{1}{2}k, (\cdot)^2\right)$. Applying Corollary 4, we may state

$$\frac{k}{2}\left[\frac{\beta - A(f)}{\beta-\alpha}\alpha^2 + \frac{A(f)-\alpha}{\beta-\alpha}\beta^2 - A\left(f^2\right)\right] \leq \frac{\beta - A(f)}{\beta-\alpha}\phi(\alpha) + \frac{A(f)-\alpha}{\beta-\alpha}\phi(\beta) - A(\phi\circ f),$$

which is clearly equivalent to (5.4)

(*ii*) Goes likewise and we omit the details.

(*iii*) Follows by (*i*) and (*ii*).

□

Another result is the following one [8].

Proposition 8. *Assume that the mapping $\phi : [\alpha,\beta] \subset (0,\infty) \to \mathbb{R}$ is twice differentiable on (α,β), Let $p \in (-\infty,0)\cup(1,\infty)$ and define $g_p : [\alpha,\beta] \to \mathbb{R}$, $g_p(t) = \phi''(t)\, t^{2-p}$.*

(*i*) *If $\inf_{t\in\mathring{I}} g_p(t) = \gamma > -\infty$, then we have the inequality*

$$\frac{\gamma}{p(p-1)}\left[pL_{p-1}^{p-1}(\alpha,\beta) A(f) - \alpha\beta(p-1) L_{p-2}^{p-2}(\alpha,\beta) - A(f^p)\right] \leq \frac{\beta - A(f)}{\beta-\alpha}\phi(\alpha) + \frac{A(f)-\alpha}{\beta-\alpha}\phi(\beta) - A(\phi\circ f), \tag{5.6}$$

provided that $\phi\circ f, f^p, f \in L$.

(ii) *If* $\sup_{t\in\mathring{I}} g_p(t) = \Gamma < \infty$, *then we have the inequality*

$$\frac{\beta - A(f)}{\beta-\alpha}\phi(\alpha) + \frac{A(f)-\alpha}{\beta-\alpha}\phi(\beta) - A(\phi\circ f) \le \frac{\Gamma}{p(p-1)}\left[pL_{p-1}^{p-1}(\alpha,\beta)A(f) - \alpha\beta(p-1)L_{p-2}^{p-2}(\alpha,\beta) - A(f^p)\right]. \tag{5.7}$$

(iii) *If* $-\infty < \gamma \le g_p(t) \le \Gamma < \infty$, $t \in \mathring{I}$, *then we have both (5.6) and (5.7).*

Proof. The proof is as follows.

(i) Consider the auxiliary mapping $h_p(t) = \phi(t) - \frac{\gamma}{p(p-1)}t^p$. Then

$$h_p''(t) = \phi''(t) - \gamma t^{p-2} = t^{p-2}\left(t^{2-p}\phi''(t) - \gamma\right) = t^{p-2}\left(g_p(t) - \gamma\right) \ge 0.$$

That is, h_p is convex, or, equivalently, $\phi \in \mathcal{L}\left(I, \frac{\gamma}{p(p-1)}, (\cdot)^p\right)$. Applying Corollary 4, we may state

$$\frac{\gamma}{p(p-1)}\left[\frac{\beta - A(f)}{\beta-\alpha}\alpha^p + \frac{A(f)-\alpha}{\beta-\alpha}\beta^p - A(f^p)\right] \le \frac{\beta - A(f)}{\beta-\alpha}\phi(\alpha) + \frac{A(f)-\alpha}{\beta-\alpha}\phi(\beta) - A(\phi\circ f),$$

which is equivalent to (5.6).

(ii) Goes likewise.

(iii) Follows by (i) and (ii).

□

The following proposition also holds [8].

Proposition 9. *Assume that the mapping* $\phi : [\alpha,\beta] \subset (0,\infty) \to \mathbb{R}$ *is twice differentiable on* (α,β). *Define* $l(t) = t^2\phi''(t)$, $t \in [\alpha,\beta]$.

(i) *If* $\inf_{t\in(\alpha,\beta)} l(t) = s > -\infty$, *then we have the inequality*

$$s\left[A(\ln f) + \ln\left[I\left(\frac{1}{\alpha},\frac{1}{\beta}\right)\right] + 1 - \frac{A(f)}{L(\alpha,\beta)}\right] \le \frac{\beta - A(f)}{\beta-\alpha}\phi(\alpha) + \frac{A(f)-\alpha}{\beta-\alpha}\phi(\beta) - A(\phi\circ f), \tag{5.8}$$

provided that $\phi\circ f, \ln f$ *and* $f \in L$, *and* $I(\cdot,\cdot)$ *denotes the identric mean, i.e., we recall it*

$$I(u,v) := \begin{cases} u & \text{if } v = u, \\ \frac{1}{e}\left(\frac{u^u}{v^v}\right)^{\frac{1}{u-v}}, & v \ne u. \end{cases}$$

(ii) *If* $\sup_{t\in(\alpha,\beta)} l(t) = S < \infty$, *then we have the inequality*

$$\frac{\beta - A(f)}{\beta-\alpha}\phi(\alpha) + \frac{A(f)-\alpha}{\beta-\alpha}\phi(\beta) - A(\phi\circ f) \le S\left[A(\ln f) + \ln\left[I\left(\frac{1}{\alpha},\frac{1}{\beta}\right)\right] + 1 - \frac{A(f)}{L(\alpha,\beta)}\right]. \tag{5.9}$$

(*iii*) *If* $-\infty < s \leq l(t) \leq S < \infty$ *for* $t \in (\alpha, \beta)$, *then both (5.8) and (5.9) hold.*

Proof. The proof is as follows.

(*i*) Define the auxiliary function $h(t) = \phi(t) + s \ln t$. Then

$$h''(t) = \phi''(t) - \frac{s}{t^2} = \frac{1}{t^2}\left(\phi''(t)\, t^2 - s\right) \geq 0,$$

showing that h is convex, or, equivalently, $\phi \in \mathcal{L}(I, s, -\ln(\cdot))$. Applying Corollary 4, we may state that:

$$s\left[\frac{\beta - A(f)}{\beta - \alpha} \cdot [-\ln(\alpha)] + \frac{A(f) - \alpha}{\beta - \alpha} \cdot [-\ln(\beta)] + A(\ln f)\right]$$
$$\leq \frac{\beta - A(f)}{\beta - \alpha}\phi(\alpha) + \frac{A(f) - \alpha}{\beta - \alpha}\phi(\beta) - A(\phi \circ f),$$

which is equivalent to (5.8).

(*ii*) Goes likewise.

(*iii*) Follows by (*i*) and (*ii*).

□

Finally, the following result also holds [8].

Proposition 10. *Assume that the mapping* $\phi : [\alpha, \beta] \subset (0, \infty) \to \mathbb{R}$ *is twice differentiable on* (α, β). *Define* $\tilde{I}(t) = t\phi''(t)$, $t \in I$.

(*i*) *If* $\inf_{t \in (\alpha, \beta)} \tilde{I}(t) = \delta > -\infty$, *then we have the inequality*

$$\delta\left[A(f)\ln I(\alpha, \beta) - \frac{G^2(\alpha, \beta)}{L(\alpha, \beta)} + A(f) - A(f \ln f)\right] \tag{5.10}$$
$$\leq \frac{\beta - A(f)}{\beta - \alpha}\phi(\alpha) + \frac{A(f) - \alpha}{\beta - \alpha}\phi(\beta) - A(\phi \circ f),$$

provided that $\phi \circ f, f \ln f, f \in L$ *and* $G(\alpha, \beta) = \sqrt{ab}$ *is the geometric mean and* $L(\alpha, \beta)$ *is the logarithmic mean, i.e., we recall it*

$$L(\alpha, \beta) := \begin{cases} \alpha & \text{if } \beta = \alpha, \\ \frac{\beta - \alpha}{\ln \beta - \ln \alpha} & \text{if } \beta \neq \alpha. \end{cases}$$

(*ii*) *If* $\sup_{t \in (\alpha, \beta)} \tilde{I}(t) = \Delta < \infty$, *then we have the inequality*

$$\frac{\beta - A(f)}{\beta - \alpha}\phi(\alpha) + \frac{A(f) - \alpha}{\beta - \alpha}\phi(\beta) - A(\phi \circ f) \tag{5.11}$$
$$\leq \Delta\left[A(f)\ln I(\alpha, \beta) - \frac{G^2(\alpha, \beta)}{L(\alpha, \beta)} + A(f) - A(f \ln f)\right]$$

(*iii*) *If* $-\infty < \delta \leq \tilde{I}(t) \leq \Delta < \infty$ *for* $t \in (\alpha, \beta)$, *then both (5.10) and (5.11) hold.*

Proof. The proof is as follows.

(*i*) Define the auxiliary mapping $h(t) = \phi(t) - \delta t \ln t$, $t \in (\alpha, \beta)$. Then

$$h''(t) = \phi''(t) - \frac{\delta}{t} = \frac{1}{t^2}\left[\phi''(t)\, t - \delta\right] = \frac{1}{t}\left[\tilde{I}(t) - \delta\right] \geq 0$$

which shows that h is convex or, equivalently, $\phi \in \mathcal{L}(I, \delta, (\cdot)\ln(\cdot))$. Applying Corollary 4, we can write

$$\delta\left[\frac{\beta - A(f)}{\beta-\alpha}\cdot[\alpha\ln\alpha] + \frac{A(f)-\alpha}{\beta-\alpha}\cdot[\beta\ln\beta] - A(f\ln f)\right] \leq \frac{\beta - A(f)}{\beta-\alpha}\phi(\alpha) + \frac{A(f)-\alpha}{\beta-\alpha}\phi(\beta) - A(\phi\circ f),$$

which is clearly equivalent to (5.10).

(ii) Goes similarly.

(iii) Follows by (i) and (ii).

□

5.2. **Applications for Hermite-Hadamard Inequalities.** The following integral inequalities were obtained in [8].

a) Assume that $\phi : [a, b] \subset \mathbb{R} \to \mathbb{R}$ is a twice differentiable function satisfying the condition $-\infty < k \leq \phi''(t) \leq K < \infty$ for $t \in (a, b)$. If in Propostion 7 we choose $A(f) := \frac{1}{b-a}\int_a^b f(t)\,dt,\ f = e$, i.e., $e(x) = x$, $x \in [a, b]$ and take into account that

$$A\left(f^2\right) = \frac{b^2+ab+a^2}{3},$$

then we may state the inequality (see also [8, p. 40])

$$\frac{k(b-a)^2}{12} \leq \frac{\phi(b)+\phi(a)}{2} - \frac{1}{b-a}\int_a^b \phi(x)\,dx \leq \frac{K(b-a)^2}{12}. \tag{5.12}$$

b) Now, if we assume that $\phi : [a, b] \subset (0,\infty) \to \mathbb{R}$ is twice differentiable on (a, b) and $-\infty < \gamma \leq t^{2-p}\phi''(t) \leq \Gamma < \infty$, $t \in (a, b)$, $p \in (-\infty, 0) \cup (1,\infty)$, then, applying Proposition 8 for integrals, we may state the inequality

$$\begin{aligned}\frac{\gamma}{p(p-1)}&\left[pL_{p-1}^{p-1}(a,b)\,A(a,b) - (p-1)\,G^2(a,b)\,L_{p-2}^{p-2}(a,b) - L_p^p(a,b)\right]\\ &\leq \frac{\phi(b)+\phi(a)}{2} - \frac{1}{b-a}\int_a^b \phi(x)\,dx\\ &\leq \frac{\Gamma}{p(p-1)}\left[pL_{p-1}^{p-1}(a,b)\,A(a,b) - (p-1)\,G^2(a,b)\,L_{p-2}^{p-2}(a,b) - L_p^p(a,b)\right].\end{aligned} \tag{5.13}$$

c) Suppose that the twice differentiable function $\phi : [a, b] \subset (0,\infty) \to \mathbb{R}$ satisfies the condition $-\infty < s \leq t^2\phi''(t) \leq S < \infty$. Then by Proposition 9 applied for the integral functional, we may state the following inequality

$$\begin{aligned}s\ln\left[\frac{I(a,b)\,I\left(\frac{1}{a},\frac{1}{b}\right)}{\exp\left(\frac{A(a,b)-L(a,b)}{L(a,b)}\right)}\right] &\leq \frac{\phi(b)+\phi(a)}{2} - \frac{1}{b-a}\int_a^b \phi(x)\,dx\\ &\leq S\ln\left[\frac{I(a,b)\,I\left(\frac{1}{a},\frac{1}{b}\right)}{\exp\left(\frac{A(a,b)-L(a,b)}{L(a,b)}\right)}\right]\end{aligned} \tag{5.14}$$

or, equivalently,

$$\left[\frac{I(a,b)\,I\left(\frac{1}{a},\frac{1}{b}\right)}{\exp\left(\frac{A(a,b)-L(a,b)}{L(a,b)}\right)}\right]^{s} \leq \frac{\exp\left[\frac{\phi(b)+\phi(a)}{2}\right]}{\exp\left[\frac{1}{b-a}\int_a^b \phi(x)\,dx\right]} \leq \left[\frac{I(a,b)\,I\left(\frac{1}{a},\frac{1}{b}\right)}{\exp\left(\frac{A(a,b)-L(a,b)}{L(a,b)}\right)}\right]^{S}. \tag{5.15}$$

d) Finally, if we assume that the twice differentiable function $\phi : [a,b] \subset (0,\infty) \to \mathbb{R}$ satisfies the condition $-\infty < \delta \leq t\phi''(t) \leq 1 < \infty$, then by Proposition 10 applied for the integral functional, we may state the following inequality:

$$\begin{aligned} &\delta A(a,b)\ln\left[\left(\frac{I(a,b)}{\sqrt{I(a^2,b^2)}}\right)\cdot\exp\left(\frac{L(a,b)\,A(a,b)-G^2(a,b)}{L(a,b)\,A(a,b)}\right)\right] \\ &\leq \frac{\phi(b)+\phi(a)}{2}-\frac{1}{b-a}\int_a^b \phi(x)\,dx \\ &\leq \Delta A(a,b)\ln\left[\left(\frac{I(a,b)}{\sqrt{I(a^2,b^2)}}\right)\cdot\exp\left(\frac{L(a,b)\,A(a,b)-G^2(a,b)}{L(a,b)\,A(a,b)}\right)\right], \end{aligned} \tag{5.16}$$

or, equivalently,

$$\begin{aligned} &\left[\left(\frac{I(a,b)}{\sqrt{I(a^2,b^2)}}\right)\cdot\exp\left(\frac{L(a,b)\,A(a,b)-G^2(a,b)}{L(a,b)\,A(a,b)}\right)\right]^{\delta A(a,b)} \\ &\leq \frac{\exp\left[\frac{\phi(b)+\phi(a)}{2}\right]}{\exp\left[\frac{1}{b-a}\int_a^b \phi(x)\,dx\right]} \\ &\leq \left[\left(\frac{I(a,b)}{\sqrt{I(a^2,b^2)}}\right)\cdot\exp\left(\frac{L(a,b)\,A(a,b)-G^2(a,b)}{L(a,b)\,A(a,b)}\right)\right]^{\Delta A(a,b)}. \end{aligned} \tag{5.17}$$

6. A Reverse Inequality

We start with the following result [6] which gives another counterpart for $A(\phi\circ f)$, as did the Lupaş-Beesack-Pečarić result.

Theorem 8 (Dragomir, 2001, [6]). *Let $\phi : (\alpha,\beta) \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable convex function on (α,β), $f : E \to (\alpha,\beta)$ such that $\phi\circ f,\ f,\ \phi'\circ f,\ \phi'\circ f\cdot f \in L$. If $A : L \to \mathbb{R}$ is an isotonic linear and normalised functional, then*

$$\begin{aligned} 0 &\leq A(\phi\circ f)-\phi(A(f)) \leq A\left(\phi'\circ f\cdot f\right)-A(f)\cdot A\left(\phi'\circ f\right) \\ &\leq \frac{1}{4}\left[\phi'(\beta)-\phi'(\alpha)\right](\beta-\alpha) \quad \text{(if } \alpha,\ \beta \text{ are finite).} \end{aligned} \tag{6.1}$$

Proof. As ϕ is differentiable convex on (α,β), we may write that

$$\phi(x)-\phi(y) \geq \phi'(y)(x-y), \ \text{ for all } x,y \in (\alpha,\beta), \tag{6.2}$$

from where we obtain

$$\phi(A(f))-(\phi\circ f)(t) \geq \left(\phi'\circ f\right)(t)\,(A(f)-f(t)) \tag{6.3}$$

for all $t \in E$, as, obviously, $A(f) \in (\alpha, \beta)$.

If we apply to (6.3) the functional A, we may write

$$\phi(A(f)) - A(\phi \circ f) \geq A(f) \cdot A\left(\phi' \circ f\right) - A\left(\phi' \circ f \cdot f\right),$$

which is clearly equivalent to the first inequality in (6.1).

It is well known that the following Grüss inequality for isotonic linear and normalised functionals holds (see [1])

$$|A(hk) - A(h) A(k)| \leq \frac{1}{4}(M-m)(N-n), \tag{6.4}$$

provided that $h, k \in L$, $hk \in L$ and $-\infty < m \leq h(t) \leq M < \infty$, $-\infty < n \leq k(t) \leq N < \infty$, for all $t \in E$.

Taking into account that for finite α,β we have $\alpha < f(t) < \beta$ with ϕ' being monotonic on (α, β), we have $\phi'(\alpha) \leq \phi' \circ f \leq \phi'(\beta)$, and then by the Grüss inequality, we may state that

$$A\left(\phi' \circ f \cdot f\right) - A(f) \cdot A\left(\phi' \circ f\right) \leq \frac{1}{4}\left[\phi'(\beta) - \phi'(\alpha)\right](\beta - \alpha)$$

and the theorem is completely proved. □

The following corollary holds [6].

Corollary 5. *Let $\phi : [a,b] \subset \mathring{I} \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable convex function on $\mathring{I}$. If ϕ, e_1, ϕ', $\phi' \cdot e_1 \in L$ ($e_1(x) = x$, $x \in [a,b]$) and $A : L \to \mathbb{R}$ is an isotonic linear and normalised functional, then:*

$$\begin{aligned} 0 &\leq A(\phi) - \phi(A(e_1)) \leq A\left(\phi' \cdot e_1\right) - A(e_1) \cdot A\left(\phi'\right) \\ &\leq \frac{1}{4}\left[\phi'(b) - \phi'(a)\right](b-a). \end{aligned} \tag{6.5}$$

There are some particular cases which can naturally be considered [6].

(1) Let $\phi(x) = \ln x$, $x > 0$. If $\ln f,\ f,\ \frac{1}{f} \in L$ and $A : L \to \mathbb{R}$ is an isotonic linear and normalised functional, then:

$$0 \leq \ln[A(f)] - A[\ln(f)] \leq A(f) A\left(\frac{1}{f}\right) - 1, \tag{6.6}$$

provided that $f(t) > 0$ for all $t \in E$ and $A(f) > 0$.

If $0 < m \leq f(t) \leq M < \infty$, $t \in E$, then, by the second part of (6.1) we have:

$$A(f) A\left(\frac{1}{f}\right) - 1 \leq \frac{(M-m)^2}{4mM} \qquad \text{(which is a known result).} \tag{6.7}$$

Note that the inequality (6.6) is equivalent to

$$1 \leq \frac{A(f)}{\exp[A[\ln(f)]]} \leq \exp\left[A(f) A\left(\frac{1}{f}\right) - 1\right]. \tag{6.8}$$

(2) Let $\phi(x) = \exp(x)$, $x \in \mathbb{R}$. If $\exp(f)$, f, $f \cdot \exp(f) \in L$ and $A : L \to \mathbb{R}$ is an isotonic linear and normalised functional, then

$$\begin{aligned} 0 &\leq A[\exp(f)] - \exp[A(f)] \leq A[f \exp(f)] - A(f) \exp[A(f)] \\ &\leq \frac{1}{4}[\exp(M) - \exp(m)](M-m) \quad (\text{if } m \leq f \leq M \text{ on } E). \end{aligned} \tag{6.9}$$

7. A Further Result for $m-\Psi-$Convex and $M-\Psi-$Convex Functions

7.1. Other General Results.

We now prove the following result [6].

Theorem 9 (Dragomir, 2001, [6])**.** *Let $\Psi : I \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable convex function and $f : E \to I$ such that $\Psi \circ f$, $\Psi' \circ f$, $\Psi' \circ f \cdot f$, $f \in L$ and $A : L \to \mathbb{R}$ be an isotonic linear and normalised functional.*

(*i*) *If ϕ is differentiable, $\phi \in \mathcal{L}\left(\mathring{I}, m, \Psi\right)$ and $\phi \circ f$, $\phi' \circ f$, $\phi' \circ f \cdot f \in L$, then we have the inequality*

$$\begin{aligned}(7.1)\qquad & m\left[A\left(\Psi' \circ f \cdot f\right) + \Psi\left(A\left(f\right)\right) - A\left(f\right) \cdot A\left(\Psi' \circ f\right) - A\left(\Psi \circ f\right)\right] \\ \leq\ & A\left(\phi' \circ f \cdot f\right) + \phi\left(A\left(f\right)\right) - A\left(f\right) \cdot A\left(\phi' \circ f\right) - A\left(\phi \circ f\right).\end{aligned}$$

(*ii*) *If ϕ is differentiable, $\phi \in \mathcal{U}\left(\mathring{I}, M, \Psi\right)$ and $\phi \circ f$, $\phi' \circ f$, $\phi' \circ f \cdot f \in L$, then we have the inequality*

$$\begin{aligned}(7.2)\qquad & A\left(\phi' \circ f \cdot f\right) + \phi\left(A\left(f\right)\right) - A\left(f\right) \cdot A\left(\phi' \circ f\right) - A\left(\phi \circ f\right) \\ \leq\ & M\left[A\left(\Psi' \circ f \cdot f\right) + \Psi\left(A\left(f\right)\right) - A\left(f\right) \cdot A\left(\Psi' \circ f\right) - A\left(\Psi \circ f\right)\right].\end{aligned}$$

(*iii*) *If ϕ is differentiable, $\phi \in \mathcal{B}\left(\mathring{I}, m, M, \Psi\right)$ and $\phi \circ f$, $\phi' \circ f$, $\phi' \circ f \cdot f \in L$, then both (7.1) and (7.2) hold.*

Proof. The proof is as follows.

(*i*) As $\phi \in \mathcal{L}\left(I, m, \Psi\right)$, then $\phi - m\Psi$ is convex and we can apply the first part of the inequality (6.1) for $\phi - m\Psi$ getting

$$\begin{aligned}(7.3)\qquad & A\left[\left(\phi - m\Psi\right) \circ f\right] - \left(\phi - m\Psi\right)\left(A\left(f\right)\right) \\ & \leq A\left[\left(\phi - m\Psi\right)' \circ f \cdot f\right] - A\left(f\right) A\left(\left(\phi - m\Psi\right)' \circ f\right).\end{aligned}$$

However,

$$A\left[\left(\phi - m\Psi\right) \circ f\right] = A\left(\phi \circ f\right) - mA\left(\Psi \circ f\right),$$
$$\left(\phi - m\Psi\right)\left(A\left(f\right)\right) = \phi\left(A\left(f\right)\right) - m\Psi\left(A\left(f\right)\right),$$
$$A\left[\left(\phi - m\Psi\right)' \circ f \cdot f\right] = A\left(\phi' \circ f \cdot f\right) - mA\left(\Psi' \circ f \cdot f\right)$$

and

$$A\left(\left(\phi - m\Psi\right)' \circ f\right) = A\left(\phi' \circ f\right) - mA\left(\Psi' \circ f\right)$$

and then, by (7.3), we deduce the desired inequality (7.1).

(*ii*) Goes likewise and we omit the details.

(*iii*) Follows by (*i*) and (*ii*).

□

The following corollary is useful in practice [6],

Corollary 6. *Let $\Psi : I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable convex function on $\mathring{I}$, $f : E \to I$ such that $\Psi \circ f$, $\Psi' \circ f$, $\Psi' \circ f \cdot f$, $f \in L$ and $A : L \to \mathbb{R}$ be an isotonic linear and normalised functional.*

(*i*) *If $\phi : I \to \mathbb{R}$ is twice differentiable, $\phi \circ f$, $\phi' \circ f$, $\phi' \circ f \cdot f \in L$ and $\phi''\left(t\right) \geq m\Psi''\left(t\right)$, $t \in \mathring{I}$, (where m is a given real number), then the inequality (7.1) holds.*

(*ii*) *With the same assumptions, but if $\phi''\left(t\right) \leq M\Psi''\left(t\right)$, $t \in \mathring{I}$, (where M is a given real number), then the inequality (7.2) holds.*

(iii) *If* $m\Psi''(t) \leq \phi''(t) \leq M\Psi''(t)$, $t \in \mathring{I}$, *then both (7.1) and (7.2) hold.*

Some particular important cases of the above corollary are embodied in the following proposition [6].

Proposition 11. *Assume that the mapping* $\phi : I \subseteq \mathbb{R} \to \mathbb{R}$ *is twice differentiable on* $\mathring{I}$.

(i) *If* $\inf_{t\in\mathring{I}} \phi''(t) = k > -\infty$, *then we have the inequality*

$$\begin{aligned} (7.4) \qquad & \frac{1}{2}k\left[A\left(f^2\right) - \left[A(f)\right]^2\right] \\ & \leq A\left(\phi' \circ f \cdot f\right) + \phi(A(f)) - A(f) \cdot A\left(\phi' \circ f\right) - A(\phi \circ f), \end{aligned}$$

provided that $\phi \circ f$, $\phi' \circ f$, $\phi' \circ f \cdot f, f^2 \in L$.

(ii) *If* $\sup_{t\in\mathring{I}} \phi''(t) = K < \infty$, *then we have the inequality*

$$\begin{aligned} (7.5) \qquad & A\left(\phi' \circ f \cdot f\right) + \phi(A(f)) - A(f) \cdot A\left(\phi' \circ f\right) - A(\phi \circ f) \\ & \leq \frac{1}{2}K\left[A\left(f^2\right) - \left[A(f)\right]^2\right]. \end{aligned}$$

(iii) *If* $-\infty < k \leq \phi''(t) \leq K < \infty$, $t \in \mathring{I}$, *then both (7.4) and (7.5) hold.*

The proof follows by Corollary 6 applied for $\Psi(t) = \frac{1}{2}t^2$ and $m = k$, $M = K$.
Another result is the following one [6].

Proposition 12. *Assume that the mapping* $\phi : I \subseteq (0, \infty) \to \mathbb{R}$ *is twice differentiable on* $\mathring{I}$. *Let* $p \in (-\infty, 0) \cup (1, \infty)$ *and define* $g_p : I \to \mathbb{R}$, $g_p(t) = \phi''(t)\, t^{2-p}$.

(i) *If* $\inf_{t\in\mathring{I}} g_p(t) = \gamma > -\infty$, *then we have the inequality*

$$\begin{aligned} & \frac{\gamma}{p(p-1)}\left[(p-1)\left[A(f^p) - [A(f)]^p\right] - pA(f)\left[A\left(f^{p-1}\right) - [A(f)]^{p-1}\right]\right] \\ (7.6) \qquad & \leq A\left(\phi' \circ f \cdot f\right) + \phi(A(f)) - A(f) \cdot A\left(\phi' \circ f\right) - A(\phi \circ f), \end{aligned}$$

provided that $\phi \circ f$, $\phi' \circ f$, $\phi' \circ f \cdot f, f^p, f^{p-1} \in L$.

(ii) *If* $\sup_{t\in\mathring{I}} g_p(t) = \Gamma < \infty$, *then we have the inequality*

$$\begin{aligned} (7.7) \quad & A\left(\phi' \circ f \cdot f\right) + \phi(A(f)) - A(f) \cdot A\left(\phi' \circ f\right) - A(\phi \circ f) \\ & \leq \frac{\Gamma}{p(p-1)}\left[(p-1)\left[A(f^p) - [A(f)]^p\right] - pA(f)\left[A\left(f^{p-1}\right) - [A(f)]^{p-1}\right]\right]. \end{aligned}$$

(iii) *If* $-\infty < \gamma \leq g_p(t) \leq \Gamma < \infty$, $t \in \mathring{I}$, *then both (7.6) and (7.7) hold.*

Proof. The proof is as follows.

(i) We have for the auxiliary mapping $h_p(t) = \phi(t) - \frac{\gamma}{p(p-1)}t^p$ that

$$\begin{aligned} h_p''(t) &= \phi''(t) - \gamma t^{p-2} = t^{p-2}\left(t^{2-p}\phi''(t) - \gamma\right) \\ &= t^{p-2}\left(g_p(t) - \gamma\right) \geq 0. \end{aligned}$$

That is, h_p is convex or, equivalently, $\phi \in \mathcal{L}\left(I, \frac{\gamma}{p(p-1)}, (\cdot)^p\right)$. Applying Corollary 6, we get

$$\frac{\gamma}{p(p-1)}\left[pA(f^p) + [A(f)]^p - pA(f)A\left(f^{p-1}\right) - A(f^p)\right]$$
$$\leq A\left(\phi' \circ f \cdot f\right) + \phi(A(f)) - A(f) \cdot A\left(\phi' \circ f\right) - A(\phi \circ f),$$

which is clearly equivalent to (7.6).

(*ii*) Goes similarly.

(*iii*) Follows by (*i*) and (*ii*).

□

The following proposition also holds [6].

Proposition 13. *Assume that the mapping $\phi : I \subseteq (0,\infty) \to \mathbb{R}$ is twice differentiable on $\mathring{I}$. Define $l(t) = t^2\phi''(t)$, $t \in I$.*

(*i*) *If $\inf_{t\in\mathring{I}} l(t) = s > -\infty$, then we have the inequality*

$$s\left[A(f)A\left(\frac{1}{f}\right) - 1 - (\ln[A(f)] - A[\ln(f)])\right] \tag{7.8}$$
$$\leq A\left(\phi' \circ f \cdot f\right) + \phi(A(f)) - A(f) \cdot A\left(\phi' \circ f\right) - A(\phi \circ f),$$

provided that $\phi \circ f, \phi^{-1} \circ f, \phi^{-1} \circ f \cdot f, \frac{1}{f}, \ln f \in L$ and $A(f) > 0$.

(*ii*) *If $\sup_{t\in\mathring{I}} l(t) = S < \infty$, then we have the inequality*

$$A\left(\phi' \circ f \cdot f\right) + \phi(A(f)) - A(f) \cdot A\left(\phi' \circ f\right) - A(\phi \circ f) \tag{7.9}$$
$$\leq S\left[A(f)A\left(\frac{1}{f}\right) - 1 - (\ln[A(f)] - A[\ln(f)])\right].$$

(*iii*) *If $-\infty < s \leq l(t) \leq S < \infty$ for $t \in \mathring{I}$, then both (7.8) and (7.9) hold.*

Proof. The proof is as follows.

(*i*) Define the auxiliary function $h(t) = \phi(t) + s\ln t$. Then

$$h''(t) = \phi''(t) - \frac{s}{t^2} = \frac{1}{t^2}\left(\phi''(t)t^2 - s\right) \geq 0$$

which shows that h is convex, or, equivalently, $\phi \in \mathcal{L}(I, s, -\ln(\cdot))$. Applying Corollary 6, we may write

$$s\left[-A(\mathbf{1}) - \ln A(f) + A(f)A\left(\frac{1}{f}\right) + A(\ln(f))\right]$$
$$\leq A\left(\phi' \circ f \cdot f\right) + \phi(A(f)) - A(f) \cdot A\left(\phi' \circ f\right) - A(\phi \circ f),$$

which is clearly equivalent to (7.8).

(*ii*) Goes similarly.

(*iii*) Follows by (*i*) and (*ii*).

□

Finally, the following result also holds [6].

Proposition 14. *Assume that the mapping $\phi : I \subseteq (0,\infty) \to \mathbb{R}$ is twice differentiable on $\mathring{I}$. Define $\tilde{I}(t) = t\phi''(t)$, $t \in I$.*

(i) *If* $\inf_{t\in\mathring{I}} \tilde{I}(t) = \delta > -\infty$, *then we have the inequality*

$$\begin{aligned}(7.10) \qquad & \delta A(f)[\ln[A(f)] - A(\ln(f))] \\ & \leq A(\phi' \circ f \cdot f) + \phi(A(f)) - A(f) \cdot A(\phi' \circ f) - A(\phi \circ f),\end{aligned}$$

provided that $\phi \circ f, \phi\prime \circ f, \phi\prime \circ f \cdot f, \ln f, f \in L$ *and* $A(f) > 0$.

(ii) *If* $\sup_{t\in\mathring{I}} \tilde{I}(t) = \Delta < \infty$, *then we have the inequality*

$$\begin{aligned}(7.11) \qquad & A(\phi' \circ f \cdot f) + \phi(A(f)) - A(f) \cdot A(\phi' \circ f) - A(\phi \circ f) \\ & \leq \Delta A(f)[\ln[A(f)] - A(\ln(f))].\end{aligned}$$

(iii) *If* $-\infty < \delta \leq \tilde{I}(t) \leq \Delta < \infty$ *for* $t \in \mathring{I}$, *then both (7.10) and (7.11) hold.*

Proof. The proof is as follows.

(i) Define the auxiliary mapping $h(t) = \phi(t) - \delta t \ln t$, $t \in I$. Then

$$h''(t) = \phi''(t) - \frac{\delta}{t} = \frac{1}{t^2}[\phi''(t)t - \delta] = \frac{1}{t}\left[\tilde{I}(t) - \delta\right] \geq 0$$

which shows that h is convex or equivalently, $\phi \in \mathcal{L}(I, \delta, (\cdot)\ln(\cdot))$. Applying Corollary 6, we get

$$\begin{aligned}& \delta[A[(\ln f + 1) f] + A(f)\ln A(f) - A(f) A(\ln f + 1) - A(f \ln f)] \\ & \leq A(\phi' \circ f \cdot f) + \phi(A(f)) - A(f) \cdot A(\phi' \circ f) - A(\phi \circ f)\end{aligned}$$

which is equivalent with (7.10).

(ii) Goes similarly.

(iii) Follows by (i) and (ii).

□

7.2. **Some Applications For Bullen's Inequality.** The following inequality is well known in the literature as Bullen's inequality (see for example [11, p. 10])

$$(7.12) \qquad \frac{1}{b-a}\int_a^b \phi(t)\,dt \leq \frac{1}{2}\left[\frac{\phi(a)+\phi(b)}{2} + \phi\left(\frac{a+b}{2}\right)\right],$$

provided that $\phi : [a,b] \to \mathbb{R}$ is a convex function on $[a,b]$. In other words, as (8.31) is equivalent to:

$$(7.13) \qquad 0 \leq \frac{1}{b-a}\int_a^b \phi(t)\,dt - \phi\left(\frac{a+b}{2}\right) \leq \frac{\phi(a)+\phi(b)}{2} - \frac{1}{b-a}\int_a^b \phi(t)\,dt$$

we can conclude that in the Hermite-Hadamard inequality

$$(7.14) \qquad \frac{\phi(a)+\phi(b)}{2} \geq \frac{1}{b-a}\int_a^b \phi(t)\,dt \geq \phi\left(\frac{a+b}{2}\right)$$

the integral mean $\frac{1}{b-a}\int_a^b \phi(t)\,dt$ is closer to $\phi\left(\frac{a+b}{2}\right)$ than to $\frac{\phi(a)+\phi(b)}{2}$.

Using some of the results pointed out in the previous sections, we may upper and lower bound the *Bullen difference:*

$$B(\phi; a, b) := \frac{1}{2}\left[\frac{\phi(a)+\phi(b)}{2} + \phi\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b \phi(t)\,dt$$

(which is positive for convex functions) for different classes of twice differentiable functions ϕ.

Now, if we assume that $A(f) := \frac{1}{b-a}\int_a^b f(t)\,dt$, then for $f = e$, $e(x) = x$, $x \in [a,b]$, we have, for a differentiable function ϕ, that

$$\begin{aligned}
&A\left(\phi' \circ f \cdot f\right) + \phi\left(A(f)\right) - A(f) \cdot A\left(\phi' \circ f\right) - A(\phi \circ f) \\
&= \frac{1}{b-a}\int_a^b x\phi'(x)\,dx + \phi\left(\frac{a+b}{2}\right) \\
&- \frac{a+b}{2} \cdot \frac{1}{b-a}\int_a^b \phi'(x)\,dx - \frac{1}{b-a}\int_a^b \phi(x)\,dx \\
&= \frac{1}{b-a}\left[b\phi(b) - a\phi(a) - \int_a^b \phi(x)\,dx\right] + \phi\left(\frac{a+b}{2}\right) \\
&- \frac{a+b}{2} \cdot \frac{\phi(b)-\phi(a)}{b-a} - \frac{1}{b-a}\int_a^b \phi(x)\,dx \\
&= \frac{\phi(a)+\phi(b)}{2} + \phi\left(\frac{a+b}{2}\right) - \frac{2}{b-a}\int_a^b \phi(x)\,dx \\
&= 2B(\phi; a, b).
\end{aligned}$$

The following integral inequalities were obtained in [6].

a) Assume that $\phi : [a,b] \subset \mathbb{R} \to \mathbb{R}$ is a twice differentiable function satisfying the property that $-\infty < k \le \phi''(t) \le K < \infty$. Then by Proposition 11, we may state the inequality

$$\frac{1}{48}(b-a)^2 k \le B(\phi; a, b) \le \frac{1}{48}(b-a)^2 K. \tag{7.15}$$

This follows by Proposition 11 on taking into account that

$$\frac{1}{b-a}\int_a^b x^2 dx - \left(\frac{1}{b-a}\int_a^b x dx\right)^2 = \frac{(b-a)^2}{12}.$$

b) Now, assume that the twice differentiable function $\phi : [a,b] \subset (0,\infty) \to \mathbb{R}$ satisfies the property that $-\infty < \gamma \le t^{2-p}\phi''(t) \le \Gamma < \infty$, $t \in (a,b)$, $p \in (-\infty, 0) \cup (1,\infty)$. Then by Proposition 12 and taking into account that

$$\begin{aligned}
A(f^p) - (A(f))^p &= \frac{1}{b-a}\int_a^b x^p dx - \left(\frac{1}{b-a}\int_a^b x dx\right)^p \\
&= L_p^p(a,b) - A^p(a,b),
\end{aligned}$$

an

$$A\left(f^{p-1}\right) - (A(f))^{p-1} = L_{p-1}^{p-1}(a,b) - A^{p-1}(a,b),$$

we may state the inequality

$$\begin{aligned}
&\frac{\gamma}{p(p-1)}\left[(p-1)\left[L_p^p(a,b) - A^p(a,b)\right] - pA(a,b)\left[L_{p-1}^{p-1}(a,b) - A^{p-1}(a,b)\right]\right] \\
&\qquad \le B(\phi; a, b) \\
&\le \frac{\Gamma}{p(p-1)}\left[(p-1)\left[L_p^p(a,b) - A^p(a,b)\right] - pA(a,b)\left[L_{p-1}^{p-1}(a,b) - A^{p-1}(a,b)\right]\right].
\end{aligned} \tag{7.16}$$

c) Assume that the twice differentiable function $\phi : [a,b] \subset (0,\infty) \to \mathbb{R}$ satisfies the prpoerty that $-\infty < s \leq t^2\phi''(t) \leq S < \infty$, $t \in (a,b)$, then by Proposition 13, and taking into account that

$$\begin{aligned} & A(f)A\left(f^{-1}\right) - 1 - \ln[A(f)] + A\ln(f) \\ &= \frac{A(a,b)}{L(a,b)} - 1 - \ln A(a,b) + I(a,b) \\ &= \ln\left[\frac{I(a,b)}{A(a,b)} \cdot \exp\left(\frac{A(a,b) - L(a,b)}{L(a,b)}\right)\right], \end{aligned}$$

we get the inequality

$$\begin{aligned} & \frac{s}{2}\ln\left[\frac{I(a,b)}{A(a,b)} \cdot \exp\left(\frac{A(a,b) - L(a,b)}{L(a,b)}\right)\right] \\ & \leq B(\phi; a,b) \\ & \leq \frac{S}{2}\ln\left[\frac{I(a,b)}{A(a,b)} \cdot \exp\left(\frac{A(a,b) - L(a,b)}{L(a,b)}\right)\right] \end{aligned} \tag{7.17}$$

d) Finally, if ϕ satisfies the condition $-\infty < \delta \leq t\phi''(t) \leq \Delta < \infty$, then by Proposition 14, we may state the inequality

$$\delta A(a,b)\ln\left[\frac{A(a,b)}{I(a,b)}\right] \leq B(\phi; a,b) \leq \Delta A(a,b)\ln\left[\frac{A(a,b)}{I(a,b)}\right]. \tag{7.18}$$

8. A Grüss Type Inequality

8.1. **A Refinement of Grüss Inequality.** In 1988, D. Andrica and C. Badea [1], proved the following generalisation of the Grüss inequality for isotonic linear functionals.

Theorem 10 (Andrica & Badea, 1988, [1])**.** *If $f,g \in L$ so that $fg \in L$ and $m \leq f \leq M$, $n \leq g \leq N$ where m,M,n,N are given real numbers, then for any normalised isotonic linear functional $A : L \to \mathbb{R}$ one has the inequality*

$$|A(fg) - A(f)A(g)| \leq \frac{1}{4}(M-m)(N-n). \tag{8.1}$$

The constant $\frac{1}{4}$ in (8.1) is best possible in the sense that it cannot be replaced by a smaller constant.

In this paper we point out a refinement of the Grüss inequality (8.1) for isotonic linear functionals. Applications for the Cauchy-Buniakowski-Schwartz and Jessen's inequality are also provided.

The following result due to author holds.

Theorem 11 (Dragomir, 2002, [9])**.** *Let $f,g \in L$ be such that $fg \in L$ and assume that there exists the real numbers n and N so that*

$$n \leq g \leq N. \tag{8.2}$$

Then for any normalised isotonic linear functional $A : L \to \mathbb{R}$ for which $|f - A(f)\cdot\mathbf{1}| \in L$ one has the inequality

$$|A(fg) - A(f)A(g)| \leq \frac{1}{2}(N-n)A(|f - A(f)\cdot\mathbf{1}|). \tag{8.3}$$

The constant $\frac{1}{2}$ in (8.3) is best possible in the sense that it cannot be replaced by a smaller constant.

Proof. Using the linearity property of A, we have

$$\begin{aligned} &A\left[(f-A(f)\cdot\mathbf{1})\left(g-\frac{n+N}{2}\cdot\mathbf{1}\right)\right] \\ &= A[(f-A(f)\cdot\mathbf{1})g]-\frac{n+N}{2}A[f-A(f)\cdot\mathbf{1}] \\ &= A(fg)-A(f)A(g)-\frac{n+N}{2}[A(f)-A(f)\cdot A(\mathbf{1})] \\ &= A(fg)-A(f)A(g) \end{aligned} \tag{8.4}$$

since, by the normality property of A, $A(\mathbf{1})=1$.

From (8.2) we may easily deduce that

$$\left|g-\frac{n+N}{2}\cdot\mathbf{1}\right| \le \frac{M-n}{2}\cdot\mathbf{1}. \tag{8.5}$$

It is known that if $h\in L$ so that $|h|\in L$, then, by the monotonicity and linearity of A, one has

$$|A(h)| \le A(|h|). \tag{8.6}$$

Using this property, the monotonicity property of A and condition (8.5), we deduce

$$\begin{aligned} &\left|A\left[(f-A(f)\cdot\mathbf{1})\left(g-\frac{n+N}{2}\cdot\mathbf{1}\right)\right]\right| \\ &\le A\left(\left|(f-A(f)\cdot\mathbf{1})\left(g-\frac{n+N}{2}\cdot\mathbf{1}\right)\right|\right) \\ &\le \frac{N-n}{2}A(|f-A(f)\cdot\mathbf{1}|). \end{aligned} \tag{8.7}$$

Utilising (8.4) and (8.7) we deduce the desired result (8.3).

To prove the sharpness of the constant $\frac{1}{2}$, we assume that (8.3) holds with a constant $c>0$ for $A=\frac{1}{b-a}\int_a^b$, $L=L[a,b]$ (the Lebesgue space of integrable functions on $[a,b]$) and g satisfying the condition (8.2) on the interval $[a,b]$, i.e., one has the inequality

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b f(x)g(x)\,dx-\frac{1}{b-a}\int_a^b f(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx\right| \\ &\le c(N-n)\cdot\frac{1}{b-a}\int_a^b\left|f(x)-\frac{1}{b-a}\int_a^b f(y)\,dy\right|dx. \end{aligned} \tag{8.8}$$

If we choose $g=f$ and $f:[a,b]\to\mathbb{R}$,

$$f(x)=\begin{cases} -1 & \text{if } x\in\left[a,\frac{a+b}{2}\right] \\ 1 & \text{if } x\in\left(\frac{a+b}{2},b\right] \end{cases}$$

then

$$\begin{aligned}\frac{1}{b-a}\int_a^b f^2(x)\,dx-\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)^2 &= 1,\\ \frac{1}{b-a}\int_a^b\left|f(x)-\frac{1}{b-a}\int_a^b f(y)\,dy\right|dx &= 1,\\ m=-1,\ M &= 1.\end{aligned}$$

and by (8.8) we deduce $c\geq\frac{1}{2}$. □

The following corollaries are natural consequences of the above result.

Corollary 7. *Let $f\in L$ be such that $f^2\in L$ and there exists the real numbers m,M so that*

$$m\leq f\leq M. \tag{8.9}$$

Then for any $A:L\rightarrow\mathbb{R}$ a normalised isotonic linear functional so that $|f-A(f)\cdot\mathbf{1}|\in L$ one has the inequality

$$0\leq A\left(f^2\right)-[A(f)]^2\leq\frac{1}{2}(M-m)\,A(|f-A(f)\cdot\mathbf{1}|). \tag{8.10}$$

The constant $\frac{1}{2}$ is sharp.

Corollary 8. *Let $f,g\in L$ so that $fg\in L$ and f satisfy (8.9) while g satisfies (8.2). Then for any normalised isotonic linear functional $A:L\rightarrow\mathbb{R}$ so that $|f-A(f)\cdot\mathbf{1}|,\ |g-A(g)\cdot\mathbf{1}|\in L$ one has the inequality:*

$$\begin{aligned}&|A(fg)-A(f)A(g)|\\ &\leq\frac{1}{2}\left[(M-m)(N-n)\right]^{\frac{1}{2}}\left[A(|f-A(f)\cdot\mathbf{1}|)\,A(|g-A(g)\cdot\mathbf{1}|)\right]^{\frac{1}{2}}.\end{aligned} \tag{8.11}$$

The constant $\frac{1}{2}$ is sharp.

Remark 6. *Using Hölder's inequality for isotonic linear functionals, we may state the following inequalities as well*

$$\begin{aligned}&|A(fg)-A(f)A(g)|\\ &\leq\frac{1}{2}(N-n)\,A(|f-A(f)\cdot\mathbf{1}|)\ \text{ if }\ |f-A(f)\cdot\mathbf{1}|\in L,\\ &\leq\frac{1}{2}(N-n)\left[A(|f-A(f)\cdot\mathbf{1}|^p)\right]^{\frac{1}{p}}\ \text{ if }\ |f-A(f)\cdot\mathbf{1}|^p\in L,\ p>1\\ &\leq\frac{1}{2}(N-n)\sup_{t\in E}|f(t)-A(f)|;\end{aligned} \tag{8.12}$$

provided $f,g\in L$ and $fg\in L$ while g satisfies the condition (8.2).

If f and g fulfill the conditions (8.9) and (8.2), then we have the following *refinement of the Grüss inequality* (8.1)

$$\begin{aligned}|A(fg)-A(f)A(g)| &\leq\frac{1}{2}(N-n)\,A(|f-A(f)\cdot\mathbf{1}|)\\ &\leq\frac{1}{2}(N-n)\left[A\left(f^2\right)-[A(f)]^2\right]^{\frac{1}{2}}\\ &\leq\frac{1}{4}(M-m)(N-n).\end{aligned} \tag{8.13}$$

The constants $\frac{1}{2}$, $\frac{1}{2}$ and $\frac{1}{4}$ are sharp in (8.13).

The following weighted version of Theorem 11 also holds.

Theorem 12 (Dragomir, 2002, [9])**.** *Let $f,g,h \in L$ be such that $h \geq 0$, fh, gh, $fgh \in L$ and there exists the real constants n,N so that (8.2) holds. Then for any $B : L \to \mathbb{R}$ an isotonic linear functional so that $B(h) > 0$, $h\left|f - \frac{1}{B(h)} \cdot \mathbf{1}\right| \in L$ one has the inequality:*

$$\begin{aligned} (8.14) \qquad & \left|\frac{B(fgh)}{B(h)} - \frac{B(fh)}{B(h)} \cdot \frac{B(gh)}{B(h)}\right| \\ & \leq \frac{1}{2}(N-n)\frac{1}{B(h)} B\left[h\left|f - \frac{1}{B(h)} B(hf) \cdot \mathbf{1}\right|\right]. \end{aligned}$$

The constant $\frac{1}{2}$ is best possible.

Proof. Apply Theorem 10 for the functional $A_h : L \to \mathbb{R}$,

$$A_h(f) := \frac{1}{B(h)} B(hf),$$

that is a normalised isotonic linear functional on L. □

Similar corollaries may be stated from the weighted inequality (8.14), but we omit the details.

8.2. **Applications for Integral and Discrete Inequalities.** Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a $\sigma-$algebra of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R} \cup \{\infty\}$.

For a $\mu-$measurable function $w : \Omega \to \mathbb{R}$ with $w(x) \geq 0$ for $\mu-$a.e. $x \in \Omega$, assume $\int_\Omega w(x) d\mu(x) > 0$. Consider the Lebesgue space $L_w(\Omega, \mu) := \{f : \Omega \to \mathbb{R},\ f$ is measurable on $\int_\Omega w(x)|f(x)| d\mu(x) < \infty\}$.

If $f, g : \Omega \to \mathbb{R}$ are $\mu-$measurable functions and $f, g, fg \in L_w(\Omega, \mu)$, then we may consider the Čebyšev functional

$$\begin{aligned} (8.15) \qquad T_w(f,g) := & \frac{1}{\int_\Omega w(x) d\mu(x)} \int_\Omega w(x) f(x) g(x) d\mu(x) \\ & - \frac{1}{\int_\Omega w(x) d\mu(x)} \int_\Omega w(x) f(x) d\mu(x) \\ & \times \frac{1}{\int_\Omega w(x) d\mu(x)} \int_\Omega w(x) g(x) d\mu(x). \end{aligned}$$

We may also consider the functional

$$\begin{aligned} D_w(f) := & \frac{1}{\int_\Omega w(x) d\mu(x)} \\ & \times \int_\Omega w(x) \left|f(x) - \frac{1}{\int_\Omega w(y) d\mu(y)} \int_\Omega w(y) f(y) d\mu(y)\right| d\mu(x). \end{aligned}$$

Applying Theorem 11 for the normalised isotonic linear functional

$$A(f) := \frac{1}{\int_\Omega w(x) d\mu(x)} \int_\Omega w(x) f(x) d\mu(x),$$

$A : L_w(\Omega, \mu) \to \mathbb{R}$, we may recapture the following result due to Cerone and Dragomir [3]. Note that the proof of this result in [3] is different to the one in Theorem 11.

Theorem 13 (Cerone & Dragomir, 2002, [3])**.** *Let $w, f, g : \Omega \to \mathbb{R}$ be $\mu-$measurable functions with $w \geq 0$ $\mu-$a.e. on Ω and $\int_\Omega w(x)\, d\mu(x) > 0$. If $f, g, fg \in L_w(\Omega, \mu)$ and there exists the constants n, N so that*

$$-\infty < n \leq g(x) \leq N < \infty \quad \text{for } \mu\text{-a.e. } x \in \Omega, \tag{8.16}$$

then we have the inequality

$$|T_w(f, g)| \leq \frac{1}{2}(N - n) D_w(f). \tag{8.17}$$

The constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller constant.

Remark 7. *If $\Omega = [a, b]$ and $w(x) = 1$ in Theorem 13, then we recapture the result obtained in* [4]

$$\left|\frac{1}{b-a}\int_a^b f(x) g(x)\, dx - \frac{1}{b-a}\int_a^b f(x)\, dx \cdot \frac{1}{b-a}\int_a^b g(x)\, dx\right| \tag{8.18}$$
$$\leq \frac{1}{2}(N - n) \cdot \frac{1}{b-a}\int_a^b \left| f(x) - \frac{1}{b-a}\int_a^b f(y)\, dy\right| dx$$

provided $n \leq g(x) \leq N$ for a.e. $x \in [a, b]$.

Note that the proof in Theorem 11 is different to the one in [4], using only the linearity and monotonicity properties of the functional A. We should also remark that in [4] the authors did not show the sharpness of the constant $\frac{1}{2}$.

Now, if we consider the normalised isotonic linear functional

$$A_{\bar{w}}(\bar{x}) := \frac{1}{W_n}\sum_{i=1}^n w_i x_i, \tag{8.19}$$

$A_{\bar{w}} : \mathbb{R}^n \to \mathbb{R}$, where $w_i \geq 0$ $\left(i = \overline{1, n}\right)$ and $W_n := \sum_{i=1}^n w_i > 0$, the by Theorem 11 we may obtain the following discrete inequality obtained by Cerone and Dragomir in [3].

Theorem 14 (Cerone & Dragomir, 2002, [3])**.** *Let $\bar{a} = (a_1, \dots, a_n), \bar{b} = (b_1, \dots, b_n) \in \mathbb{R}$ be such that there exists the constants $b, B \in \mathbb{R}$ so that*

$$b \leq b_i \leq B \text{ for each } i \in \{1, \dots, n\}. \tag{8.20}$$

Then one has the inequality

$$\left|\frac{1}{W_n}\sum_{i=1}^n w_i a_i b_i - \frac{1}{W_n}\sum_{i=1}^n w_i a_i \cdot \frac{1}{W_n}\sum_{i=1}^n w_i b_i\right| \tag{8.21}$$
$$\leq \frac{1}{2}(B - b)\frac{1}{W_n}\sum_{i=1}^n w_i \left| a_i - \frac{1}{W_n}\sum_{j=1}^n w_j a_j\right|.$$

The constant $\frac{1}{2}$ is sharp in (8.21).

8.3. **A Counterpart of the (CBS)-Inequality.** The following inequality is known in the literature as the Cauchy-Buniakowski-Schwartz's inequality for isotonic linear functionals or the (CBS)-inequality, for short,

$$[A(fg)]^2 \le A\left(f^2\right) A\left(g^2\right), \tag{8.22}$$

provided $f, g : E \to \mathbb{R}$ are with the property that $fg, f^2, g^2 \in L$ and $A : L \to \mathbb{R}$ is any isotonic linear functional.

Making use of the Grüss inequality (8.14), we may prove the following counterpart of the (CBS)-inequality for isotonic linear functionals.

Theorem 15 (Dragomir, 2002, [9]). *Let $k, l : E \to \mathbb{R}$ be such that $k^2, l^2, kl \in L$ and there exists the real constants $\gamma, \Gamma \in \mathbb{R}$ so that*

$$\gamma \le \frac{k}{l} \le \Gamma. \tag{8.23}$$

Then for any isotonic linear functional $A : L \to \mathbb{R}$ so that $|l| \left|A\left(l^2\right)k - A(kl)\,l\right| \in L$, one has the inequality:

$$\begin{aligned} 0 &\le A\left(k^2\right) A\left(l^2\right) - [A(kl)]^2 \\ &\le \frac{1}{2}(\Gamma - \gamma) A\left[|l| \left|A\left(l^2\right)k - A(kl)\,l\right|\right]. \end{aligned} \tag{8.24}$$

The constant $\frac{1}{2}$ is sharp.

Proof. We choose in (8.14) $f = g = \frac{k}{l}$, $h = l^2$ and $B = A$ to get

$$\begin{aligned} 0 &\le \frac{A\left(k^2\right)}{A\left(l^2\right)} - \frac{[A(kl)]^2}{\left[A\left(l^2\right)\right]^2} \\ &\le \frac{1}{2}(\Gamma - \gamma)\frac{1}{A\left(l^2\right)} A\left[l^2 \left|\frac{k}{l} - \frac{1}{A\left(l^2\right)} A(kl)\right|\right], \end{aligned}$$

provided $A\left(l^2\right) \neq 0$, which is equivalent to

$$\begin{aligned} 0 &\le A\left(k^2\right) A\left(l^2\right) - [A(kl)]^2 \\ &\le \frac{1}{2}(\Gamma - \gamma) A\left(l^2\right) A\left[\left|kl - \frac{l^2}{A\left(l^2\right)} A(kl)\right|\right], \end{aligned}$$

which is clearly equivalent to (8.24). □

The following integral inequality holds.

Corollary 9. *Let $w, f, g : \Omega \to \mathbb{R}$ be a $\mu-$measurable function with $w \ge 0$ μ-a.e. on Ω. If $f, g \in L_w^2(\Omega, \mu) := \left\{f : \Omega \to \mathbb{R},\ \int_\Omega w(y) f^2(y)\, d\mu(y) < \infty\right\}$ and there exists γ, Γ so that*

$$-\infty < \gamma \le \frac{f}{g} \le \Gamma < \infty \quad \text{for } \mu\text{-a.e. } x \in \Omega, \tag{8.25}$$

then one has the inequality:

$$0 \leq \int_\Omega w(x) f^2(x) d\mu(x) \int_\Omega w(x) g^2(x) d\mu(x) - \left[\int_\Omega w(x) f(x) g(x) d\mu(x)\right]^2$$
$$\leq \frac{1}{2}(\Gamma - \gamma) \int_\Omega w(x) |g(x)| \left| \left(\int_\Omega w(y) g^2(y) d\mu(y)\right) f(x) - g(x) \int_\Omega w(y) f(y) g(y) d\mu(y) \right| d\mu(x)$$
$$= \frac{1}{2}(\Gamma - \gamma) \int_\Omega w(x) |g(x)| \left| \int_\Omega w(y) g(y) \left| \begin{array}{cc} f(x) & g(x) \\ f(y) & g(y) \end{array} \right| d\mu(y) \right| d\mu(x). \tag{8.26}$$

The constant $\frac{1}{2}$ *is sharp.*

Remark 8. *In particular, if* $f, g \in L^2(\Omega, \mu)$ *and the condition (8.25) holds, then*

$$0 \leq \int_\Omega f^2(x) d\mu(x) \int_\Omega g^2(x) d\mu(x) - \left[\int_\Omega f(x) g(x) d\mu(x)\right]^2$$
$$\leq \frac{1}{2}(\Gamma - \gamma) \int_\Omega |g(x)| \left| \int_\Omega g(y) \left| \begin{array}{cc} f(x) & g(x) \\ f(y) & g(y) \end{array} \right| d\mu(y) \right| d\mu(x). \tag{8.27}$$

The constant $\frac{1}{2}$ *is sharp.*

The following discrete inequality also holds.

Corollary 10. *Let* $\bar{a} = (a_1, \dots, a_n)$, $\bar{b} = (b_1, \dots, b_n)$ *and* $\bar{w} = (w_1, \dots, w_n)$ *be the sequences of real numbers so that* $w_i \geq 0$ $(i = 1, \dots, n)$, $W_n := \sum_{i=1}^n w_i > 0$ *and*

$$\gamma \leq \frac{a_i}{b_i} \leq \Gamma \quad \text{for each } i \in \{1, \dots, n\}. \tag{8.28}$$

Then one has the inequality

$$0 \leq \sum_{i=1}^n w_i a_i^2 \sum_{i=1}^n w_i b_i^2 - \left(\sum_{i=1}^n w_i a_i b_i\right)^2$$
$$\leq \frac{1}{2}(\Gamma - \gamma) \sum_{i=1}^n w_i b_i \left| \sum_{j=1}^n w_j b_j \left| \begin{array}{cc} a_i & b_i \\ a_j & b_j \end{array} \right| \right|. \tag{8.29}$$

The constant $\frac{1}{2}$ *is sharp.*

Remark 9. *If* $\bar{a}, \bar{b}$ *satisfy (8.28), then one has the inequality*

$$0 \leq \sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 - \left(\sum_{i=1}^n a_i b_i\right)^2$$
$$\leq \frac{1}{2}(\Gamma - \gamma) \sum_{i=1}^n b_i \left| \sum_{j=1}^n b_j \left| \begin{array}{cc} a_i & b_i \\ a_j & b_j \end{array} \right| \right|. \tag{8.30}$$

The constant $\frac{1}{2}$ *is sharp.*

8.4. **A Converse for Jessen's Inequality.** In [6], the author has proved the following converse of Jessen's inequality for normalized isotonic linear functionals.

Theorem 16. *Let $\Phi : (\alpha, \beta) \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable convex function on (α, β), $f : E \to (\alpha, \beta)$ so that $\Phi \circ f,\ f,\ \Phi' \circ f,\ (\Phi' \circ f) \cdot f \in L$. If $A : L \to \mathbb{R}$ is an isotonic linear and normalised functional, then*

$$\begin{aligned} 0 &\leq A(\Phi \circ f) - \Phi(A(f)) \\ &\leq A[(\Phi' \circ f) \cdot f] - A(f) A(\Phi' \circ f) \\ &\leq \frac{1}{4} [\Phi'(\beta) - \Phi'(\alpha)] (\beta - \alpha) \quad \textit{(if } \alpha, \beta \textit{ are finite).} \end{aligned} \tag{8.31}$$

We can state the following result improving the inequality (8.31).

Theorem 17 (Dragomir, 2001, [6])**.** *Let $\Phi : [\alpha, \beta] \to \mathbb{R}$ with $-\infty < \alpha < \beta < \infty$, and f, A are as in Theorem 16, then one has the inequality*

$$\begin{aligned} 0 &\leq A(\Phi \circ f) - \Phi(A(f)) \\ &\leq A[(\Phi' \circ f) \cdot f] - A(f) A(\Phi' \circ f) \\ &\leq \frac{1}{2} [\Phi'(\beta) - \Phi'(\alpha)] A(|f - A(f) \cdot \mathbf{1}|), \end{aligned} \tag{8.32}$$

provided $|f - A(f) \cdot \mathbf{1}| \in L$.

Proof. Taking into account that $\alpha \leq f \leq \beta$ and Φ' is monotonic on $[\alpha, \beta]$, we have $\Phi'(\alpha) \leq \Phi' \circ f \leq \Phi'(\beta)$. Applying Theorem 11, we deduce

$$\begin{aligned} &A[(\Phi' \circ f) \cdot f] - A(f) A(\Phi' \circ f) \\ &\leq \frac{1}{2} [\Phi'(\beta) - \Phi'(\alpha)] A(|f - A(f) \cdot \mathbf{1}|), \end{aligned}$$

and the theorem is proved. □

The following corollary addressing the integral case also holds.

Corollary 11. *Let $\Phi : [\alpha, \beta] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable convex function on (α, β) and $f : \Omega \to [\alpha, \beta]$ so that $\Phi \circ f,\ f,\ \Phi' \circ f,\ (\Phi' \circ f) \cdot f \in L_w(\Omega, \mu)$, where $w \geq 0$ μ-a.e. on Ω with $\int_\Omega w(x) \, d\mu(x) > 0$. Then we have the inequality:*

$$\begin{aligned} 0 &\leq \frac{1}{\int_\Omega w(x) \, d\mu(x)} \int_\Omega w(x) \Phi(f(x)) \, d\mu(x) \\ &\quad - \Phi\left(\frac{1}{\int_\Omega w(x) \, d\mu(x)} \int_\Omega w(x) f(x) \, d\mu(x) \right) \\ &\leq \frac{1}{\int_\Omega w(x) \, d\mu(x)} \int_\Omega w(x) \Phi'(f(x)) f(x) \, d\mu(x) \\ &\quad - \frac{1}{\int_\Omega w(x) \, d\mu(x)} \int_\Omega w(x) \Phi'(f(x)) \, d\mu(x) \\ &\quad \times \frac{1}{\int_\Omega w(x) \, d\mu(x)} \int_\Omega w(x) f(x) \, d\mu(x) \\ &\leq \frac{1}{2} [\Phi'(\beta) - \Phi'(\alpha)] \frac{1}{\int_\Omega w(x) \, d\mu(x)} \\ &\quad \times \int_\Omega w(x) \left| f(x) - \frac{1}{\int_\Omega w(y) \, d\mu(y)} \int_\Omega w(y) f(y) \, d\mu(y) \right| d\mu(x). \end{aligned} \tag{8.33}$$

Remark 10. *If $\mu(\Omega) < \infty$ and $\Phi \circ f,\ f,\ \Phi' \circ f,\ (\Phi' \circ f) \cdot f \in L(\Omega, \mu)$, then we have the inequality:*

$$
\begin{aligned}
(8.34) \qquad 0 &\leq \frac{1}{\mu(\Omega)} \int_{\Omega} \Phi(f(x))\, d\mu(x) - \Phi\left(\frac{1}{\mu(\Omega)} \int_{\Omega} f(x)\, d\mu(x)\right) \\
&\leq \frac{1}{\mu(\Omega)} \int_{\Omega} \Phi'(f(x))\, f(x)\, d\mu(x) \\
&\quad - \frac{1}{\mu(\Omega)} \int_{\Omega} \Phi'(f(x))\, d\mu(x) \cdot \frac{1}{\mu(\Omega)} \int_{\Omega} f(x)\, d\mu(x) \\
&\leq \frac{1}{2}\left[\Phi'(\beta) - \Phi'(\alpha)\right] \frac{1}{\mu(\Omega)} \int_{\Omega} \left| f(x) - \frac{1}{\mu(\Omega)} \int_{\Omega} f(y)\, d\mu(y) \right| d\mu(x).
\end{aligned}
$$

The case of functions of a real variable is embodied in the following inequality that provides a counterpart for the Jensen's integral inequality

$$
\begin{aligned}
(8.35) \qquad 0 &\leq \frac{1}{b-a} \int_a^b \Phi(f(x))\, dx - \Phi\left(\frac{1}{b-a} \int_a^b f(x)\, dx\right) \\
&\leq \frac{1}{b-a} \int_a^b \Phi'(f(x))\, f(x)\, dx \\
&\quad - \frac{1}{b-a} \int_a^b \Phi'(f(x))\, dx \cdot \frac{1}{b-a} \int_a^b f(x)\, dx \\
&\leq \frac{1}{2}\left[\Phi'(\beta) - \Phi'(\alpha)\right] \frac{1}{b-a} \int_a^b \left| f(x) - \frac{1}{b-a} \int_a^b f(y)\, dy \right| dx.
\end{aligned}
$$

The following discrete inequality is valid as well.

Corollary 12. *Let $\Phi : [\alpha, \beta] \to \mathbb{R}$ be a differentiable convex function on (α, β). If $x_i \in [\alpha, \beta]$ and $w_i \geq 0$ $(i = 1, \dots, n)$ with $W_n > 0$, then one has the counterpart of Jensen's discrete inequality:*

$$
\begin{aligned}
(8.36) \qquad 0 &\leq \frac{1}{W_n} \sum_{i=1}^{n} w_i \Phi(x_i) - \Phi\left(\frac{1}{W_n} \sum_{i=1}^{n} w_i x_i\right) \\
&\leq \frac{1}{W_n} \sum_{i=1}^{n} w_i \Phi'(x_i)\, x_i - \frac{1}{W_n} \sum_{i=1}^{n} w_i \Phi'(x_i)\, \frac{1}{W_n} \sum_{i=1}^{n} w_i x_i \\
&\leq \frac{1}{2}\left[\Phi'(\beta) - \Phi'(\alpha)\right] \frac{1}{W_n} \sum_{i=1}^{n} w_i \left| x_i - \frac{1}{W_n} \sum_{j=1}^{n} w_j x_j \right|.
\end{aligned}
$$

Remark 11. *In particular, we get the discrete inequality:*

$$
\begin{aligned}
(8.37) \qquad 0 &\leq \frac{1}{n} \sum_{i=1}^{n} \Phi(x_i) - \Phi\left(\frac{1}{n} \sum_{i=1}^{n} x_i\right) \\
&\leq \frac{1}{n} \sum_{i=1}^{n} \Phi'(x_i)\, x_i - \frac{1}{n} \sum_{i=1}^{n} \Phi'(x_i)\, \frac{1}{n} \sum_{i=1}^{n} x_i \\
&\leq \frac{1}{2}\left[\Phi'(\beta) - \Phi'(\alpha)\right] \frac{1}{n} \sum_{i=1}^{n} \left| x_i - \frac{1}{n} \sum_{j=1}^{n} x_j \right|.
\end{aligned}
$$

9. Generalizations of the Hermite-Hadamard Inequality for Isotonic Sublinear Functionals and Related Results

9.1. **Isotonic Sublinear Functionals.** Let L be a linear class of real-valued functions $g : E \to \mathbb{R}$ having the properties:

(L1) $f, g \in L$ imply $(\alpha f + \beta g) \in L$ for all $\alpha, \beta \in \mathbb{R}$;
(L2) $\mathbb{I} \in L$, i.e., if $f(t) = 1$ for all $t \in E$, then $f \in L$.

An *isotonic linear functional* $A : L \to \mathbb{R}$ is a functional satisfying the conditions:

(A1) $A(\alpha f + \beta g) = \alpha A(f) + \beta A(g)$ for all $f, g \in L$ and $\alpha, \beta \in \mathbb{R}$;
(A2) If $f \in L$ and $f \geq 0$, then $A(f) \geq 0$.
The mapping A is said to be *normalized* if
(A3) $A(\mathbb{I}) = 1$.

Isotonic, that is, order-preserving, linear functionals are natural objects in analysis which enjoy a number of convenient properties. Thus, they provide, for example, Jensen's inequality, which is a functional form of Jensen's inequality (see [13, p. 97]) and a functional Hermite-Hadamard inequality.
In this section we show that these ideas carry over to a sublinear setting [12].
Let E be a non-empty set and K a class of real-valued functions $g : E \to \mathbb{R}$ having the properties

(K1) $\mathbb{I} \in K$;
(K2) $f, g \in K$ imply $f + g \in K$;
(K3) $f \in K$ implies $\alpha \cdot \mathbb{I} + \beta \cdot f \in K$ for all $\alpha, \beta \in \mathbb{R}$.

We define the family of *isotonic sublinear functionals* $S : K \to \mathbb{R}$ by the properties

(S1) $S(f + g) \leq S(f) + S(g)$ for all $f, g \in K$;
(S2) $S(\alpha f) = \alpha S(f)$ for all $\alpha \geq 0$ and $f \in K$;
(S3) If $f \geq g, f, g \in K$, then $S(f) \geq S(g)$.
An isotonic sublinear functional is said to be *normalized* if
(S4) $S(\mathbb{I}) = 1$
and *totally normalized* if, in addition,
(S5) $S(-\mathbb{I}) = -1$.
We note some immediate consequences. from (K2) and (K3), $f - g$ belongs to K whenever $f, g \in K$, so that from (S1)

$$S(f) = S((f - g) + g) \leq S(f - g) + S(g)$$

and hence
(S6) $S(f - g) \geq S(f) - S(g)$ if $f, g \in K$.
Moreover, if S is a totally normalized isotonic sublinear functional, then we have
(S7) $S(\alpha \cdot \mathbb{I}) = \alpha$ for all $\alpha \in \mathbb{R}$
and
(S8) $S(f + \alpha \cdot \mathbb{I}) = S(f) + \alpha$ for all $\alpha \in \mathbb{R}$.

Equation (S7) is immediate from (S2) when $\alpha \geq 0$. When $\alpha < 0$ we have

$$S(\alpha \cdot \mathbb{I}) = S((-\alpha) \cdot (-\mathbb{I})) = (-\alpha) S(-\mathbb{I}) = (-\alpha)(-1) = \alpha.$$

Also, by (S6) and S(7), we have for $\alpha \in \mathbb{R}$

$$S(f - \alpha \cdot \mathbb{I}) \geq S(f) + S(-\alpha \cdot \mathbb{I}) = S(f) - \alpha,$$

which by S1) and S(7)

$$S(f-\alpha\cdot\mathbb{I})\leq S(f)+S(-\alpha\cdot\mathbb{I})=S(f)-\alpha,$$

so that

$$S(f-\alpha\cdot\mathbb{I})=S(f)-\alpha.$$

Since this holds for all $\alpha\in\mathbb{R}$, we have (S8).

It is clear that every normalized isotonic linear functional is a totally normalized isotonic sublinear functional.

In what follows, we shall present some simple examples of sublinear functionals that are not linear.

Example 1. *Let $A_1,...,A_n:L\to\mathbb{R}$ be normalized isotonic linear functionals and $p_{i,j}\in\mathbb{R}$ $(i,j\in\{1,...,n\})$ such that*

$$p_{i,j}\geq 0 \text{ for all } i,j\in\{1,...,n\} \text{ and } \sum_{i=1}^{n}p_{i,j}=1 \text{ for all } j\in\{1,...,n\}.$$

Define the mapping $S:L\to\mathbb{R}$ by

$$S(f)=\max_{1\leq j\leq n}\left\{\sum_{i=1}^{n}p_{i,j}A_i(f)\right\}.$$

Then S is a totally normalized isotonic sublinear functional on L. As particular cases of this functional, we have the mappings

$$S_0(f):=\max_{1\leq j\leq n}\{A_i(f)\}$$

and

$$S_Q(f):=\max_{1\leq j\leq n}\left\{\frac{1}{Q_j}\sum_{i=1}^{n}q_iA_i(f)\right\}$$

where $q_i\geq 0$ for all $i\in\{1,...,n\}$ and $Q_j>0$ for $j=1,...,n$. If we choose $q_i=1$ for all $i\in\{1,...,n\}$, we also have that

$$S_1(f):=\max_{1\leq j\leq n}\left\{\frac{1}{j}\sum_{i=1}^{n}A_i(f)\right\}$$

is a totally normalized isotonic sublinear functional on L.

Example 2. *If $A_1,...,A_n$ are as above and $A:L\to\mathbb{R}$ is also a normalized isotonic linear functional, then the mapping*

$$S_A(f):=\frac{1}{P_n}\sum_{i=1}^{n}p_i\max\{A(f),A_i(f)\}$$

where $p_i\geq 0$ $(1\leq i\leq n)$ with $P_n=\sum_{i=1}^{n}p_i>0$, is also a totally normalized isotonic sublinear functional.

The following provide concrete examples.

Example 3. *Suppose $x=(x_1,...,x_n)$ and $y=(y_1,...,y_n)$ are points in $\mathbb{R}^n$. Then the mappings*

$$S(x):=\max_{1\leq j\leq n}\left\{\sum_{i=1}^{n}p_{i,j}x_i\right\},$$

where $p_i \geq 0$ *and* $\sum_{i=1}^{n} p_{i,j} = 1$ *for* $j \in \{1, ..., n\}$,

$$S_0(x) := \max_{1 \leq i \leq n} \{x_i\}$$

and

$$S_Q(x) := \max_{1 \leq j \leq n} \left\{ \frac{1}{Q_j} \sum_{i=1}^{j} q_i A_x \right\}$$

where $q_i \geq 0$ *and* $Q_j > 0$ *for all* $i, j \in \{1, ..., n\}$, *are totally normalized isotonic sublinear functionals on* $\mathbb{R}^n$.
Suppose $i_0 \in \{1, .., n\}$ *is fixed and* $p_i \geq 0$ *for all* $i \in \{1, ..., n\}$, *with* $P_n > 0$.
Then the mapping

$$S_{i0}(x) := \frac{1}{P_n} \sum_{i=1}^{n} p_i \max\{x_{i0}, x_i\}$$

is also totally normalized.

Example 4. *Denote by* $R[a, b]$ *the linear space of Riemann integrable functions on* $[a, b]$. *Suppose that* $p \in R[a, b]$ *with* $p(t) > 0$ *for all* $t \in [a, b]$. *Then the mappings*

$$S_p(f) := \sup_{x \in (a,b]} \left[\frac{\int_a^x p(t) f(t) \, dt}{\int_a^x p(t) \, dt} \right]$$

and

$$s_1(f) := \sup_{x \in (a,b]} \left[\frac{1}{x - a} \int_a^x f(t) \, dt \right]$$

are totally normalized isotonic sublinear functionals on $R[a, b]$.
If $C \in [a, b]$, *then*

$$S_{c,p}(f) := \frac{\int_a^b p(t) \max(f(c), f(t)) \, dt}{\int_a^b p(t) \, dt}$$

and

$$s_c(f) := \frac{1}{b - a} \int_a^b \max(f(c), f(t)) \, dt$$

are also totally normalized on $R[a, b]$.

9.2. **Jessen Type Inequalities for Sublinear Functionals.** We can give the following generalization of the well-known Jensen's inequality due to S. S. Dragomir, C. E. M. Pearce and J. E. Pečarić [12]:

Theorem 18 (Dragomir, Pearce & Pečarić, 1995, [12])**.** *Let* $\phi : [\alpha, \beta] \subset \mathbb{R} \to \mathbb{R}$ *be a continuous convex function and* $f : E \to [\alpha, \beta]$ *such that* $f, \phi \circ f \in K$. *Then, if* S *is a totally normalized isotonic sublinear functional on* K, *we have* $S(f) \in [\alpha, \beta]$ *and:*

$$S(\phi \circ f) \geq \phi(S(f)). \tag{9.1}$$

Proof. By (S3) and (S7), $\alpha \cdot \mathbb{I} \leq f \leq \beta \cdot \mathbb{I}$ implies

$$\alpha = S(\alpha \cdot \mathbb{I}) \leq S(f) \leq S(\beta \cdot \mathbb{I}) = \beta$$

so that $S(f) \in [\alpha, \beta]$.
Set $l_1(x) = x$ for all $x \in [\alpha, \beta]$. For an arbitrary but fixed $q > 0$, we have by convexity of ϕ that there exist real numbers $u, v \in \mathbb{R}$ such that

(i) $p \leq \phi$ and

(ii) $p(S(f)) \geq \phi(S(f)) - q$

where

$$p(t) = u \cdot \mathbb{I} + v \cdot l_1(t).$$

If $\alpha < S(f) < \beta$ or if ϕ has a finite derivative in $[\alpha, \beta]$, we can replace (ii) by $p(S(f)) = \phi(S(f))$. Now (i) implies $p \circ f \leq \phi \circ f$. Hence, by (S3)

$$S(\phi \circ f) \geq S(p \circ f) = S(u \cdot \mathbb{I} + v \cdot f).$$

If $v \geq 0$, by (S8) and (S2) we have

$$S(u \cdot \mathbb{I} + v \cdot f) = u + vS(f) = p(S(f)),$$

while if $v < 0$, by (S6), (S7) and (S2) we have

$$\begin{aligned} S(u \cdot \mathbb{I} + v \cdot f) &= S(u \cdot \mathbb{I} - |v| f) \geq u - S(|v| f) \\ &= u - |v| S(f) = u + vS(f) = p(S(f)). \end{aligned}$$

Therefore, we have in either case

$$S(\phi \circ f) \geq \phi(S(f)) - q.$$

Since q is arbitrary, the proof is complete.

□

Remark 12. *If $S = A$, a normalized isotonic linear functional on L, then* (9.1) *becomes the well-known Jessen's inequality.*

The following generalizations of Jensen's inequality for isotonic linear functionals also hold:

Let $A_1, ..., A_n : L \to \mathbb{R}$ be normalized isotonic linear functionals and $p_{i,j} \in \mathbb{R}$ be such that:

$$p_{i,j} \geq 0 \text{ and } \sum_{i=1}^{n} p_{i,j} = 1 \text{ for all } i, j \in \{1, ..., n\}.$$

If $\phi : [\alpha, \beta] \to \mathbb{R}$ is convex and $f : E \to [\alpha, \beta]$ is such that $f, \phi \circ f \in L$ then:

$$\max_{1 \leq j \leq n} \left\{ \sum_{i=1}^{n} p_{i,j} A_i(\phi \circ f) \right\} \geq \phi\left(\max_{1 \leq j \leq n} \left\{ \sum_{i=1}^{n} p_{i,j} A_i(f) \right\} \right).$$

The proof follows by Theorem 18 applied for the following mapping

$$S(f) := \max_{1 \leq j \leq n} \left\{ \sum_{i=1}^{n} p_{i,j} A_i(f) \right\},$$

which is a totally normalized isotonic sublinear functional on L.

Remark 13. *If $A_1, ..., A_n, \phi$ and f are as above, then*

$$\max_{1 \leq j \leq n} \{A_i(\phi \circ f)\} \geq \phi\left(\max_{1 \leq j \leq n} \{A_i(f)\} \right)$$

and

$$\max_{1 \leq j \leq n} \left\{ \frac{1}{Q_j} \sum_{i=1}^{j} q_i A_i(\phi \circ f) \right\} \geq \phi\left(\max_{1 \leq j \leq n} \left\{ \frac{1}{Q_j} \sum_{i=1}^{j} q_i A_i(f) \right\} \right)$$

where $q_i \geq 0$ with $Q_j > 0$ for all $i, j \in \{1, ..., n\}$.

Corollary 13. *If $A_1, ..., A_n, \phi$ and f are as shown, $p_i \geq 0, i \in \{1, ..., n\}, P_n > 0$ and $A : L \to \mathbb{R}$ is also a normalized isotonic linear functional, then we have the inequality*

$$\frac{1}{P_n} \sum_{i=1}^{n} p_i \max \{A (\phi \circ f), A_i (\phi \circ f)\} \geq \phi \left(\frac{1}{P_n} \sum_{i=1}^{n} p_i \max \{A (f), A_i (f)\} \right).$$

The following reverse of Jensen's inequality for sublinear functionals was proved by S. S. Dragomir and C. E. M. Pearce in [12]:

Theorem 19 (Dragomir, Pearce & Pečarić, 1995, [12])**.** *Let $\phi : [\alpha, \beta] \subset \mathbb{R} \to \mathbb{R}$ be a convex function ($\alpha < \beta$) and $f : E \to [\alpha, \beta]$ such that $\phi \circ f, f \in K$. Let $\lambda = sgn (\phi (\beta) - \phi (\alpha))$. Then, if S is a totally normalized isotonic sublinear functional on K we have*

$$S (\phi \circ f) \leq \frac{\beta \phi (\alpha) - \alpha \phi (\beta)}{\beta - \alpha} + \frac{|\phi (\beta) - \phi (\alpha)|}{\beta - \alpha} S (\lambda f). \tag{9.2}$$

Proof. Since ϕ is convex on $[\alpha, \beta]$ we have:

$$\phi (v) \leq \frac{w - v}{w - u} \phi (u) + \frac{v - u}{w - u} \phi (w),$$

where $u \leq v \leq w$ and $u < w$ (see also [13, p. 2]).
Set $u = \alpha, v = f (t), w = \beta$. Then

$$\phi (f (t)) \leq \frac{\beta - f (t)}{\beta - \alpha} \phi (\alpha) + \frac{f (t) - \alpha}{\beta - \alpha} \phi (\beta), t \in E,$$

or, alternatively,

$$\phi \circ f \leq \frac{\beta \phi (\alpha) - \alpha \phi (\beta)}{\beta - \alpha} \cdot \mathbb{I} + \frac{\phi (\beta) - \phi (\alpha)}{\beta - \alpha} \cdot f.$$

Applying the functional S and using its properties we have

$$\begin{aligned} S (\phi \circ f) &\leq S \left(\frac{\beta \phi (\alpha) - \alpha \phi (\beta)}{\beta - \alpha} \cdot \mathbb{I} + \frac{\phi (\beta) - \phi (\alpha)}{\beta - \alpha} \cdot f \right) \\ &= \frac{\beta \phi (\alpha) - \alpha \phi (\beta)}{\beta - \alpha} + S \left(\frac{\phi (\beta) - \phi (\alpha)}{\beta - \alpha} \cdot f \right) \\ &= \frac{\beta \phi (\alpha) - \alpha \phi (\beta)}{\beta - \alpha} + \frac{|\phi (\beta) - \phi (\alpha)|}{\beta - \alpha} S (\lambda f). \end{aligned}$$

Hence, the theorem is proved. □

Remark 14. *If $S = A$, and A is a normalized isotonic linear functional, then, by* (9.2) *we deduce the inequality*

$$A (\phi (f)) \leq \frac{\{(\beta - A (f)) \phi (\alpha) + (A (f) - \alpha) \phi (\beta)\}}{(\beta - \alpha)}.$$

That is, the result of Lemma 1 from [13]*. Note that this last inequality is a generalization of the inequality*

$$A (\phi) \leq \frac{\{(b - A (l_1)) \phi (a) + (A (l_1) - a) \phi (b)\}}{(b - a)}$$

due to A. Lupas [13, Theorem A]*. Here, $E = [a, b]$ $(-\infty < a < b < \infty)$, L satisfies (L1), (L2), $A : L \to \mathbb{R}$ satisfies (A1), (A2), $A (\mathbb{I}) = 1, \phi$ is convex on E and $\phi \in L, l_1 \in L$, where $l_1 (x) = x, x \in [a, b]$.*

By the use of Jensen's and Lupas' inequalities for totally normalized sublinear functionals, we can state the following generalization of the classical Hermite-Hadamard's integral inequality due to S.S. Dragomir, C.E.M. Pearce and J.E. Pečarić [12].

Theorem 20 (Dragomir, Pearce & Pečarić, 1995, [12])**.** *Let $\phi : [\alpha, \beta] \to \mathbb{R}$ be a convex function and $e : E \to [\alpha, \beta]$ a mapping such that $\phi \circ e$ and e belong to K and let $\lambda := sgn(\phi(\beta) - \phi(\alpha))$. If S is a totally normalized isotonic sublinear functional on K with*

$$S(\lambda e) = \lambda \cdot \frac{\alpha + \beta}{2} \text{ and } S(e) = \frac{\alpha + \beta}{2},$$

then we have the inequality

$$\phi\left(\frac{\alpha + \beta}{2}\right) \le S(\phi \circ e) \le \frac{\phi(\alpha) + \phi(\beta)}{2}. \tag{9.3}$$

Proof. The first inequality in (9.3) follows by Jensen's inequality (9.1) applied to the mapping e.
By inequality (9.2), we have

$$\begin{aligned} S(\phi \circ e) &\le \frac{\beta\phi(\alpha) - \alpha\phi(\beta)}{\beta - \alpha} + \frac{(\phi(\beta) - \phi(\alpha))(\beta + \alpha)}{2(\beta - \alpha)} \\ &= \frac{\phi(\alpha) + \phi(\beta)}{2}, \end{aligned}$$

and the statement is proved. □

Remark 15. *If $S = A, \phi$ is as above and $e : E \to [\alpha, \beta]$ is such that $\phi \circ e, e \in L$ and $A(e) = \frac{\alpha+\beta}{2}$, then the Hermite-Hadamard inequality*

$$\phi\left(\frac{\alpha + \beta}{2}\right) \le A(\phi \circ e) \le \frac{\phi(\alpha) + \phi(\beta)}{2},$$

holds for normalized isotonic linear functionals (see also [16] *and* [5]*).*

Remark 16. *If in the above theorem we assume that $\phi(\beta) \ge \phi(\alpha)$, then we can drop the assumption $S(\lambda e) = \lambda \cdot \frac{\alpha+\beta}{2}$.*

Theorem 21 (Dragomir, Pearce & Pečarić, 1995, [12])**.** *Let ϕ, f and S be defined as in Theorem 19 with $\phi(\beta) \ge \phi(\alpha)$. Then*

$$S(\phi(f)) \le \frac{\{(\beta - S(f))\phi(\alpha) + (S(f) - \alpha)\phi(\beta)\}}{\beta - \alpha}. \tag{9.4}$$

The proof is a simple consequence of Theorem 19.

Finally, we have the following result [12]:

Theorem 22 (Dragomir, Pearce & Pečarić, 1995, [12])**.** *Let the hypothesis of Theorem 21 be fulfilled and let T be an interval which is such that $T \supset \phi([\alpha, \beta])$. If $F(u, v)$ is a real-valued function defined on $T \times T$ and increasing in u, then*

$$\begin{aligned} &F[S(\phi(f)), \phi(S(f))] \\ &\le \max_{x \in [a,b]} F\left[\frac{\beta - x}{\beta - a}\phi(\alpha) + \frac{x - \alpha}{\beta - \alpha}\phi(\beta), \phi(x)\right] \\ &= \max_{\theta \in [0,1]} F[\theta\phi(\alpha) + (1 - \theta)\phi(\beta), \phi(\theta\alpha + (1 - \theta)\beta)]. \end{aligned} \tag{9.5}$$

Proof. By (9.4) and the increasing property of $F(\cdot, y)$ we have

$$F[S(\phi(f)), \phi(S(f))] \leq F\left[\frac{\beta - S(f)}{\beta - a}\phi(\alpha) + \frac{S(f) - \alpha}{\beta - \alpha}\phi(\beta, \phi(S(f)))\right]$$
$$\leq \max_{x \in [a,b]} F\left[\frac{\beta - x}{\beta - a}\phi(\alpha) + \frac{x - \alpha}{\beta - \alpha}\phi(\beta), \phi(x)\right].$$

Of course the equality in (9.5) follows immediately from the change of variable $\theta = \frac{\beta - x}{\beta - a}$, so that $x = \theta\alpha + (1 - \theta)\beta$ with $0 \leq \theta \leq 1$. □

9.3. **Applications for Special Means.**

(1) Suppose that $e \in K, p \geq 1, e^p \in K$ and S is as above. We can define the mean
$$L_p(s, e) := [S(e^p)]^{\frac{1}{p}}.$$
By the use of Theorem 20 we have the inequality
$$A(\alpha, \beta) \leq L_p(s, e) \leq [A(\alpha^p, \beta^p)]^{\frac{1}{p}},$$
provided that
$$S(e) = \frac{\alpha + \beta}{2}.$$
A particular case which generates in its turn the classical L_p-mean is where $S = A$, where A is a linear isotonic functional defined on K.

(2) Now, if $e \in K$ is such that $e^{-1} \in K$, we can define the mean as
$$L(s, e) := \left[S\left(e^{-1}\right)\right]^{-1}.$$
If we assume that $S(-e) = -\frac{\alpha+\beta}{2}$ and $S(e) = \frac{\alpha+\beta}{2}$, then, by Theorem 20 we have the inequality:
$$H(\alpha, \beta) \leq L(S, e) \leq A(\alpha, \beta).$$
A particular case which generalizes in its turn the classical logarithmic mean is where $S = A$, where A is as above.

(3) Finally, if we suppose that $e \in K$ is such that $\ln e \in K$, we can also define the mean
$$I(S, e) := \exp[-S(-\ln e)].$$
Now, if we assume that $S(-e) = -\frac{\alpha+\beta}{2}$ and $S(e) = \frac{\alpha+\beta}{2}$, then, by Theorem 20 we get the inequality:
$$G(\alpha, \beta) \leq I(S, e) \leq A(\alpha, \beta),$$
which generalizes the corresponding inequality for the identric mean.

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