

APPROXIMATING THE RIEMANN-STIELTJES INTEGRAL BY A TRAPEZOIDAL QUADRATURE RULE WITH APPLICATIONS

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ABSTRACT. In this paper we provide sharp bounds for the error in approximating the Riemann-Stieltjes integral $\int_a^b f(t) du(t)$ by the trapezoidal rule

$$\frac{f(a) + f(b)}{2} \cdot [u(b) - u(a)]$$

under various assumptions for the integrand f and the integrator u for which the above integral exists. Applications for continuous functions of selfadjoint operators in Hilbert spaces are provided as well.

1. INTRODUCTION

In Classical Analysis, a *trapezoidal type inequality* is an inequality that provides upper and/or lower bounds for the quantity

$$\frac{f(a) + f(b)}{2} (b - a) - \int_a^b f(t) dt,$$

that is the error in approximating the integral by a trapezoidal rule, for various classes of integrable functions f defined on the compact interval $[a, b]$.

In the following we recall some trapezoidal inequalities for various classes of scalar functions of interest, such as: functions of bounded variation, monotonic, Lipschitzian, absolutely continuous or convex functions.

The case of functions of bounded variation was obtained in [6] (see also [5, p. 68]):

Theorem 1. *Let $f : [a, b] \rightarrow \mathbb{C}$ be a function of bounded variation. We have the inequality*

$$(1.1) \quad \left| \int_a^b f(t) dt - \frac{f(a) + f(b)}{2} (b - a) \right| \leq \frac{1}{2} (b - a) \bigvee_a^b(f),$$

where $\bigvee_a^b(f)$ denotes the total variation of f on the interval $[a, b]$. The constant $\frac{1}{2}$ is the best possible one.

This result may be improved if one assumes the monotonicity of f as follows (see [5, p. 76]):

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Theorem 2. Let $f : [a, b] \rightarrow \mathbb{R}$ be a monotonic nondecreasing function on $[a, b]$. Then we have the inequalities:

$$(1.2) \quad \left| \int_a^b f(t) dt - \frac{f(a) + f(b)}{2} (b - a) \right| \\ \leq \frac{1}{2} (b - a) [f(b) - f(a)] - \int_a^b \operatorname{sgn} \left(t - \frac{a+b}{2} \right) f(t) dt \\ \leq \frac{1}{2} (b - a) [f(b) - f(a)].$$

The above inequalities are sharp.

If the mapping is Lipschitzian, then the following result holds as well [9] (see also [5, p. 82]).

Theorem 3. Let $f : [a, b] \rightarrow \mathbb{C}$ be an L -Lipschitzian function on $[a, b]$, i.e., f satisfies the condition:

$$(L) \quad |f(s) - f(t)| \leq L |s - t| \quad \text{for any } s, t \in [a, b] \quad (L > 0 \text{ is given}).$$

Then we have the inequality:

$$(1.3) \quad \left| \int_a^b f(t) dt - \frac{f(a) + f(b)}{2} (b - a) \right| \leq \frac{1}{4} (b - a)^2 L.$$

The constant $\frac{1}{4}$ is best in (1.3).

If we would assume absolute continuity for the function f , then the following estimates in terms of the Lebesgue norms of the derivative f' hold [5, p. 93]:

Theorem 4. Let $f : [a, b] \rightarrow \mathbb{C}$ be an absolutely continuous function on $[a, b]$. Then we have

$$(1.4) \quad \left| \int_a^b f(t) dt - \frac{f(a) + f(b)}{2} (b - a) \right| \\ \leq \begin{cases} \frac{1}{4} (b - a)^2 \|f'\|_\infty & \text{if } f' \in L_\infty[a, b]; \\ \frac{1}{2(q+1)^{\frac{1}{q}}} (b - a)^{1+1/q} \|f'\|_p & \text{if } f' \in L_p[a, b], \\ & p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{2} (b - a) \|f'\|_1, & \end{cases}$$

where $\|\cdot\|_p$ ($p \in [1, \infty]$) are the Lebesgue norms, i.e.,

$$\|f'\|_\infty = \operatorname{ess\,sup}_{s \in [a, b]} |f'(s)|$$

and

$$\|f'\|_p := \left(\int_a^b |f'(s)|^p ds \right)^{\frac{1}{p}}, \quad p \geq 1.$$

The case of convex functions is as follows [13]:

Theorem 5. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a convex function on $[a, b]$. Then we have the inequalities*

$$(1.5) \quad \begin{aligned} & \frac{1}{8} (b-a)^2 \left[f'_+ \left(\frac{a+b}{2} \right) - f'_- \left(\frac{a+b}{2} \right) \right] \\ & \leq \frac{f(a) + f(b)}{2} (b-a) - \int_a^b f(t) dt \\ & \leq \frac{1}{8} (b-a)^2 [f'_-(b) - f'_+(a)]. \end{aligned}$$

The constant $\frac{1}{8}$ is sharp in both sides of (1.5).

For other scalar trapezoidal type inequalities, see [5].

Motivated by the above results, we endeavour in the following to provide sharp bounds for the error in approximating the Riemann-Stieltjes integral $\int_a^b f(t) du(t)$ by the trapezoidal rule

$$(1.6) \quad \frac{f(a) + f(b)}{2} \cdot [u(b) - u(a)]$$

under various assumptions for the integrand f and the integrator u for which the above integral exists. Applications for continuous functions of selfadjoint operators in Hilbert spaces are provided as well.

The above quadrature (1.6) is different from the one considered in the papers [2], [4], [12] and [14] where error bounds in approximating the Riemann-Stieltjes integral $\int_a^b f(t) du(t)$ by the *generalized trapezoidal formula*

$$\left[u(b) - u \left(\frac{a+b}{2} \right) \right] f(b) + \left[u \left(\frac{a+b}{2} \right) - u(a) \right] f(a)$$

were provided.

In [21], P.R. Mercer has obtained some Hadamard's type inequalities for the Riemann-Stieltjes integral when the integrand is convex while in [24] M. Munteanu has provided error bounds in approximating the Riemann-Stieltjes integral by the use of Weyl derivatives and the method of approximation.

For other results and techniques that are different from the ones outlined below, we recommend the papers [1], [3], [7], [8] and the classical paper on the closed Newton-Cotes quadrature rules for the Riemann-Stieltjes integrals [26].

2. THE CASE OF HÖLDER-CONTINUOUS INTEGRANDS

2.1. The Case of Bounded Variation Integrators. The following theorem generalizing the classical trapezoid inequality for integrators of bounded variation and Hölder-continuous integrands was obtained by the author in 2001, see [11]. For the sake of completeness and since parts of it will be used in the proofs of other results, we will present it here as well.

Theorem 6 (Dragomir, 2001, [11]). *Let $f : [a, b] \rightarrow \mathbb{C}$ be a p -Hölder type function, that is, it satisfies the condition*

$$(2.1) \quad |f(x) - f(y)| \leq H |x - y|^p \text{ for all } x, y \in [a, b],$$

where $H > 0$ and $p \in (0, 1]$ are given, and $u : [a, b] \rightarrow \mathbb{C}$ is a function of bounded variation on $[a, b]$. Then we have the inequality:

$$(2.2) \quad \left| \frac{f(a) + f(b)}{2} \cdot [u(b) - u(a)] - \int_a^b f(t) du(t) \right| \leq \frac{1}{2^p} H (b-a)^p \bigvee_a^b(u).$$

The constant $C = 1$ on the right hand side of (2.2) cannot be replaced by a smaller quantity.

Proof. Using the inequality for the Riemann-Stieltjes integral of continuous integrands and bounded variation integrators, we have

$$(2.3) \quad \begin{aligned} & \left| \frac{f(a) + f(b)}{2} \cdot [u(b) - u(a)] - \int_a^b f(t) du(t) \right| \\ &= \left| \int_a^b \left(\frac{f(a) + f(b)}{2} - f(t) \right) du(t) \right| \\ &\leq \sup_{t \in [a, b]} \left| \frac{f(a) + f(b)}{2} - f(t) \right| \bigvee_a^b(u). \end{aligned}$$

As f is of p - H -Hölder type, then we have

$$(2.4) \quad \begin{aligned} \left| \frac{f(a) + f(b)}{2} - f(t) \right| &= \left| \frac{f(a) - f(t) + f(b) - f(t)}{2} \right| \\ &\leq \frac{1}{2} |f(a) - f(t)| + \frac{1}{2} |f(b) - f(t)| \\ &\leq \frac{1}{2} H [(t-a)^p + (b-t)^p], \end{aligned}$$

for any $t \in [a, b]$.

Now, consider the mapping $\gamma(t) = (t-a)^p + (b-t)^p$, $t \in [a, b]$, $p \in (0, 1]$. Then $\gamma'(t) = p(t-a)^{p-1} - p(b-t)^{p-1} = 0$ iff $t = \frac{a+b}{2}$ and $\gamma'(t) > 0$ on $(a, \frac{a+b}{2})$, $\gamma'(t) < 0$ on $(\frac{a+b}{2}, b)$, which shows that its maximum is realized at $t = \frac{a+b}{2}$ and

$$\max_{t \in [a, b]} \gamma(t) = \gamma\left(\frac{a+b}{2}\right) = 2^{1-p} (b-a)^p.$$

Consequently, by (2.4), we have

$$\sup_{t \in [a, b]} \left| \frac{f(a) + f(b)}{2} - f(t) \right| \leq H \left(\frac{b-a}{2} \right)^p.$$

Using (2.3) we obtain the desired inequality (2.2).

To prove the sharpness of the constant 1, assume that (2.2) holds with a constant $C > 0$. That is

$$(2.5) \quad \left| \frac{f(a) + f(b)}{2} \cdot [u(b) - u(a)] - \int_a^b f(t) du(t) \right| \leq \frac{C}{2^p} H (b-a)^p \bigvee_a^b(u).$$

Choose $f : [0, 1] \rightarrow \mathbb{R}$, $f(t) = t^p$, $p \in (0, 1]$ and $u(t) = t$, $t \in [0, 1]$. We observe that f is of p - H -Hölder type with $H = 1$ and u is of bounded variation, then, by (2.5) we obtain

$$\left| \frac{1}{2} - \frac{1}{p+1} \right| \leq \frac{C}{2^p}, \text{ for any } p \in (0, 1].$$

That is,

$$C \geq \frac{1-p}{p+1} \cdot 2^p, \text{ for any } p \in (0, 1].$$

Letting $p \rightarrow 0+$, we get $C \geq 1$ and the theorem is completely proved. \square

Remark 1. We notice that, if f is H -Lipschitzian, then (2.2) becomes

$$(2.6) \quad \left| \frac{f(a) + f(b)}{2} \cdot [u(b) - u(a)] - \int_a^b f(t) du(t) \right| \leq \frac{1}{2} H (b-a) \bigvee_a^b(u).$$

The constant $\frac{1}{2}$ is best possible in (2.6).

Remark 2. If we assume that $g : [a, b] \rightarrow \mathbb{C}$ is Lebesgue integrable on $[a, b]$, then $u(x) = \int_a^x g(t) dt$ is differentiable almost everywhere, $u(b) = \int_a^b g(t) dt$, $u(a) = 0$ and $\bigvee_a^b(u) = \int_a^b |g(t)| dt$. Consequently, by (2.2) we obtain

$$(2.7) \quad \left| \frac{f(a) + f(b)}{2} \cdot \int_a^b g(t) dt - \int_a^b f(t) g(t) dt \right| \leq \frac{1}{2^p} H (b-a)^p \int_a^b |g(t)| dt.$$

From (2.7) we get a weighted version of the trapezoid inequality,

$$(2.8) \quad \left| \frac{f(a) + f(b)}{2} - \frac{1}{\int_a^b g(t) dt} \cdot \int_a^b f(t) g(t) dt \right| \leq \frac{1}{2^p} H (b-a)^p,$$

provided that $g(t) \geq 0$, for almost every $t \in [a, b]$ and $\int_a^b g(t) dt \neq 0$.

Example 1. By applying the inequality (2.8), we give now some examples of weighted trapezoid inequalities for some of the most popular weights.

- a) (Legendre) If $g(t) = 1$ and $t \in [a, b]$, then we get the following trapezoid inequality for Hölder type mappings f :

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{1}{2^p} H (b-a)^p.$$

- b) (Logarithm) If $g(t) = \ln\left(\frac{1}{t}\right)$, $t \in (0, 1]$, f is of p -Hölder type on $[0, 1]$ and the integral $\int_0^1 f(t) \ln\left(\frac{1}{t}\right) dt$ is finite, then we have

$$\left| \frac{f(0) + f(1)}{2} - \int_0^1 f(t) \ln\left(\frac{1}{t}\right) dt \right| \leq \frac{1}{2^p} H.$$

- c) (Jacobi) If $g(t) = \frac{1}{\sqrt{t}}$, $t \in (0, 1]$, f is as above and the integral $\int_0^1 \frac{f(t)}{\sqrt{t}} dt$ is finite, then we obtain

$$\left| \frac{f(0) + f(1)}{2} - \frac{1}{2} \int_0^1 \frac{f(t)}{\sqrt{t}} dt \right| \leq \frac{1}{2^p} H.$$

- d) (Chebychev) If $g(t) = \frac{1}{\sqrt{1-t^2}}$, $t \in (-1, 1)$, f is of p -Hölder type on $(-1, 1)$ and the integral $\int_{-1}^1 \frac{f(t)}{\sqrt{1-t^2}} dt$ is finite, then

$$\left| \frac{f(-1) + f(1)}{2} - \frac{1}{\pi} \int_{-1}^1 \frac{f(t)}{\sqrt{1-t^2}} dt \right| \leq H.$$

2.2. The Case of Lipschitzian Integrators. The case when the integrator is Lipschitzian is as follows:

Theorem 7. *Let $f : [a, b] \rightarrow \mathbb{C}$ be a p - H -Hölder type mapping where $H > 0$ and $p \in (0, 1]$ are given, and $u : [a, b] \rightarrow \mathbb{C}$ is a Lipschitzian function on $[a, b]$, this means that*

$$(2.9) \quad |u(x) - u(y)| \leq L|x - y| \text{ for all } x, y \in [a, b],$$

where $L > 0$ is given. Then we have the inequality:

$$(2.10) \quad \left| \frac{f(a) + f(b)}{2} \cdot [u(b) - u(a)] - \int_a^b f(t) du(t) \right| \leq \frac{1}{p+1} HL(b-a)^{p+1}.$$

Proof. Using the inequality for the Riemann-Stieltjes integral of Riemann integrable integrands and Lipschitzian integrators, we have

$$(2.11) \quad \begin{aligned} & \left| \frac{f(a) + f(b)}{2} \cdot [u(b) - u(a)] - \int_a^b f(t) du(t) \right| \\ &= \left| \int_a^b \left(\frac{f(a) + f(b)}{2} - f(t) \right) du(t) \right| \leq L \int_a^b \left| \frac{f(a) + f(b)}{2} - f(t) \right| dt. \end{aligned}$$

On utilizing the inequality (2.4) we have

$$(2.12) \quad \begin{aligned} & \int_a^b \left| \frac{f(a) + f(b)}{2} - f(t) \right| dt \\ & \leq \frac{1}{2} H \int_a^b [(t-a)^p + (b-t)^p] dt = \frac{1}{p+1} H(b-a)^{p+1}, \end{aligned}$$

which, by (2.11), provides the desired result (2.10). \square

Remark 3. *If we assume that $g : [a, b] \rightarrow \mathbb{C}$ is Lebesgue measurable on $[a, b]$ with $\|g\|_\infty := \text{ess sup}_{t \in [a, b]} |g(t)| < \infty$, then on choosing $u(x) = \int_a^x g(t) dt$ we get that u is Lipschitzian with the constant $L = \|g\|_\infty$. Consequently, by (2.10) we obtain*

$$(2.13) \quad \left| \frac{f(a) + f(b)}{2} \cdot \int_a^b g(t) dt - \int_a^b f(t) g(t) dt \right| \leq \frac{1}{p+1} H \|g\|_\infty (b-a)^{p+1}.$$

Remark 4. *We observe that if the function $u : [a, b] \rightarrow \mathbb{R}$ is a Lipschitzian function on $[a, b]$ with the constant L then it is of bounded variation, and, obviously, $\bigvee_a^b(u) \leq (b-a)L$. On utilizing the inequality (2.2) we deduce*

$$(2.14) \quad \left| \frac{f(a) + f(b)}{2} \cdot [u(b) - u(a)] - \int_a^b f(t) du(t) \right| \leq \frac{1}{2^p} HL(b-a)^{p+1}.$$

Now, in order to compare which one of the inequalities (2.10) and (2.14) is better, we consider the auxiliary function $\rho(p) := 2^p - p - 1$ and $p \geq 0$. We observe that $\rho'(p) = 2^p \ln 2 - 1$ and the equation $\rho'(p) = 0$ has a unique solution $p_0 = -\log_2(\ln 2) \in (0, 1)$. Since $\rho'(p) < 0$ on $(0, p_0)$ and $\rho'(p) > 0$ on (p_0, ∞) it follows that $\min_{p \in [0, \infty)} \rho(p) = \rho(p_0) = \frac{1}{\ln 2} - \ln 2 - 1$. Also, we observe that $\rho(p) < 0$ on $(0, 1)$, $\rho(p) > 0$ on $(1, \infty)$ and $\rho(0) = \rho(1) = 0$. In conclusion

$$\frac{1}{p+1} < \frac{1}{2^p} \text{ for all } p \in (0, 1),$$

showing that the inequality (2.10) is always better than the inequality (2.14).

Remark 5. We notice that, if f is H -Lipschitzian, then (2.10) becomes

$$\left| \frac{f(a) + f(b)}{2} \cdot [u(b) - u(a)] - \int_a^b f(t) du(t) \right| \leq \frac{1}{2}HL(b-a)^2.$$

2.3. The Case of Monotonic Nondecreasing Integrators. In the case when u is monotonic nondecreasing, we have the following result as well:

Theorem 8. Let $f : [a, b] \rightarrow \mathbb{C}$ be a p - H -Hölder type mapping where $H > 0$ and $p \in (0, 1]$ are given, and $u : [a, b] \rightarrow \mathbb{R}$ a monotonic nondecreasing function on $[a, b]$. Then we have the inequality:

$$(2.15) \quad \left| \frac{f(a) + f(b)}{2} \cdot [u(b) - u(a)] - \int_a^b f(t) du(t) \right| \\ \leq \frac{1}{2}H \left\{ (b-a)^p [u(b) - u(a)] - p \int_a^b \left[\frac{(b-t)^{1-p} - (t-a)^{1-p}}{(b-t)^{1-p}(t-a)^{1-p}} \right] u(t) dt \right\} \\ \leq \frac{1}{2^p}H(b-a)^p [u(b) - u(a)].$$

The inequalities in (2.15) are sharp.

Proof. Using the inequality for the Riemann-Stieltjes integral of continuous integrands and monotonic integrators, we have

$$(2.16) \quad \left| \frac{f(a) + f(b)}{2} \cdot [u(b) - u(a)] - \int_a^b f(t) du(t) \right| \\ \leq \int_a^b \left| \frac{f(a) + f(b)}{2} - f(t) \right| du(t).$$

Utilising (2.4) we then have

$$(2.17) \quad \int_a^b \left| \frac{f(a) + f(b)}{2} - f(t) \right| du(t) \leq \frac{1}{2}H \int_a^b [(t-a)^p + (b-t)^p] du(t).$$

Integrating by parts in the Riemann-Stieltjes integral we get

$$(2.18) \quad \int_a^b [(t-a)^p + (b-t)^p] du(t) \\ = [(t-a)^p + (b-t)^p] u(t) \Big|_a^b - p \int_a^b \left[(t-a)^{p-1} - (b-t)^{p-1} \right] u(t) dt \\ = (b-a)^p [u(b) - u(a)] - p \int_a^b \left[\frac{(b-t)^{1-p} - (t-a)^{1-p}}{(b-t)^{1-p}(t-a)^{1-p}} \right] u(t) dt,$$

which together with (2.7) produces the first part of the inequality (2.15).

Now, since

$$[(t-a)^p + (b-t)^p] \leq 2^{1-p}(b-a)^p$$

for any $t \in [a, b]$, then

$$\int_a^b [(t-a)^p + (b-t)^p] u(t) dt \leq 2^{1-p}(b-a)^p [u(b) - u(a)]$$

and the last part of (2.15) is also proved.

Choose $f : [0, 1] \rightarrow \mathbb{R}$, $f(t) = t$, and

$$u(t) = \begin{cases} 0 & \text{for } t \in [0, 1), \\ 1 & \text{for } t = 1. \end{cases}$$

We observe that f is of H -Hölder type with $H = 1$ and u is monotonic nondecreasing on $[0, 1]$, then we obtain in all sides of the inequality (2.15) the same quantity $\frac{1}{2}$. \square

Remark 6. We notice that if f is H -Lipschitzian, then (2.15) reduces to

$$(2.19) \quad \left| \frac{f(a) + f(b)}{2} \cdot [u(b) - u(a)] - \int_a^b f(t) du(t) \right| \leq \frac{1}{2} H (b - a) [u(b) - u(a)].$$

The constant $\frac{1}{2}$ is best possible in (2.19).

3. THE CASE OF HÖLDER-CONTINUOUS INTEGRATORS

3.1. The Case of Bounded Variation Integrands.

Theorem 9 (Dragomir, 2001, [11]). Let $f : [a, b] \rightarrow \mathbb{C}$ be a function of bounded variation on $[a, b]$ and $u : [a, b] \rightarrow \mathbb{C}$ a p - K -Hölder continuous function on that interval. Then we have the inequality:

$$(3.1) \quad \left| \frac{f(a) + f(b)}{2} [u(b) - u(a)] - \int_a^b f(t) du(t) \right| \leq \frac{1}{2^p} K (b - a)^p \bigvee_a^b(f).$$

The constant $C = 1$ on the right hand side of (3.1) cannot be replaced by a smaller quantity.

Proof. Using the integration by parts formula for the Riemann-Stieltjes integrals, we have the following equality of interest

$$(3.2) \quad \begin{aligned} & \int_a^b \left[u(t) - \frac{u(a) + u(b)}{2} \right] df(t) \\ &= \left[u(t) - \frac{u(a) + u(b)}{2} \right] f(t) \Big|_a^b - \int_a^b f(t) du(t) \\ &= \frac{f(a) + f(b)}{2} [u(b) - u(a)] - \int_a^b f(t) du(t). \end{aligned}$$

Utilising a similar approach to the one in the proof of Theorem 6 we deduce the desired inequality (3.1).

To prove the sharpness of the constant 1, assume that (3.1) holds with a constant $C > 0$, i.e.,

$$(3.3) \quad \left| \frac{f(a) + f(b)}{2} [u(b) - u(a)] - \int_a^b f(t) du(t) \right| \leq \frac{C}{2^p} K (b - a)^p \bigvee_a^b(f).$$

Choose $u(t) = t^p$, $p \in (0, 1]$, $t \in [0, 1]$ which is of p -Hölder type with the constant $K = 1$ and $f : [0, 1] \rightarrow \mathbb{R}$ given by:

$$f(t) = \begin{cases} 0 & \text{if } t \in [0, 1), \\ 1 & \text{if } t = 1; \end{cases}$$

which is of bounded variation on $[0, 1]$.

Substituting in (3.3) we obtain

$$\left| \frac{1}{2} - \int_0^1 pt^{p-1} f(t) dt \right| \leq \frac{C}{2^p} (b-a) \bigvee_0^1(f).$$

However,

$$\int_0^1 t^{p-1} f(t) dt = 0 \text{ and } \bigvee_0^1(f) = 1,$$

and then $C \geq 2^{p-1}$ for all $p \in (0, 1]$. Choosing $p = 1$, we deduce $C \geq 1$ and the theorem is completely proved. \square

Remark 7. Let $f : [a, b] \rightarrow \mathbb{C}$ be as in Theorem 9 and u be a K -Lipschitzian mapping on $[a, b]$, where $K > 0$ is given. Then we have the inequality

$$(3.4) \quad \left| \frac{f(a) + f(b)}{2} [u(b) - u(a)] - \int_a^b f(t) du(t) \right| \leq \frac{1}{2} K (b-a) \bigvee_a^b(f).$$

The constant $\frac{1}{2}$ in (3.4) is best possible.

We now point out some results in estimating the integral of a product.

Corollary 1. Let $f : [a, b] \rightarrow \mathbb{C}$ be a mapping of bounded variation on $[a, b]$ and g be Lebesgue measurable on $[a, b]$. Put $\|g\|_\infty := \text{ess sup}_{t \in [a, b]} |f(t)|$, and if $\|g\|_\infty < \infty$, then we have the inequality:

$$(3.5) \quad \left| \frac{f(a) + f(b)}{2} \int_a^b g(s) ds - \int_a^b f(t) g(t) dt \right| \leq \frac{1}{2} \|g\|_\infty (b-a) \bigvee_a^b(f).$$

Proof. Define the mapping $u : [a, b] \rightarrow \mathbb{C}$, $u(t) := \int_a^t g(s) ds$. Then u is L -Lipschitzian with the constant $L = \|g\|_\infty$. Therefore, by the properties of Riemann-Stieltjes integrals, we have

$$\int_a^b f(t) du(t) = \int_a^b f(t) g(t) dt,$$

and then, by (3.4) we deduce the desired result (3.5). \square

The following corollary is also a natural consequence of Theorem 9.

Corollary 2. Let $f : [a, b] \rightarrow \mathbb{C}$ be a mapping of bounded variation on $[a, b]$ and g be Lebesgue measurable on $[a, b]$. Put

$$\|g\|_q := \left(\int_a^b |g(s)|^q ds \right)^{\frac{1}{q}}; q > 1.$$

If $\|g\|_q < \infty$, then we have the inequality

$$(3.6) \quad \left| \frac{f(a) + f(b)}{2} \int_a^b g(s) ds - \int_a^b f(t) g(t) dt \right| \leq \frac{1}{2^{\frac{q-1}{q}}} \|g\|_q (b-a)^{\frac{q-1}{q}} \bigvee_a^b(f).$$

Proof. Consider the mapping u as in the proof of Corollary 1. Then, by Hölder's integral inequality, we can state that

$$|u(t) - u(s)| = \left| \int_s^t g(z) dz \right| \leq |t-s|^{\frac{q-1}{q}} \left| \int_s^t |g(z)|^q dz \right|^{\frac{1}{q}} \leq |t-s|^{\frac{q-1}{q}} \|g\|_q,$$

for all $t, s \in [a, b]$, which shows that the mapping u is of $r - K$ -Hölder type with $r := \frac{q-1}{q} \in (0, 1)$ and $K = \|g\|_q < \infty$. Applying Theorem 9 we deduce the desired result (3.6). \square

Example 2. We give now some examples of weighted trapezoid inequalities for some of the most popular weights.

- a) (Legendre). If $g(t) = 1, t \in [a, b]$ then by (3.5) and (3.6) we get the trapezoid inequalities

$$\left| \frac{f(a) + f(b)}{2} (b-a) - \int_a^b f(t) dt \right| \leq \frac{1}{2} (b-a) \mathcal{V}_a^b(f)$$

and

$$\left| \frac{f(a) + f(b)}{2} (b-a) - \int_a^b f(t) dt \right| \leq \frac{1}{2^{1-1/q}} (b-a) \mathcal{V}_a^b(f), q > 1.$$

We remark that the first inequality is better than the second one.

- b) (Jacobi). If $g(t) = \frac{1}{\sqrt{t}}, t \in (0, 1]$, then obviously $\|g\|_\infty = +\infty$, so we cannot apply the inequality (3.5). If we assume that $q \in (1, 2)$, then we have

$$\|g\|_q = \left[\int_0^1 \left(\frac{1}{\sqrt{t}} \right)^q dt \right]^{1/q} = \left(\frac{2}{2-q} \right)^{1/q}$$

and applying the inequality (3.6) we deduce

$$(3.7) \quad \left| \frac{f(0) + f(1)}{2} - \frac{1}{2} \int_0^1 \frac{1}{\sqrt{t}} f(t) dt \right| \leq \frac{1}{4^{(q-1)/q}} \cdot \frac{1}{(2-q)^{1/q}} \mathcal{V}_0^1(f)$$

for all $q \in (1, 2)$.

- c) (Chebychev). If $g(t) = \frac{1}{\sqrt{1-t^2}}, t \in (-1, 1)$, then obviously $\|g\|_\infty = +\infty$, so we cannot apply the inequality (3.5). If we assume that $q \in (1, 2)$ then we have

$$\begin{aligned} \|g\|_q &= \left[\int_{-1}^1 \left(\frac{1}{\sqrt{1-t^2}} \right)^q dt \right]^{1/q} \\ &= \left[\int_{-1}^1 (t+1)^{\frac{2-q}{2}-1} (1-t)^{\frac{2-q}{2}-1} dt \right]^{1/q} \\ &= 2^{2-1/q} \left[B \left(\frac{2-q}{2}, \frac{2-q}{2} \right) \right]^{1/q}. \end{aligned}$$

Applying the inequality (3.6) we deduce

$$(3.8) \quad \left| \frac{f(-1) + f(1)}{2} - \frac{1}{\pi} \int_{-1}^1 \frac{f(t)}{\sqrt{1-t^2}} dt \right| \leq \frac{2}{\pi} \left[B \left(\frac{2-q}{2}, \frac{2-q}{2} \right) \right]^{1/q} \mathcal{V}_0^1(f)$$

for all $q \in (1, 2)$.

3.2. The Case of Lipschitzian Integrands.

Theorem 10. *Let $f : [a, b] \rightarrow \mathbb{C}$ be a Lipschitzian function with the constant $S > 0$ on $[a, b]$ and $u : [a, b] \rightarrow \mathbb{C}$ a $p - K$ -Hölder continuous function on that interval. Then we have the inequality:*

$$(3.9) \quad \left| \frac{f(a) + f(b)}{2} [u(b) - u(a)] - \int_a^b f(t) du(t) \right| \leq \frac{1}{p+1} KS (b-a)^{p+1}.$$

Proof. Is similar to the one from the proof of Theorem 7 and we omit the details. \square

Corollary 3. *Let $f : [a, b] \rightarrow \mathbb{C}$ be a Lipschitzian function with the constant $S > 0$ on $[a, b]$ and g be Lebesgue measurable on $[a, b]$. If $\|g\|_q < \infty$, where $q \geq 1$, then we have the inequality*

$$(3.10) \quad \left| \frac{f(a) + f(b)}{2} \int_a^b g(s) ds - \int_a^b f(t) g(t) dt \right| \leq \frac{q}{2q-1} \|g\|_q S (b-a)^{\frac{2q-1}{q}}.$$

Some applications for various classes of weights may be provided, however the details are left to the interested reader.

3.3. The Case of Monotonic Nondecreasing Integrands.

Theorem 11. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a monotonic nondecreasing function on $[a, b]$ and $u : [a, b] \rightarrow \mathbb{K}$ a $p - K$ -Hölder continuous function on that interval. Then we have the inequalities:*

$$(3.11) \quad \begin{aligned} & \left| \frac{f(a) + f(b)}{2} [u(b) - u(a)] - \int_a^b f(t) du(t) \right| \\ & \leq \frac{1}{2} K \left[(b-a)^p [f(b) - f(a)] - p \int_a^b \left[\frac{(b-t)^{1-p} - (t-a)^{1-p}}{(b-t)^{1-p} (t-a)^{1-p}} \right] f(t) dt \right] \\ & \leq \frac{1}{2^p} K (b-a)^p [f(b) - f(a)]. \end{aligned}$$

The inequalities in (3.11) are sharp.

Proof. Is similar to the one from the proof of Theorem 8 and we omit the details. \square

Remark 8. *We observe that if u is K -Lipschitzian, then (3.11) reduces to*

$$(3.12) \quad \left| \frac{f(a) + f(b)}{2} [u(b) - u(a)] - \int_a^b f(t) du(t) \right| \leq \frac{1}{2} K (b-a) [f(b) - f(a)]$$

and the constant $\frac{1}{2}$ is best possible in (3.12).

4. OTHER TRAPEZOIDAL INEQUALITIES

4.1. The Case of Bounded Variation Integrands and Integrators. The case when both the integrand and the integrator are of bounded variation is as follows:

Theorem 12. *Let $f, u : [a, b] \rightarrow \mathbb{C}$ be of bounded variation on $[a, b]$. If one of them is continuous on $[a, b]$, then the Riemann-Stieltjes integral $\int_a^b f(t) du(t)$ exists and we have the inequality*

$$(4.1) \quad \left| \frac{f(a) + f(b)}{2} [u(b) - u(a)] - \int_a^b f(t) du(t) \right| \leq \frac{1}{2} \bigvee_a^b(f) \bigvee_a^b(u).$$

The constant $\frac{1}{2}$ is best possible in (4.1).

Proof. Since f is of bounded variation, then we have

$$\begin{aligned} \left| \frac{f(a) + f(b)}{2} - f(t) \right| &\leq \frac{1}{2} [|f(b) - f(t)| + |f(t) - f(a)|] \\ &\leq \frac{1}{2} \bigvee_a^b(f) \end{aligned}$$

for all $t \in [a, b]$.

If we assume that f is continuous on $[a, b]$, then it follows that the Riemann-Stieltjes integral exists and

$$\begin{aligned} (4.2) \quad &\left| \frac{f(a) + f(b)}{2} \cdot [u(b) - u(a)] - \int_a^b f(t) du(t) \right| \\ &= \left| \int_a^b \left(\frac{f(a) + f(b)}{2} - f(t) \right) du(t) \right| \\ &\leq \max_{t \in [a, b]} \left| \frac{f(a) + f(b)}{2} - f(t) \right| \bigvee_a^b u(t) \leq \frac{1}{2} \bigvee_a^b(f) \bigvee_a^b u(t), \end{aligned}$$

which proves the desired result (4.1).

Now, if we choose in (4.1) $f(t) = t$, then we get the inequality

$$(4.3) \quad \left| \frac{a+b}{2} [u(b) - u(a)] - \int_a^b t du(t) \right| \leq \frac{1}{2} (b-a) \bigvee_a^b(u),$$

that is of interest in itself as well. We show that the constant $\frac{1}{2}$ is best possible in this inequality.

Assume that (4.3) with a constant $E > 0$, i.e.

$$(4.4) \quad \left| \frac{a+b}{2} [u(b) - u(a)] - \int_a^b t du(t) \right| \leq E (b-a) \bigvee_a^b(u),$$

for any function of bounded variation $u : [a, b] \rightarrow \mathbb{C}$.

If we choose the function $u : [a, b] \rightarrow \mathbb{R}$, with

$$u(t) = \begin{cases} 1, & t = a, \\ 0, & t \in (a, b), \\ 1, & t = b, \end{cases}$$

then we observe that u is of bounded variation and $\bigvee_a^b(u) = 2$. Also, integrating by parts in the Riemann-Stieltjes integral we have

$$\int_a^b t du(t) = tu(t)|_a^b - \int_a^b u(t) dt = b - a$$

and by (4.4) we deduce that $b - a \leq 2E(b - a)$, which implies that $E \geq \frac{1}{2}$, and the sharpness of the constant in (4.1) is proven.

If u is continuous, then on utilizing the integration by parts formula for the Riemann-Stieltjes integral, we have the following equality of interest

$$(4.5) \quad \begin{aligned} & \frac{f(a) + f(b)}{2} \cdot [u(b) - u(a)] - \int_a^b f(t) du(t) \\ &= \int_a^b \left(\frac{u(a) + u(b)}{2} - u(t) \right) df(t), \end{aligned}$$

which, as above, gives that

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} \cdot [u(b) - u(a)] - \int_a^b f(t) du(t) \right| \\ & \leq \max_{t \in [a, b]} \left| \frac{u(a) + u(b)}{2} - u(t) \right| \bigvee_a^b f(t). \end{aligned}$$

On utilizing the same argument as in the first part, we deduce the desired result (4.1).

If in (4.1) we take $u(t) = t$, then we get the inequality of interest

$$\left| \frac{f(a) + f(b)}{2} (b - a) - \int_a^b f(t) dt \right| \leq \frac{1}{2} (b - a) \bigvee_a^b (f),$$

that holds for any function of bounded variation $f : [a, b] \rightarrow \mathbb{C}$ for which $\frac{1}{2}$ is the best possible constant.

Note that this results was obtained for the first time in [10]. \square

Remark 9. *If we assume that $g : [a, b] \rightarrow \mathbb{C}$ is Lebesgue integrable on $[a, b]$, then $u(x) = \int_a^x g(t) dt$ is differentiable almost everywhere, $u(b) = \int_a^b g(t) dt$, $u(a) = 0$ and $\bigvee_a^b (u) = \int_a^b |g(t)| dt$. Consequently, by (4.1) we obtain*

$$(4.6) \quad \left| \frac{f(a) + f(b)}{2} \cdot \int_a^b g(t) dt - \int_a^b f(t) g(t) dt \right| \leq \frac{1}{2} \bigvee_a^b (f) \int_a^b |g(t)| dt.$$

From (4.6) we get a weighted version of the trapezoid inequality,

$$(4.7) \quad \left| \frac{f(a) + f(b)}{2} - \frac{1}{\int_a^b g(t) dt} \cdot \int_a^b f(t) g(t) dt \right| \leq \frac{1}{2} \bigvee_a^b (f),$$

provided that $g(t) \geq 0$, for almost every $t \in [a, b]$ and $\int_a^b g(t) dt \neq 0$.

4.2. The Case of End-point Lipschitzian Functions. In this subsection we consider the case when the function $f : [a, b] \rightarrow \mathbb{C}$ satisfies the end-point Lipschitzian conditions

$$(4.8) \quad |f(t) - f(a)| \leq L_a (t - a)^\alpha \quad \text{and} \quad |f(b) - f(t)| \leq L_b (b - t)^\beta$$

for any $t \in (a, b)$ where the constants $L_a, L_b > 0$ and $\alpha, \beta > -1$ are given.

Theorem 13. *Assume that the function f satisfies the condition (4.8).*

a) If $u : [a, b] \rightarrow \mathbb{C}$ is Lipschitzian with the constant $K > 0$, then we have the inequality

$$(4.9) \quad \left| \frac{f(a) + f(b)}{2} [u(b) - u(a)] - \int_a^b f(t) du(t) \right| \\ \leq \frac{1}{2} K \left[\frac{L_a}{\alpha + 1} (b - a)^{\alpha+1} + \frac{L_b}{\beta + 1} (b - a)^{\beta+1} \right].$$

b) If $\alpha, \beta > 0$ and $u : [a, b] \rightarrow \mathbb{R}$ is monotonic nondecreasing on $[a, b]$, then

$$(4.10) \quad \left| \frac{f(a) + f(b)}{2} [u(b) - u(a)] - \int_a^b f(t) du(t) \right| \\ \leq \frac{1}{2} L_a \left[(b - a)^\alpha u(b) - \alpha \int_a^b (t - a)^{\alpha-1} u(t) dt \right] \\ + \frac{1}{2} L_b \left[\beta \int_a^b (b - t)^{\beta-1} u(t) dt - (b - a)^\beta u(a) \right] \\ \leq \frac{1}{2} \left[L_a (b - a)^\alpha + L_b (b - a)^\beta \right] [u(b) - u(a)].$$

Proof. a). Since $u : [a, b] \rightarrow \mathbb{C}$ is Lipschitzian with the constant $K > 0$, then we have

$$\left| \frac{f(a) + f(b)}{2} [u(b) - u(a)] - \int_a^b f(t) du(t) \right| \\ \leq \frac{1}{2} K \int_a^b [|f(b) - f(t)| + |f(t) - f(a)|] dt \\ \leq \frac{1}{2} K \int_a^b [L_b (b - t)^\beta + L_a (t - a)^\alpha] dt \\ = \frac{1}{2} K \left[\frac{L_a}{\alpha + 1} (b - a)^{\alpha+1} + \frac{L_b}{\beta + 1} (b - a)^{\beta+1} \right]$$

and the inequality (4.9) is obtained.

b) Since $u : [a, b] \rightarrow \mathbb{R}$ is monotonic nondecreasing on $[a, b]$, then

$$(4.11) \quad \left| \frac{f(a) + f(b)}{2} [u(b) - u(a)] - \int_a^b f(t) du(t) \right| \\ \leq \frac{1}{2} \int_a^b [|f(b) - f(t)| + |f(t) - f(a)|] du(t) \\ \leq \frac{1}{2} \int_a^b [L_b (b - t)^\beta + L_a (t - a)^\alpha] du(t).$$

Utilising the integration by parts formula for the Riemann-Stieltjes integral and the fact that u is monotonic nondecreasing on $[a, b]$ we have that

$$(4.12) \quad \begin{aligned} \int_a^b (t-a)^\alpha du(t) &= (b-a)^\alpha u(b) - \alpha \int_a^b (t-a)^{\alpha-1} u(t) dt \\ &\leq (b-a)^\alpha u(b) - \alpha u(a) \int_a^b (t-a)^{\alpha-1} dt \\ &= (b-a)^\alpha [u(b) - u(a)] \end{aligned}$$

and

$$(4.13) \quad \begin{aligned} \int_a^b (b-t)^\beta du(t) &= -(b-a)^\beta u(a) + \beta \int_a^b (b-t)^{\beta-1} u(t) dt \\ &\leq \beta u(b) \int_a^b (b-t)^{\beta-1} dt - (b-a)^\beta u(a) \\ &= (b-a)^\beta [u(b) - u(a)]. \end{aligned}$$

On making use of (4.11)-(4.13) we deduce the desired inequality (4.10). \square

Remark 10. We notice that the dual case, i.e., when the integrator satisfies the condition (4.8) and the integrand f is either Lipschitzian or monotonic nondecreasing produces similar inequalities. However they will not be stated here.

5. A QUADRATURE RULE FOR THE RIEMANN-STIELTJES INTEGRAL

Consider the partition $I_n : a = t_0 < t_1 < \dots < t_{n-1} < t_n = b$ of the interval $[a, b]$, and define $h_i := t_{i+1} - t_i$ ($i = 0, \dots, n-1$), $\nu(h) := \max \{h_i | i = 0, \dots, n-1\}$ and the *generalized trapezoidal quadrature rule*

$$(5.1) \quad T_n(f, u, I_n) := \sum_{i=0}^{n-1} \frac{f(t_i) + f(t_{i+1})}{2} \times [u(t_{i+1}) - u(t_i)].$$

The following result for the numerical approximation of the Riemann-Stieltjes integral holds.

Theorem 14. Let $f : [a, b] \rightarrow \mathbb{C}$ be a p -Hölder continuous function on $[a, b]$ ($p \in (0, 1]$) and $u : [a, b] \rightarrow \mathbb{C}$ be a function of bounded variation on $[a, b]$. Then

$$(5.2) \quad \int_a^b f(t) du(t) = T_n(f, u, I_n) + R_n(f, u, I_n),$$

where $T_n(f, u, I_n)$ is the generalized trapezoidal formula given by (5.1), and the remainder $R(f, u, I_n)$ satisfies the estimate

$$(5.3) \quad |R_n(f, u, I_n)| \leq \frac{1}{2^p} H [\nu(h)]^p \bigvee_a^b(u).$$

If the integrator $u : [a, b] \rightarrow \mathbb{C}$ is a Lipschitzian function with the constant $L > 0$, then the remainder $R(f, u, I_n)$ satisfies the bound

$$(5.4) \quad |R_n(f, u, I_n)| \leq \frac{1}{p+1} HL(b-a) [\nu(h)]^p.$$

Proof. We apply Theorem 6 on every subinterval $[t_i, t_{i+1}]$ ($i = 0, \dots, n-1$) to obtain

$$(5.5) \quad \left| \frac{f(t_i) + f(t_{i+1})}{2} \times [u(t_{i+1}) - u(t_i)] - \int_{t_i}^{t_{i+1}} f(t) du(t) \right| \\ \leq \frac{1}{2^p} H h_i^p \bigvee_{t_i}^{t_{i+1}}(u).$$

Summing the inequalities (5.5) over i from 0 to $n-1$ and using the generalized triangle inequality, we obtain

$$|R(f, u, I_n)| \leq \sum_{i=0}^{n-1} \left| \frac{f(t_i) + f(t_{i+1})}{2} \times [u(t_{i+1}) - u(t_i)] - \int_{t_i}^{t_{i+1}} f(t) du(t) \right| \\ \leq \frac{1}{2^p} H \sum_{i=0}^{n-1} h_i^p \bigvee_{t_i}^{t_{i+1}}(u) \leq \frac{1}{2^p} H [\nu(h)]^p \sum_{i=0}^{n-1} \bigvee_{t_i}^{t_{i+1}}(u) \\ = \frac{1}{2^p} H [\nu(h)]^p \bigvee_a^b(u),$$

and the bound (5.3) is proved.

The case of Lipschitzian integrators follows from Theorem 7 and the details are omitted. \square

Let us assume that $g : [a, b] \rightarrow \mathbb{C}$ is Lebesgue integrable and $f : [a, b] \rightarrow \mathbb{C}$ is of $r-H$ -Hölder type on $[a, b]$. For a given partition I_n of the interval $[a, b]$, consider the quadrature

$$(5.6) \quad WT_n(f, g, I_n) := \sum_{i=0}^{n-1} \frac{f(t_i) + f(t_{i+1})}{2} \times \int_{t_i}^{t_{i+1}} g(s) ds.$$

We can state the following corollary:

Corollary 4. *Let $f : [a, b] \rightarrow \mathbb{C}$ be of $r-H$ -Hölder type and $g : [a, b] \rightarrow \mathbb{C}$ be Lebesgue integrable on $[a, b]$. Then we have the formula*

$$(5.7) \quad \int_a^b g(t) f(t) dt = WT_n(f, g, I_n) + WR_n(f, g, I_n),$$

where the remainder term $WR_n(f, g, I_n)$ satisfies the estimate in terms of the integral of $|g|$

$$(5.8) \quad |WR_n(f, g, I_n)| \leq \frac{1}{2^p} H [\nu(h)]^p \int_a^b |g(s)| ds$$

and the estimate in terms of the essential supremum of $|g|$

$$(5.9) \quad |WR_n(f, g, I_n)| \leq \frac{1}{p+1} H (b-a) [\nu(h)]^p \|g\|_\infty,$$

provided that, in this case, $\|g\|_\infty < \infty$.

The previous corollary allows us to obtain adaptive quadrature formulae for different weighted integrals.

Example 3. *We point out only a few examples for the most common weights.*

- a) (*Legendre*) If $g(t) = 1$, and $t \in [a, b]$, then we get for the mapping $f : [a, b] \rightarrow \mathbb{C}$ of $p - H$ -Hölder type:

$$\int_a^b f(t) dt = T(f, I_n) + R(f, I_n),$$

where $T(f, I_n)$ is the usual trapezoidal quadrature rule

$$T(f, I_n) = \sum_{i=0}^{n-1} \frac{f(t_i) + f(t_{i+1})}{2} \cdot h_i$$

and the remainder satisfies the estimate

$$|R(f, I_n)| \leq \frac{1}{2^p} H (b - a) [\nu(h)]^p.$$

- b) (*Logarithm*) If $g(t) = \ln\left(\frac{1}{t}\right)$, $t \in (a, b] \subseteq (0, 1]$, f is of $p - H$ -Hölder type and the integral $\int_a^b f(t) \ln\left(\frac{1}{t}\right) dt < \infty$, then we have the generalized trapezoid formula:

$$\int_a^b f(t) \ln\left(\frac{1}{t}\right) dt = T_L(f, I_n) + R_L(f, I_n),$$

where $T_L(f, I_n)$ is the “Logarithm-Trapezoid” quadrature rule

$$T_L(f, I_n) = \sum_{i=0}^{n-1} \frac{f(t_i) + f(t_{i+1})}{2} \times \left[t_i \ln\left(\frac{t_i}{e}\right) - t_{i+1} \ln\left(\frac{t_{i+1}}{e}\right) \right]$$

and the remainder term $R_L(f, I_n)$ satisfies the estimate

$$|R_L(f, I_n)| \leq \frac{1}{2^p} H [\nu(h)]^p \left[a \ln\left(\frac{a}{e}\right) - b \ln\left(\frac{b}{e}\right) \right].$$

- c) (*Jacobi*) If $g(t) = \frac{1}{\sqrt{t}}$, $t \in (a, b] \subset (0, \infty)$, f is of $p - H$ -Hölder type and $\int_a^b \frac{f(t)}{\sqrt{t}} dt < \infty$, then we have the generalized trapezoid formula

$$\int_a^b \frac{f(t)}{\sqrt{t}} dt = T_J(f, I_n) + R_J(f, I_n),$$

where $T_J(f, I_n)$ is the “Jacobi-Trapezoid” quadrature rule

$$T_J(f, I_n) = \frac{1}{2} \sum_{i=0}^{n-1} \frac{f(t_i) + f(t_{i+1})}{2} \times \left(\sqrt{t_{i+1}} - \sqrt{t_i} \right)$$

and the remainder term $R_J(f, I_n)$ satisfies the estimate

$$|R_J(f, I_n)| \leq \frac{1}{2^{p+1}} H [\nu(h)]^p \left(\sqrt{b} - \sqrt{a} \right).$$

- d) If $g(t) = \frac{1}{\sqrt{1-t^2}}$, $t \in (a, b] \subseteq (-1, 1)$, f is of $p - H$ -Hölder type and $\int_a^b \frac{f(t)}{\sqrt{1-t^2}} dt < \infty$, then we have the generalized trapezoid formula

$$\int_a^b \frac{f(t)}{\sqrt{1-t^2}} dt = T_C(f, I_n) + R_C(f, I_n)$$

where $T_C(f, I_n)$ is the ‘‘Chebychev-Trapezoid’’ quadrature rule

$$T_C(f, I_n) := \sum_{i=0}^{n-1} \frac{f(t_i) + f(t_{i+1})}{2} \times [\arcsin(t_{i+1}) - \arcsin(t_i)]$$

and the remainder term $R_C(f, I_n)$ satisfies the estimate

$$|R_C(f, I_n)| \leq \frac{1}{2^p} H [\nu(h)]^p [\arcsin(b) - \arcsin(a)].$$

5.1. More Error Bounds.

Theorem 15. Let $u : [a, b] \rightarrow \mathbb{C}$ be a p - K -Hölder continuous function on $[a, b]$ ($p \in (0, 1]$) and $f : [a, b] \rightarrow \mathbb{C}$ be a function of bounded variation on $[a, b]$. Then

$$(5.10) \quad \int_a^b f(t) du(t) = T_n(f, u, I_n) + R_n(f, u, I_n),$$

where $T_n(f, u, I_n)$ is the generalized trapezoidal formula given by (5.1), and the remainder $R(f, u, I_n)$ satisfies the estimate

$$(5.11) \quad |R_n(f, u, I_n)| \leq \frac{1}{2^p} K [\nu(h)]^p \bigvee_a^b(f).$$

If the integrand $f : [a, b] \rightarrow \mathbb{C}$ is a Lipschitzian function with the constant $S > 0$, then the reminder $R(f, u, I_n)$ satisfies the bound

$$(5.12) \quad |R_n(f, u, I_n)| \leq \frac{1}{p+1} KS(b-a) [\nu(h)]^p.$$

Proof. Follows from Theorems 9 and 10 and the details are omitted. \square

The case of weighted integrals is as follows:

Corollary 5. Let $f : [a, b] \rightarrow \mathbb{C}$ be a mapping of bounded variation on $[a, b]$ and g be Lebesgue measurable on $[a, b]$. Then we have the formula

$$(5.13) \quad \int_a^b g(t) f(t) dt = WT_n(f, g, I_n) + WR_n(f, g, I_n),$$

where the remainder term $WR_n(f, g, I_n)$ satisfies the estimate

$$(5.14) \quad |WR_n(f, g, I_n)| \leq \frac{1}{2^{\frac{q-1}{q}}} \|g\|_q [\nu(h)]^{\frac{q-1}{q}} \bigvee_a^b(f),$$

provided that $\|g\|_q < \infty$, $q \geq 1$.

Let $f : [a, b] \rightarrow \mathbb{C}$ be a Lipschitzian function with the constant $S > 0$ on $[a, b]$ and g be Lebesgue measurable on $[a, b]$. If $\|g\|_q < \infty$, where $q \geq 1$, then we have the inequality

$$(5.15) \quad |WR_n(f, g, I_n)| \leq \frac{q}{2q-1} \|g\|_q S [\nu(h)]^{\frac{2q-1}{q}}.$$

Remark 11. Various particular quadratures and their errors bounds for the usual weights can be provided, however the details are left to the interested reader.

5.2. Other Error Bounds.

Theorem 16. *Let $f : [a, b] \rightarrow \mathbb{C}$ be continuous and of bounded variation on $[a, b]$ and $u : [a, b] \rightarrow \mathbb{C}$ be a function of bounded variation on $[a, b]$. Then*

$$(5.16) \quad \int_a^b f(t) du(t) = T_n(f, u, I_n) + R_n(f, u, I_n),$$

where $T_n(f, u, I_n)$ is the generalized trapezoidal formula and the remainder $R(f, u, I_n)$ satisfies the estimate

$$(5.17) \quad |R_n(f, u, I_n)| \leq \frac{1}{2} \max_{i \in \{0, \dots, n-1\}} \left\{ \bigvee_{t_i}^{t_{i+1}}(f) \right\} \bigvee_a^b(u).$$

In particular, if f is Lipschitzian with the constant $L > 0$, then

$$(5.18) \quad |R_n(f, u, I_n)| \leq \frac{1}{2} L \nu(h) \bigvee_a^b(u).$$

Proof. We apply Theorem 12 on every subinterval $[t_i, t_{i+1}]$ ($i = 0, \dots, n-1$) to obtain

$$(5.19) \quad \left| \frac{f(t_i) + f(t_{i+1})}{2} \times [u(t_{i+1}) - u(t_i)] - \int_{t_i}^{t_{i+1}} f(t) du(t) \right| \\ \leq \frac{1}{2} \bigvee_{t_i}^{t_{i+1}}(f) \bigvee_{t_i}^{t_{i+1}}(u).$$

Summing the inequalities (5.19) over i from 0 to $n-1$ and using the generalized triangle inequality, we obtain

$$|R(f, u, I_n)| \leq \sum_{i=0}^{n-1} \left| \frac{f(t_i) + f(t_{i+1})}{2} \times [u(t_{i+1}) - u(t_i)] - \int_{t_i}^{t_{i+1}} f(t) du(t) \right| \\ \leq \frac{1}{2} \sum_{i=0}^{n-1} \bigvee_{t_i}^{t_{i+1}}(f) \bigvee_{t_i}^{t_{i+1}}(u) \leq \frac{1}{2} \max_{i \in \{0, \dots, n-1\}} \left\{ \bigvee_{t_i}^{t_{i+1}}(f) \right\} \sum_{i=0}^{n-1} \bigvee_{t_i}^{t_{i+1}}(u) \\ = \frac{1}{2} \max_{i \in \{0, \dots, n-1\}} \left\{ \bigvee_{t_i}^{t_{i+1}}(f) \right\} \bigvee_a^b(u),$$

and the bound (5.17) is proved.

The second part follows from the fact that if f is L -Lipschitzian, then $\bigvee_{t_i}^{t_{i+1}}(f) \leq Lh_i$ for all $i \in \{0, \dots, n-1\}$. \square

In particular we have:

Corollary 6. *Let $f : [a, b] \rightarrow \mathbb{C}$ be continuous and of bounded variation on $[a, b]$ and $g : [a, b] \rightarrow \mathbb{C}$ be Lebesgue integrable on $[a, b]$. Then we have the formula*

$$(5.20) \quad \int_a^b g(t) f(t) dt = WT_n(f, g, I_n) + WR_n(f, g, I_n),$$

where the remainder term $WR_n(f, g, I_n)$ satisfies the estimate

$$(5.21) \quad |WR_n(f, g, I_n)| \leq \frac{1}{2} \max_{i \in \{0, \dots, n-1\}} \left\{ \bigvee_{t_i}^{t_{i+1}}(f) \right\} \int_a^b |g(s)| ds.$$

In particular, if f is Lipschitzian with the constant $L > 0$, then

$$(5.22) \quad |R_n(f, u, I_n)| \leq \frac{1}{2} L \nu(h) \bigvee_a^b(u).$$

6. APPLICATIONS FOR FUNCTIONS OF SELFADJOINT OPERATORS

Let A be a selfadjoint linear operator on a complex Hilbert space $(H; \langle \cdot, \cdot \rangle)$. The *Gelfand map* establishes a $*$ -isometrically isomorphism Φ between the set $C(Sp(A))$ of all *continuous functions* defined on the *spectrum* of A , denoted $Sp(A)$, and the C^* -algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows (see for instance [19, p. 3]):

For any $f, g \in C(Sp(A))$ and any $\alpha, \beta \in \mathbb{C}$ we have

- (i) $\Phi(\alpha f + \beta g) = \alpha \Phi(f) + \beta \Phi(g)$;
- (ii) $\Phi(fg) = \Phi(f)\Phi(g)$ and $\Phi(\bar{f}) = \Phi(f)^*$;
- (iii) $\|\Phi(f)\| = \|f\| := \sup_{t \in Sp(A)} |f(t)|$;
- (iv) $\Phi(f_0) = 1_H$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$, for $t \in Sp(A)$.

With this notation we define

$$f(A) := \Phi(f) \text{ for all } f \in C(Sp(A))$$

and we call it the *continuous functional calculus* for a selfadjoint operator A .

If A is a selfadjoint operator and f is a real valued continuous function on $Sp(A)$, then $f(t) \geq 0$ for any $t \in Sp(A)$ implies that $f(A) \geq 0$, *i.e.* $f(A)$ is a *positive operator* on H . Moreover, if both f and g are real valued functions on $Sp(A)$ then the following important property holds:

$$(P) \quad f(t) \geq g(t) \text{ for any } t \in Sp(A) \text{ implies that } f(A) \geq g(A)$$

in the operator order of $B(H)$.

For a recent monograph devoted to various inequalities for continuous functions of selfadjoint operators, see [19] and the references therein.

For other recent results see [15], [16], [17], [20], [22], [23] and [25].

Let U be a selfadjoint operator on the complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$ with the spectrum $Sp(U)$ included in the interval $[m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its *spectral family*. Then for any continuous function $f : [m, M] \rightarrow \mathbb{R}$, it is well known that we have the following *spectral representation in terms of the Riemann-Stieltjes integral*:

$$(6.1) \quad \langle f(U)x, y \rangle = \int_{m-0}^M f(\lambda) d(\langle E_\lambda x, y \rangle),$$

for any $x, y \in H$. The function $g_{x,y}(\lambda) := \langle E_\lambda x, y \rangle$ is of *bounded variation* on the interval $[m, M]$ and

$$g_{x,y}(m-0) = 0 \text{ and } g_{x,y}(M) = \langle x, y \rangle$$

for any $x, y \in H$. It is also well known that $g_x(\lambda) := \langle E_\lambda x, x \rangle$ is *monotonic nondecreasing* and *right continuous* on $[m, M]$.

Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its *spectral family*.

Consider the partition $I_n : m = t_0 < t_1 < \dots < t_{n-1} < t_n = M$ of the interval $[m, M]$, and define $h_i := t_{i+1} - t_i$ ($i = 0, \dots, n-1$), $\nu(h) := \max\{h_i | i = 0, \dots, n-1\}$

and the *generalized trapezoidal quadrature rule* associated to the continuous function $f : [m, M] \rightarrow \mathbb{C}$, selfadjoint operator A and the vectors $x, y \in H$

$$(6.2) \quad T_n(f, A, I_n; x, y) := \sum_{i=0}^{n-1} \frac{f(t_i) + f(t_{i+1})}{2} \langle (E_{t_{i+1}} - E_{t_i}) x, y \rangle.$$

Theorem 17. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family.*

a) *If $f : [m, M] \rightarrow \mathbb{C}$ is continuous and with bounded variation on $[m, M]$, then for any $x, y \in H$*

$$(6.3) \quad \langle f(A) x, y \rangle = T_n(f, A, I_n; x, y) + R_n(f, A, I_n; x, y)$$

and the remainder $R_n(f, A, I_n; x, y)$ satisfies the error bounds

$$(6.4) \quad |R_n(f, A, I_n; x, y)| \leq \frac{1}{2} \max_{i \in \{0, \dots, n-1\}} \left\{ \bigvee_{t_i}^{t_{i+1}}(f) \right\} \bigvee_m^M (\langle E_{(\cdot)} x, y \rangle) \\ \leq \frac{1}{2} \max_{i \in \{0, \dots, n-1\}} \left\{ \bigvee_{t_i}^{t_{i+1}}(f) \right\} \|x\| \|y\|.$$

b) *Let $f : [m, M] \rightarrow \mathbb{C}$ be a p - H -Hölder continuous function on $[m, M]$, then for any $x, y \in H$ we have the equality (6.3) and the remainder $R_n(f, A, I_n; x, y)$ satisfies the error bounds*

$$(6.5) \quad |R_n(f, A, I_n; x, y)| \leq \frac{1}{2^p} H[\nu(h)]^p \bigvee_m^M (\langle E_{(\cdot)} x, y \rangle) \\ \leq \frac{1}{2^p} H[\nu(h)]^p \|x\| \|y\|.$$

Proof. The first inequalities in (6.4) and (6.5) follow from Theorem 14 and Theorem 16 applied for the integrator of bounded variation $u(t) = \langle E_t x, y \rangle$ with $t \in [m, M]$.

If P is a nonnegative operator on H , i.e., $\langle Px, x \rangle \geq 0$ for any $x \in H$, then the following inequality is a generalization of the Schwarz inequality in H

$$(6.6) \quad |\langle Px, y \rangle|^2 \leq \langle Px, x \rangle \langle Py, y \rangle$$

for any $x, y \in H$.

Further, if $d : m = t_0 < t_1 < \dots < t_{n-1} < t_n = M$ is an arbitrary partition of the interval $[m, M]$, then we have by Schwarz's inequality for nonnegative operators that

$$\bigvee_m^M (\langle E_{(\cdot)} x, y \rangle) \\ = \sup_d \left\{ \sum_{i=0}^{n-1} |\langle (E_{t_{i+1}} - E_{t_i}) x, y \rangle| \right\} \\ \leq \sup_d \left\{ \sum_{i=0}^{n-1} \left[\langle (E_{t_{i+1}} - E_{t_i}) x, x \rangle^{1/2} \langle (E_{t_{i+1}} - E_{t_i}) y, y \rangle^{1/2} \right] \right\} := I.$$

By the Cauchy-Buniakovski-Schwarz inequality for sequences of real numbers we also have that

$$\begin{aligned}
I &\leq \sup_d \left\{ \left[\sum_{i=0}^{n-1} \langle (E_{t_{i+1}} - E_{t_i}) x, x \rangle \right]^{1/2} \left[\sum_{i=0}^{n-1} \langle (E_{t_{i+1}} - E_{t_i}) y, y \rangle \right]^{1/2} \right\} \\
&\leq \sup_d \left\{ \left[\sum_{i=0}^{n-1} \langle (E_{t_{i+1}} - E_{t_i}) x, x \rangle \right]^{1/2} \sup_d \left[\sum_{i=0}^{n-1} \langle (E_{t_{i+1}} - E_{t_i}) y, y \rangle \right]^{1/2} \right\} \\
&= \left[\bigvee_m^M (\langle E_{(\cdot)} x, x \rangle) \right]^{1/2} \left[\bigvee_m^M (\langle E_{(\cdot)} y, y \rangle) \right]^{1/2} = \|x\| \|y\|
\end{aligned}$$

for any $x, y \in H$. These prove the last parts of (6.4) and (6.5). \square

Remark 12. *In the particular case when the partition reduces to the whole interval $[m, M]$ we have the following trapezoidal inequalities for any $x, y \in H$*

$$\begin{aligned}
(6.7) \quad & \left| \frac{f(M) + f(m)}{2} \cdot \langle x, y \rangle - \langle f(A) x, y \rangle \right| \\
& \leq \frac{1}{2} \bigvee_m^M (f) \bigvee_m^M (\langle E_{(\cdot)} x, y \rangle) \leq \frac{1}{2} \bigvee_m^M (f) \|x\| \|y\|,
\end{aligned}$$

when $f : [m, M] \rightarrow \mathbb{C}$ is continuous and with bounded variation on $[m, M]$, and

$$\begin{aligned}
(6.8) \quad & \left| \frac{f(M) + f(m)}{2} \cdot \langle x, y \rangle - \langle f(A) x, y \rangle \right| \\
& \leq \frac{1}{2^p} H (M - m)^p \bigvee_m^M (\langle E_{(\cdot)} x, y \rangle) \leq \frac{1}{2^p} H (M - m)^p \|x\| \|y\|,
\end{aligned}$$

when $f : [m, M] \rightarrow \mathbb{C}$ is a $p - H$ -Hölder continuous function on $[m, M]$.

Moreover, if we use the inequality (2.4) for the monotonic nondecreasing function $u(t) = \langle E_t x, x \rangle$ with $x \in H$, then we get

$$\begin{aligned}
(6.9) \quad & \left| \frac{f(M) + f(m)}{2} \cdot \|x\|^2 - \langle f(A) x, x \rangle \right| \\
& \leq \frac{1}{2} H \left\{ (M - m)^p \|x\|^2 - p \int_{m-0}^M \left[\frac{(M - t)^{1-p} - (t - m)^{1-p}}{(M - t)^{1-p} (t - m)^{1-p}} \right] \langle E_t x, x \rangle dt \right\} \\
& \leq \frac{1}{2^p} H (M - m)^p \|x\|^2,
\end{aligned}$$

for any $x \in H$ where $f : [m, M] \rightarrow \mathbb{C}$ is a $p - H$ -Hölder continuous function on $[m, M]$.

Finally, if the function $f : [m, M] \rightarrow \mathbb{C}$ satisfies the end-point Lipschitzian conditions

$$|f(t) - f(m)| \leq L_m (t - m)^\alpha \quad \text{and} \quad |f(M) - f(t)| \leq L_M (M - t)^\beta$$

for any $t \in (m, M)$ where the constants $L_m, L_M > 0$ and $\alpha, \beta > 0$ are given, then on applying the inequality (4.10) we have

$$\begin{aligned}
 (6.10) \quad & \left| \frac{f(M) + f(m)}{2} \cdot \|x\|^2 - \langle f(A)x, x \rangle \right| \\
 & \leq \frac{1}{2} L_m \left[(M - m)^\alpha \|x\|^2 - \alpha \int_{m-0}^M (t - m)^{\alpha-1} \langle E_t x, x \rangle dt \right] \\
 & + \frac{1}{2} L_M \beta \int_{m-0}^M (M - t)^{\beta-1} \langle E_t x, x \rangle dt \\
 & \leq \frac{1}{2} \left[L_m (M - m)^\alpha + L_M (M - m)^\beta \right] \|x\|^2.
 \end{aligned}$$

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