NORM AND NUMERICAL RADIUS INEQUALITIES FOR TWO LINEAR OPERATORS IN HILBERT SPACES, A SURVEY OF RECENT RESULTS

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Dedicated to the Memory of the 100th Anniversary of S.M.Ulam

ABSTRACT. The main aim of this paper is to survey some recent norm and numerical radius inequalities obtained by the author for composite operators generated by a pair of operators (A, B) in complex Hilbert spaces under various assumptions. Applications in connection with classical results are also provided.

1. INTRODUCTION

Let $(H; \langle \cdot, \cdot \rangle)$ be a complex Hilbert space. The *numerical range* of an operator T is the subset of the complex numbers $\mathbb{C}$ given by $W(T) = \{\langle Tx, x \rangle, \ x \in H, \ \|x\| = 1\}$, see for instance [26, p. 1]. It is well known that (see [26]):

(i) The numerical range of an operator is convex;
(ii) The spectrum of an operator is contained in the closure of its numerical range;
(iii) T is self-adjoint if and only if $W(T)$ is real.

The *numerical radius* $w(T)$ of an operator T on H is defined by $w(T) := \sup\{|\lambda|, \lambda \in W(T)\} = \sup\{|\langle Tx, x \rangle|, \|x\| = 1\}$, [26, p. 8]. It is well known that $w(\cdot)$ is a norm on the Banach algebra $B(H)$ of all bounded linear operators acting on H and the following inequality holds true:

$$w(T) \leq \|T\| \leq 2w(T). \tag{1.1}$$

We recall some classical results involving the numerical radius of two linear operators A, B.

The following general result for the product of two operators holds [26, p. 37]:

Theorem 1. *If A, B are two bounded linear operators on the Hilbert space $(H, \langle \cdot, \cdot \rangle)$, then $w(AB) \leq 4w(A)w(B)$. In the case that $AB = BA$, then $w(AB) \leq 2w(A)w(B)$.*

The following results are also well known [26, p. 38].

Theorem 2. *If A is a unitary operator that commutes with another operator B, then*

$$w(AB) \leq w(B). \tag{1.2}$$

If A is an isometry and $AB = BA$, then (1.2) also holds true.

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We say that A and B *double commute* if $AB = BA$ and $AB^* = B^*A$. The following result holds [26, p. 38].

Theorem 3 (Double commute)**.** *If the operators A and B double commute, then* $w(AB) \le w(B) \|A\|$.

As a consequence of the above, we have [26, p. 39]:

Corollary 1. *Let A be a normal operator commuting with B. Then* $w(AB) \le w(A) w(B)$.

For other results and historical comments on the above see [26, p. 39–41]. For more results on the numerical radius, see [27].

The main aim of this paper is to survey some recent inequalities obtained by the author in [19], [21], [22], [20], [18] and [9] for composite operators generated by a pair of operators (A, B) under various assumptions. For related results on operator and vector inequalities in Hilbert spaces see, [1]-[23].

1.1. General Inequalities for the Norm and Numerical Radius.

1.2. Some Preliminary Results.

The following result may be stated:

Theorem 4 (Dragomir, 2008 [19])**.** *Let $A, B : H \to H$ be two bounded linear operators on the Hilbert space $(H, \langle \cdot, \cdot \rangle)$. If $r > 0$ and*

$$\|A - B\| \le r, \tag{1.3}$$

then

$$\left\| \frac{A^*A + B^*B}{2} \right\| \le w(B^*A) + \frac{1}{2} r^2. \tag{1.4}$$

Proof. For any $x \in H$, $\|x\| = 1$, we have from (1.3) that

$$\|Ax\|^2 + \|Bx\|^2 \le 2 \operatorname{Re} \langle Ax, Bx \rangle + r^2. \tag{1.5}$$

However

$$\|Ax\|^2 + \|Bx\|^2 = \langle (A^*A + B^*B) x, x \rangle$$

and by (1.5) we obtain

$$\langle (A^*A + B^*B) x, x \rangle \le 2 |\langle (B^*A) x, x \rangle| + r^2 \tag{1.6}$$

for any $x \in H$, $\|x\| = 1$.

Taking the supremum over $x \in H$, $\|x\| = 1$ in (1.6) we get

$$w(A^*A + B^*B) \le 2w(B^*A) + r^2 \tag{1.7}$$

and since the operator $A^*A + B^*B$ is self-adjoint, hence $w(A^*A + B^*B) = \|A^*A + B^*B\|$ and by (1.7) we deduce the desired inequality (1.4). □

Remark 1. *We observe that, from the proof of the above theorem, we have the inequalities*

$$0 \le \left\| \frac{A^*A + B^*B}{2} \right\| - w(B^*A) \le \frac{1}{2} \|A - B\|^2, \tag{1.8}$$

provided that A, B are bounded linear operators in H.

The second inequality in (1.8) is obvious while the first inequality follows by the fact that

$$\langle (A^*A + B^*B) x, x \rangle = \|Ax\|^2 + \|Bx\|^2 \ge 2 |\langle (B^*A) x, x \rangle|$$

for any $x \in H$.

The inequality (1.4) is obviously a rich source of particular inequalities of interest.

Indeed, if we assume, for $\lambda \in \mathbb{C}$ and a bounded linear operator T, that we have $\|T - \lambda T^*\| \le r$, for a given positive number r, then by (1.8) we deduce the inequality

$$0 \le \left\| \frac{T^*T + |\lambda|^2 TT^*}{2} \right\| - |\lambda| \, w\left(T^2\right) \le \frac{1}{2} r^2. \tag{1.9}$$

Now, if we assume that for $\lambda \in \mathbb{C}$ and a bounded linear operator V we have that $\|V - \lambda I\| \le r$, where I is the identity operator on H, then by (1.4) we deduce the inequality

$$0 \le \left\| \frac{V^*V + |\lambda|^2 I}{2} \right\| - |\lambda| \, w(V) \le \frac{1}{2} r^2.$$

As a dual approach, the following result may be noted as well:

Theorem 5 (Dragomir, 2008 [19])**.** *Let* $A, B : H \to H$ *be two bounded linear operators on the Hilbert space* H*. Then*

$$\left\| \frac{A+B}{2} \right\|^2 \le \frac{1}{2} \left[\left\| \frac{A^*A + B^*B}{2} \right\| + w(B^*A) \right]. \tag{1.10}$$

Proof. We obviously have

$$\begin{aligned} \|Ax + Bx\|^2 &= \|Ax\|^2 + 2 \operatorname{Re} \langle Ax, Bx \rangle + \|Bx\|^2 \\ &\le \langle (A^*A + B^*B) x, x \rangle + 2 \left| \langle (B^*A) x, x \rangle \right| \end{aligned}$$

for any $x \in H$.

Taking the supremum over $x \in H$, $\|x\| = 1$, we get $\|A + B\|^2 \le \|A^*A + B^*B\| + 2w(B^*A)$, from where we get the desired inequality (1.10). □

Remark 2. *The inequality (1.10) can generate some interesting particular results such as the following inequality*

$$\left\| \frac{T+T^*}{2} \right\|^2 \le \frac{1}{2} \left[\left\| \frac{T^*T + TT^*}{2} \right\| + w\left(T^2\right) \right], \tag{1.11}$$

holding for each bounded linear operator $T : H \to H$.

The following result concerning powers may be stated as well.

Theorem 6 (Dragomir, 2008 [19])**.** *Let* $A, B : H \to H$ *be two bounded linear operators on the Hilbert space* H *and* $p \ge 2$*. Then*

$$\left\| \frac{A^*A + B^*B}{2} \right\|^{\frac{p}{2}} \le \frac{1}{4} \left[\|A - B\|^p + \|A + B\|^p \right]. \tag{1.12}$$

Proof. We use the following inequality for vectors in inner product spaces obtained by Dragomir and Sándor in [23]:

$$2\left(\|a\|^p + \|b\|^p\right) \le \|a + b\|^p + \|a - b\|^p \tag{1.13}$$

for any $a, b \in H$ and $p \ge 2$.

Utilising (1.13) we may write

$$2\left(\|Ax\|^p + \|Bx\|^p\right) \le \|Ax + Bx\|^p + \|Ax - Bx\|^p \tag{1.14}$$

for any $x \in H$.

Now, observe that $\|Ax\|^p + \|Bx\|^p = \left(\|Ax\|^2\right)^{\frac{p}{2}} + \left(\|Bx\|^2\right)^{\frac{p}{2}}$ and by the elementary inequality $\frac{\alpha^q+\beta^q}{2} \geq \left(\frac{\alpha+\beta}{2}\right)^q, \quad \alpha, \beta \geq 0$ and $q \geq 1$ we have

$$\left(\|Ax\|^2\right)^{\frac{p}{2}} + \left(\|Bx\|^2\right)^{\frac{p}{2}} \geq 2^{1-\frac{p}{2}} \left[\langle (A^*A + B^*B)\, x, x\rangle\right]^{\frac{p}{2}}. \tag{1.15}$$

Combining (1.14) with (1.15) we get

$$\frac{1}{4}\left[\|Ax - Bx\|^p + \|Ax + Bx\|^p\right] \geq \left|\left\langle \left(\frac{A^*A + B^*B}{2}\right) x, x\right\rangle\right|^{\frac{p}{2}} \tag{1.16}$$

for any $x \in H$, $\|x\| = 1$. Taking the supremum over $x \in H$, $\|x\| = 1$, and taking into account that $w\left(\frac{A^*A+B^*B}{2}\right) = \left\|\frac{A^*A+B^*B}{2}\right\|$, we deduce the desired result (1.12). □

Remark 3. *If $p = 2$, then we have the inequality: $\left\|\frac{A^*A+B^*B}{2}\right\| \leq \left\|\frac{A-B}{2}\right\|^2 + \left\|\frac{A+B}{2}\right\|^2$, for any A, B bounded linear operators. This result can be obtained directly on utilising the parallelogram identity as well. We also should observe that for $A = T$ and $B = T^*$, T a normal operator, the inequality (1.12) becomes $\|T\|^p \leq \frac{1}{4}\left[\|T - T^*\|^p + \|T + T^*\|^p\right]$, where $p \geq 2$.*

The following result may be stated as well.

Theorem 7 (Dragomir, 2008 [19]). *Let $A, B : H \to H$ be two bounded linear operators on the Hilbert space H and $r \geq 1$. If $A^*A \geq B^*B$ in the operator order or, equivalently, $\|Ax\| \geq \|Bx\|$ for any $x \in H$, then:*

$$\left\|\frac{A^*A + B^*B}{2}\right\|^r \leq \|A\|^{r-1}\|B\|^{r-1} w\left(B^*A\right) + \frac{1}{2}r^2 \|A\|^{2r-2}\|A - B\|^2. \tag{1.17}$$

Proof. We use the following inequality for vectors in inner product spaces due to Goldstein, Ryff and Clarke [24]:

$$\|a\|^{2r} + \|b\|^{2r} \leq 2\|a\|^{r-1}\|b\|^{r-1} \operatorname{Re}\langle a, b\rangle + r^2\|a\|^{2r-2}\|a - b\|^2, \tag{1.18}$$

where $r \geq 1$, $a, b \in H$ and $\|a\| \geq \|b\|$.

Utilising (1.18) we can state that:

$$\|Ax\|^{2r} + \|Bx\|^{2r} \leq 2\|Ax\|^{r-1}\|Bx\|^{r-1}|\langle Ax, Bx\rangle| + r^2\|Ax\|^{2r-2}\|Ax - Bx\|^2, \tag{1.19}$$

for any $x \in H$. As in the proof of Theorem 6, we also have

$$2^{1-r}\left[\langle (A^*A + B^*B)\, x, x\rangle\right]^r \leq \|Ax\|^{2r} + \|Bx\|^{2r}, \tag{1.20}$$

for any $x \in H$. Therefore, by (1.19) and (1.20) we deduce

$$\left[\left\langle \left(\frac{A^*A + B^*B}{2}\right) x, x\right\rangle\right]^r \leq \|Ax\|^{r-1}\|Bx\|^{r-1}|\langle Ax, Bx\rangle| + \frac{1}{2}r^2\|A\|^{2r-2}\|Ax - Bx\|^2 \tag{1.21}$$

for any $x \in H$.

Taking the supremum in (1.21) we obtain the desired result (1.17). □

Remark 4. *Following* [26, p. 156], *we recall that the bounded linear operator V is hyponormal, if $\|V^*x\| \leq \|Vx\|$ for all $x \in H$. Now, if we choose in (1.17) $A = V$ and $B = V^*$, then, on taking into account that for hyponormal operators $w(V^2) = \|V\|^2$, we get the inequality*

$$\left\|\frac{V^*V + VV^*}{2}\right\|^r \leq \|V\|^{2r-2}\left[\|V\|^2 + \frac{1}{2}r^2\|V - V^*\|^2\right], \tag{1.22}$$

holding for any hyponormal operator V and any $r \geq 1$.

1.3. **Further Inequalities for an Invertible Operator.** In this section we assume that $B : H \to H$ is an invertible bounded linear operator and let $B^{-1} : H \to H$ be its inverse. Then, obviously,

$$\|Bx\| \geq \frac{1}{\|B^{-1}\|}\|x\| \quad \text{for any} \quad x \in H, \tag{1.23}$$

where $\|B^{-1}\|$ denotes the norm of the inverse B^{-1}.

Theorem 8 (Dragomir, 2008 [19])**.** *Let $A, B : H \to H$ be two bounded linear operators on H and B is invertible such that, for a given $r > 0$,*

$$\|A - B\| \leq r. \tag{1.24}$$

Then:

$$\|A\| \leq \|B^{-1}\|\left[w(B^*A) + \frac{1}{2}r^2\right]. \tag{1.25}$$

Proof. The condition (1.24) is obviously equivalent to:

$$\|Ax\|^2 + \|Bx\|^2 \leq 2\,\mathrm{Re}\,\langle(B^*A)\,x, x\rangle + r^2 \tag{1.26}$$

for any $x \in H$, $\|x\| = 1$. Since, by (1.23), $\|Bx\|^2 \geq \frac{1}{\|B^{-1}\|^2}\|x\|^2, \quad x \in H$ and

$$\mathrm{Re}\,\langle(B^*A)\,x, x\rangle \leq |\langle(B^*A)\,x, x\rangle|,$$

hence by (1.26) we get

$$\|Ax\|^2 + \frac{\|x\|^2}{\|B^{-1}\|^2} \leq 2\,|\langle(B^*A)\,x, x\rangle| + r^2 \tag{1.27}$$

for any $x \in H$, $\|x\| = 1$. Taking the supremum over $x \in H$, $\|x\| = 1$ in (1.27), we have

$$\|A\|^2 + \frac{1}{\|B^{-1}\|^2} \leq 2w(B^*A) + r^2. \tag{1.28}$$

By the elementary inequality

$$\frac{2\|A\|}{\|B^{-1}\|} \leq \|A\|^2 + \frac{1}{\|B^{-1}\|^2} \tag{1.29}$$

and by (1.28) we then deduce the desired result (1.25). □

Remark 5. *If we choose above $B = \lambda I$, $\lambda \neq 0$, then we get the inequality*

$$(0 \leq)\,\|A\| - w(A) \leq \frac{1}{2|\lambda|}r^2, \tag{1.30}$$

provided $\|A - \lambda I\| \leq r$. This result has been obtained in the earlier paper [8].

Also, if we assume that $B = \lambda A^*$, A is invertible, then we obtain

$$\|A\| \leq \left\|A^{-1}\right\| \left[w\left(A^2\right) + \frac{1}{2|\lambda|}r^2\right], \tag{1.31}$$

provided $\|A - \lambda A^*\| \leq r$, $\lambda \neq 0$.

The following result may be stated as well:

Theorem 9 (Dragomir, 2008 [19])**.** *Let $A, B : H \to H$ be two bounded linear operators on H. If B is invertible and for $r > 0$,*

$$\|A - B\| \leq r, \tag{1.32}$$

then

$$(0 \leq)\ \|A\|\,\|B\| - w\left(B^*A\right) \leq \frac{1}{2}r^2 + \frac{\|B\|^2 \left\|B^{-1}\right\|^2 - 1}{2\left\|B^{-1}\right\|^2}. \tag{1.33}$$

Proof. The condition (1.32) is obviously equivalent to

$$\|Ax\|^2 + \|Bx\|^2 \leq 2\operatorname{Re}\langle Ax, Bx\rangle + r^2$$

for any $x \in H$, which is clearly equivalent to

$$\|Ax\|^2 + \|B\|^2 \leq 2\operatorname{Re}\langle B^*Ax, x\rangle + r^2 + \|B\|^2 - \|Bx\|^2. \tag{1.34}$$

Since

$$\operatorname{Re}\langle B^*Ax, x\rangle \leq |\langle B^*Ax, x\rangle|, \quad \|Bx\|^2 \geq \frac{1}{\|B^{-1}\|^2}\|x\|^2$$

and

$$\|Ax\|^2 + \|B\|^2 \geq 2\|B\|\,\|Ax\|$$

for any $x \in H$, hence by (1.34) we get

$$2\|B\|\,\|Ax\| \leq 2|\langle B^*Ax, x\rangle| + r^2 + \frac{\|B\|^2 \left\|B^{-1}\right\|^2 - 1}{\|B^{-1}\|^2} \tag{1.35}$$

for any $x \in H$, $\|x\| = 1$. Taking the supremum over $x \in H$, $\|x\| = 1$ we deduce the desired result (1.33). □

Remark 6. *If we choose in Theorem 9, $B = \lambda A^*$, $\lambda \neq 0$, A is invertible, then we get the inequality:*

$$(0 \leq)\ \|A\|^2 - w\left(A^2\right) \leq \frac{1}{2|\lambda|}r^2 + |\lambda| \cdot \frac{\|A\|^2 \left\|A^{-1}\right\|^2 - 1}{2\left\|A^{-1}\right\|^2} \tag{1.36}$$

provided $\|A - \lambda A^*\| \leq r$.

The following result may be stated as well.

Theorem 10 (Dragomir, 2008 [19])**.** *Let $A, B : H \to H$ be two bounded linear operators on H. If B is invertible and for $r > 0$ we have*

$$\|A - B\| \leq r < \|B\|, \tag{1.37}$$

then

$$\|A\| \leq \frac{1}{\sqrt{\|B\|^2 - r^2}}\left(w\left(B^*A\right) + \frac{\|B\|^2 \left\|B^{-1}\right\|^2 - 1}{2\left\|B^{-1}\right\|^2}\right). \tag{1.38}$$

Proof. The first part of condition (1.37) is obviously equivalent to

$$\|Ax\|^2 + \|Bx\|^2 \le 2\operatorname{Re}\langle Ax, Bx\rangle + r^2$$

for any $x \in H$, which is clearly equivalent to

$$\|Ax\|^2 + \|B\|^2 - r^2 \le 2\operatorname{Re}\langle B^*Ax, x\rangle + \|B\|^2 - \|Bx\|^2 . \tag{1.39}$$

Since

$$\operatorname{Re}\langle B^*Ax, x\rangle \le |\langle B^*Ax, x\rangle|, \|Bx\|^2 \ge \frac{1}{\|B^{-1}\|^2}\|x\|^2$$

and, by the second part of (1.37),

$$\|Ax\|^2 + \|B\|^2 - r^2 \ge 2\sqrt{\|B\|^2 - r^2}\,\|Ax\|,$$

for any $x \in H$, hence by (1.39) we get

$$2\|Ax\|\sqrt{\|B\|^2 - r^2} \le 2|\langle B^*Ax, x\rangle| + \frac{\|B\|^2\|B^{-1}\|^2 - 1}{\|B^{-1}\|^2} \tag{1.40}$$

for any $x \in H$, $\|x\| = 1$. Taking the supremum over $x \in H$, $\|x\| = 1$ in (1.40), we deduce the desired inequality (1.38). □

Remark 7. *The above Theorem 10 has some particular cases of interest. For instance, if we choose $B = \lambda I$, with $|\lambda| > r$, then (1.37) is obviously fulfilled and by (1.38) we get*

$$\|A\| \le \frac{w(A)}{\sqrt{1 - \left(\frac{r}{|\lambda|}\right)^2}}, \tag{1.41}$$

provided $\|A - \lambda I\| \le r$. This result has been obtained in the earlier paper [8].

On the other hand, if in the above we choose $B = \lambda A^*$ with $\|A\| \ge \frac{r}{|\lambda|}$ $(\lambda \ne 0)$, then by (1.38) we get

$$\|A\| \le \frac{1}{\sqrt{\|A\|^2 - \left(\frac{r}{|\lambda|}\right)^2}}\left[w\left(A^2\right) + |\lambda| \cdot \frac{\|A\|^2\|A^{-1}\|^2 - 1}{2\|A^{-1}\|^2}\right], \tag{1.42}$$

provided $\|A - \lambda A^*\| \le r$.

The following result may be stated as well.

Theorem 11 (Dragomir, 2008 [19])**.** *Let A, B and r be as in Theorem 8. Moreover, if*

$$\|B^{-1}\| < \frac{1}{r}, \tag{1.43}$$

then

$$\|A\| \le \frac{\|B^{-1}\|}{\sqrt{1 - r^2\|B^{-1}\|^2}}\, w(B^*A). \tag{1.44}$$

Proof. Observe that, by (1.28) we have

$$\|A\|^2 + \frac{1 - r^2 \|B^{-1}\|^2}{\|B^{-1}\|^2} \le 2w(B^*A). \tag{1.45}$$

Utilising the elementary inequality

$$2\frac{\|A\|}{\|B^{-1}\|}\sqrt{1 - r^2 \|B^{-1}\|^2} \le \|A\|^2 + \frac{1 - r^2 \|B^{-1}\|^2}{\|B^{-1}\|^2}, \tag{1.46}$$

which can be stated since (1.43) is assumed to be true, hence by (1.45) and (1.46) we deduce the desired result (1.44). □

Remark 8. *If we assume that $B = \lambda A^*$ with $\lambda \neq 0$ and A an invertible operator, then, by applying Theorem 11, we get the inequality:*

$$\|A\| \le \frac{\|A^{-1}\| w(A^2)}{\sqrt{|\lambda|^2 - r^2 \|A^{-1}\|^2}}, \tag{1.47}$$

provided $\|A - \lambda A^*\| \le r$ and $\|A^{-1}\| \le \frac{|\lambda|}{r}$.

The following result may be stated as well.

Theorem 12 (Dragomir, 2008 [19])**.** *Let $A, B : H \to H$ be two bounded linear operators. If $r > 0$ and B is invertible with the property that $\|A - B\| \le r$ and*

$$\frac{1}{\sqrt{r^2+1}} \le \|B^{-1}\| < \frac{1}{r}, \tag{1.48}$$

then

$$\|A\|^2 \le w^2(B^*A) + 2w(B^*A) \cdot \frac{\|B^{-1}\| - \sqrt{1 - r^2 \|B^{-1}\|^2}}{\|B^{-1}\|}. \tag{1.49}$$

Proof. Let $x \in H$, $\|x\| = 1$. Then by (1.27) we have

$$\|Ax\|^2 + \frac{1}{\|B^{-1}\|^2} \le 2|\langle B^*Ax, x\rangle| + r^2, \tag{1.50}$$

and since $\frac{1}{\|B^{-1}\|^2} - r^2 > 0$, we can conclude that $|\langle B^*Ax, x\rangle| > 0$ for any $x \in H$, $\|x\| = 1$.

Dividing (1.50) throughout by $|\langle B^*Ax, x\rangle| > 0$, we obtain

$$\frac{\|Ax\|^2}{|\langle B^*Ax, x\rangle|} \le 2 + \frac{r^2}{|\langle B^*Ax, x\rangle|} - \frac{1}{\|B^{-1}\|^2 |\langle B^*Ax, x\rangle|}. \tag{1.51}$$

Subtracting $|\langle B^*Ax, x\rangle|$ from both sides of (1.51), we get

$$
\begin{aligned}
(1.52)\qquad & \frac{\|Ax\|^2}{|\langle B^*Ax, x\rangle|} - |\langle B^*Ax, x\rangle| \\
& \leq 2 - |\langle B^*Ax, x\rangle| - \frac{1 - r^2 \left\|B^{-1}\right\|^2}{|\langle B^*Ax, x\rangle| \, \|B^{-1}\|^2} \\
& = 2 - \frac{2\sqrt{1 - r^2 \|B^{-1}\|^2}}{\|B^{-1}\|} - \left(\sqrt{|\langle B^*Ax, x\rangle|} - \frac{\sqrt{1 - r^2 \|B^{-1}\|^2}}{\|B^{-1}\| \sqrt{|\langle B^*Ax, x\rangle|}} \right)^2 \\
& \leq 2 \left(\frac{\left\|B^{-1}\right\| - \sqrt{1 - r^2 \|B^{-1}\|^2}}{\|B^{-1}\|} \right),
\end{aligned}
$$

which gives:

$$
(1.53)\qquad \|Ax\|^2 \leq |\langle B^*Ax, x\rangle|^2 + 2\,|\langle B^*Ax, x\rangle| \, \frac{\left\|B^{-1}\right\| - \sqrt{1 - r^2 \|B^{-1}\|^2}}{\|B^{-1}\|}.
$$

We also remark that, by (1.48) the quantity

$$
\left\|B^{-1}\right\| - \sqrt{1 - r^2 \|B^{-1}\|^2} \geq 0,
$$

hence, on taking the supremum in (1.53) over $x \in H$, $\|x\| = 1$, we deduce the desired inequality. □

Remark 9. *It is interesting to remark that if we assume* $\lambda \in \mathbb{C}$ *with* $0 < r \leq |\lambda| \leq \sqrt{r^2 + 1}$ *and* $\|A - \lambda I\| \leq r$*, then by (1.24) we can state the following inequality:*

$$
(1.54)\qquad \|A\|^2 \leq |\lambda|^2 w\left(A^2\right) + 2\,|\lambda| \left(1 - \sqrt{|\lambda|^2 - r^2}\right) w\left(A\right).
$$

Also, if $\|A - A^*\| \leq r$*,* A *is invertible and* $\frac{1}{\sqrt{r^2+1}} \leq \left\|A^{-1}\right\| \leq \frac{1}{r}$*, then, by (1.49) we also have*

$$
(1.55)\qquad \|A\|^2 \leq w^2\left(A^2\right) + 2w\left(A^2\right) \cdot \frac{\left\|A^{-1}\right\| - \sqrt{1 - r^2 \|A^{-1}\|^2}}{\|A^{-1}\|}.
$$

One can also prove the following result.

Theorem 13 (Dragomir, 2008 [19])**.** *Let* $A, B : H \to H$ *be two bounded linear operators. If* $r > 0$ *and* B *is invertible with the property that* $\|A - B\| \leq r$ *and* $\left\|B^{-1}\right\| \leq \frac{1}{r}$*, then*

$$
\begin{aligned}
(1.56)\qquad & (0 \leq) \|A\|^2 \|B\|^2 - w^2\left(B^*A\right) \\
& \leq 2w\left(B^*A\right) \cdot \frac{\|B\|}{\|B^{-1}\|} \left(\|B\| \left\|B^{-1}\right\| - \sqrt{1 - r^2 \|B^{-1}\|^2} \right).
\end{aligned}
$$

Proof. We subtract the quantity $\frac{|\langle B^*Ax,x\rangle|}{\|B\|^2}$ from both sides of (1.51) to obtain

$$0 \le \frac{\|Ax\|^2}{|\langle B^*Ax,x\rangle|} - \frac{|\langle B^*Ax,x\rangle|}{\|B\|^2} \le 2 - 2\cdot\frac{\sqrt{1-r^2\|B^{-1}\|^2}}{\|B\|\|B^{-1}\|} - \left(\frac{\sqrt{|\langle B^*Ax,x\rangle|}}{\|B\|} - \frac{\sqrt{1-r^2\|B^{-1}\|^2}}{\sqrt{|\langle B^*Ax,x\rangle|}\,\|B^{-1}\|}\right)^2 \le 2\cdot\frac{\left(\|B\|\|B^{-1}\| - \sqrt{1-r^2\|B^{-1}\|^2}\right)}{\|B\|\|B^{-1}\|}, \tag{1.57}$$

which is equivalent with

$$(0\le)\ \|Ax\|^2\|B\|^2 - |\langle B^*Ax,x\rangle|^2 \le 2\frac{\|B\|}{\|B^{-1}\|}|\langle B^*Ax,x\rangle|\left(\|B\|\|B^{-1}\| - \sqrt{1-r^2\|B^{-1}\|^2}\right) \tag{1.58}$$

for any $x\in H$, $\|x\|=1$.

The inequality (1.58) also shows that $\|B\|\|B^{-1}\| \ge \sqrt{1-r^2\|B^{-1}\|^2}$ and then, by (1.58), we get

$$\|Ax\|^2\|B\|^2 \le |\langle B^*Ax,x\rangle|^2 + 2\frac{\|B\|}{\|B^{-1}\|}|\langle B^*Ax,x\rangle|\left(\|B\|\|B^{-1}\| - \sqrt{1-r^2\|B^{-1}\|^2}\right) \tag{1.59}$$

for any $x\in X$, $\|x\|=1$. Taking the supremum in (1.59) we deduce the desired inequality (1.56). □

Remark 10. *The above Theorem 13 has some particular instances of interest as follows. If, for instance, we choose $B=\lambda I$ with $|\lambda|\ge r>0$ and $\|A-\lambda I\|\le r$, then by (1.56) we obtain the inequality*

$$(0\le)\ \|A\|^2 - w^2(A) \le 2|\lambda|\,w(A)\left(1-\sqrt{1-\frac{r^2}{|\lambda|^2}}\right). \tag{1.60}$$

Also, if A is invertible, $\|A-\lambda A^*\|\le r$ and $\|A^{-1}\|\le\frac{|\lambda|}{r}$, then by (1.56) we can state:

$$(0\le)\ \|A\|^4 - w^2(A^2) \le 2|\lambda|\,w(A^2)\cdot\frac{\|A\|}{\|A^{-1}\|}\left(\|A\|\|A^{-1}\| - \sqrt{1-\frac{r^2}{|\lambda|^2}\|A^{-1}\|^2}\right). \tag{1.61}$$

2. Other Inequalities for a Product of Two Linear Operators

2.1. **Some Preliminary Results.** For the complex numbers α,β and the bounded linear operator T we define the following transform (see [16]):

$$C_{\alpha,\beta}(T) := (T^* - \overline{\alpha}I)(\beta I - T), \tag{2.1}$$

where by T^* we denote the adjoint of T.

We list some properties of the transform $C_{\alpha,\beta}(\cdot)$ that are of interest:

(i) For any $\alpha, \beta \in \mathbb{C}$ and $T \in B(H)$ we have:

$$C_{\alpha,\beta}(I) = (1-\overline{\alpha})(\beta - 1) I, \qquad C_{\alpha,\alpha}(T) = -(\alpha I - T)^* (\alpha I - T), \tag{2.2}$$

$$C_{\alpha,\beta}(\gamma T) = |\gamma|^2 C_{\frac{\alpha}{\gamma},\frac{\beta}{\gamma}}(T) \quad \text{for each } \gamma \in \mathbb{C} \setminus \{0\}, \tag{2.3}$$

$$[C_{\alpha,\beta}(T)]^* = C_{\beta,\alpha}(T) \tag{2.4}$$

and

$$C_{\overline{\beta},\overline{\alpha}}(T^*) - C_{\alpha,\beta}(T) = T^*T - TT^*. \tag{2.5}$$

(ii) The operator $T \in B(H)$ is normal if and only if $C_{\overline{\beta},\overline{\alpha}}(T^*) = C_{\alpha,\beta}(T)$ for each $\alpha, \beta \in \mathbb{C}$.

We recall that a bounded linear operator T on the complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$ is called *accretive* if $\operatorname{Re}\langle Ty, y \rangle \geq 0$ for any $y \in H$.

Utilizing the following identity

$$\begin{aligned} \operatorname{Re}\langle C_{\alpha,\beta}(T)x, x\rangle &= \operatorname{Re}\langle C_{\beta,\alpha}(T)x, x\rangle \\ &= \frac{1}{4}|\beta - \alpha|^2 - \left\| \left(T - \frac{\alpha+\beta}{2} I \right) x \right\|^2 \end{aligned} \tag{2.6}$$

that holds for any scalars α, β and any vector $x \in H$ with $\|x\| = 1$ we can give a simple characterization result that is useful in the following:

Lemma 1 (Dragomir, 2009 [21])**.** *For $\alpha, \beta \in \mathbb{C}$ and $T \in B(H)$ the following statements are equivalent:*

(i) The transform $C_{\alpha,\beta}(T)$ (or, equivalently, $C_{\beta,\alpha}(T)$) is accretive;

(ii) The transform $C_{\overline{\alpha},\overline{\beta}}(T^*)$ $\left(\text{or, equivalently, } C_{\overline{\beta},\bar{\alpha}}(T^*)\right)$ is accretive;

(iii) We have the norm inequality

$$\left\| T - \frac{\alpha+\beta}{2} \cdot I \right\| \leq \frac{1}{2}|\beta - \alpha| \tag{2.7}$$

or, equivalently,

$$\left\| T^* - \frac{\bar{\alpha}+\bar{\beta}}{2} \cdot I \right\| \leq \frac{1}{2}|\beta - \alpha|. \tag{2.8}$$

Remark 11. *In order to give examples of operators $T \in B(H)$ and numbers $\alpha, \beta \in \mathbb{C}$ such that the transform $C_{\alpha,\beta}(T)$ is accretive, it suffices to select a bounded linear operator S and the complex numbers z, w with the property that $\|S - zI\| \leq |w|$ and, by choosing $T = S$, $\alpha = \frac{1}{2}(z+w)$ and $\beta = \frac{1}{2}(z-w)$ we observe that T satisfies (2.7), i.e., $C_{\alpha,\beta}(T)$ is accretive.*

2.2. **Other Norm and Numerical Radius Inequalities.** In light of the above results it is then natural to compare the quantities $\|AB\|$ and $w(A)w(B) + w(A)\|B\| + \|A\| w(B)$ provided that some information about the transforms $C_{\alpha,\beta}(A)$ and $C_{\gamma,\delta}(B)$ are available, where $\alpha, \beta, \gamma, \delta \in \mathbb{K}$.

Theorem 14 (Dragomir, 2008, [21])**.** *Let $A, B \in B(H)$ and $\alpha, \beta, \gamma, \delta \in \mathbb{K}$ be such that the transforms $C_{\alpha,\beta}(A)$ and $C_{\gamma,\delta}(B)$ are accretive, then*

$$\|BA\| \leq w(A)w(B) + w(A)\|B\| + \|A\| w(B) + \frac{1}{4}|\beta - \alpha||\gamma - \delta|. \tag{2.9}$$

Proof. Since $C_{\alpha,\beta}(A)$ and $C_{\gamma,\delta}(B)$ are accretive, then, on making use of Lemma 1 we have that

$$\left\| Ax - \frac{\alpha+\beta}{2}x \right\| \le \frac{1}{2}|\beta-\alpha| \quad \text{and} \quad \left\| B^*x - \frac{\bar{\gamma}+\bar{\delta}}{2}x \right\| \le \frac{1}{2}\left|\bar{\gamma}-\bar{\delta}\right|,$$

for any $x \in H$, $\|x\| = 1$.

Utilizing the Schwarz inequality we may write that

$$\text{(2.10)} \quad \begin{aligned} &|\langle Ax - \langle Ax,x\rangle x, B^*y - \langle B^*y,y\rangle y\rangle| \\ &\qquad \le \|Ax - \langle Ax,x\rangle x\| \|B^*y - \langle B^*y,y\rangle y\|, \end{aligned}$$

for any $x, y \in H$, with $\|x\| = \|y\| = 1$.

Since for any vectors $u, f \in H$ with $\|f\| = 1$ we have $\|u - \langle u,f\rangle f\| = \inf_{\mu\in\mathbb{K}}\|u - \mu f\|$, then obviously

$$\|Ax - \langle Ax,x\rangle x\| \le \left\| Ax - \frac{\alpha+\beta}{2}x \right\| \le \frac{1}{2}|\beta-\alpha|$$

and

$$\|B^*y - \langle B^*y,y\rangle y\| \le \left\| B^*y - \frac{\bar{\gamma}+\bar{\delta}}{2}y \right\| \le \frac{1}{2}|\gamma-\delta|$$

producing the inequality

$$\text{(2.11)} \quad \|Ax - \langle Ax,x\rangle x\| \|B^*y - \langle B^*y,y\rangle y\| \le \frac{1}{4}|\beta-\alpha||\gamma-\delta|.$$

Now, observe that

$$\begin{aligned} &\langle Ax - \langle Ax,x\rangle x, B^*y - \langle B^*y,y\rangle y\rangle \\ &\qquad = \langle BAx,y\rangle + \langle Ax,x\rangle\langle By,y\rangle\langle x,y\rangle - \langle Ax,x\rangle\langle Bx,y\rangle - \langle Ax,y\rangle\langle By,y\rangle, \end{aligned}$$

for any $x, y \in H$, with $\|x\| = \|y\| = 1$.

Taking the modulus in the equality and utilizing its properties we have successively

$$\begin{aligned} &|\langle Ax - \langle Ax,x\rangle x, B^*y - \langle B^*y,y\rangle y\rangle| \\ &\ge |\langle BAx,y\rangle| - |\langle Ax,x\rangle\langle Bx,y\rangle + \langle Ax,y\rangle\langle By,y\rangle - \langle Ax,x\rangle\langle By,y\rangle\langle x,y\rangle| \\ &\ge |\langle BAx,y\rangle| - |\langle Ax,x\rangle\langle Bx,y\rangle| - |\langle Ax,y\rangle\langle By,y\rangle| - |\langle Ax,x\rangle\langle By,y\rangle\langle x,y\rangle| \end{aligned}$$

which is equivalent with

$$\text{(2.12)} \quad \begin{aligned} &|\langle Ax - \langle Ax,x\rangle x, B^*y - \langle B^*y,y\rangle y\rangle| \\ &\quad + |\langle Ax,x\rangle\langle Bx,y\rangle| + |\langle Ax,y\rangle\langle By,y\rangle| + |\langle Ax,x\rangle\langle By,y\rangle\langle x,y\rangle| \\ &\qquad \ge |\langle BAx,y\rangle|, \end{aligned}$$

for any $x, y \in H$, with $\|x\| = \|y\| = 1$.

Finally, on making use of the inequalities (2.10)-(2.12) we can state that

$$\text{(2.13)} \quad \begin{aligned} &\frac{1}{4}|\beta-\alpha||\gamma-\delta| \\ &\quad + |\langle Ax,x\rangle\langle Bx,y\rangle| + |\langle Ax,y\rangle\langle By,y\rangle| + |\langle Ax,x\rangle\langle By,y\rangle\langle x,y\rangle| \\ &\qquad \ge |\langle BAx,y\rangle|, \end{aligned}$$

for any $x, y \in H$, with $\|x\| = \|y\| = 1$.

Taking the supremum in (2.13) over $\|x\|=\|y\|=1$ and noticing that

$$\sup_{\|x\|=1}|\langle Ax,x\rangle|=w(A), \quad \sup_{\|x\|=\|y\|=1}|\langle Ax,y\rangle|=\|A\|, \quad \sup_{\|y\|=1}|\langle By,y\rangle|=w(B),$$

$$\sup_{\|x\|=\|y\|=1}|\langle Bx,y\rangle|=\|B\|, \quad \sup_{\|x\|=\|y\|=1}|\langle x,y\rangle|=1 \quad \text{and}$$
$$\sup_{\|x\|=\|y\|=1}|\langle BAx,y\rangle|=\|BA\|,$$

we deduce the desired result (2.9). □

Remark 12. *It is an open problem whether or not the constant* $\frac{1}{4}$ *is best possible in the inequality (2.9).*

A different approach is considered in the following result:

Theorem 15 (Dragomir, 2008, [21])**.** *With the assumptions from Theorem 14 we have the inequality*

$$\|BA\|\le w(A)\|B\|+\frac{1}{4}|\beta-\alpha|(|\gamma+\delta|+|\gamma-\delta|). \tag{2.14}$$

Proof. By the Schwarz inequality and taking into account the assumptions for the operators A and B we may state that

$$\begin{aligned}\left|\left\langle Ax-\langle Ax,x\rangle x, B^*y-\frac{\bar\gamma+\bar\delta}{2}y\right\rangle\right| &\le \|Ax-\langle Ax,x\rangle x\|\left\|B^*y-\frac{\bar\gamma+\bar\delta}{2}y\right\| \\ &\le \left\|Ax-\frac{\alpha+\beta}{2}x\right\|\left\|B^*y-\frac{\bar\gamma+\bar\delta}{2}y\right\| \\ &\le \frac{1}{4}|\beta-\alpha||\gamma-\delta|,\end{aligned} \tag{2.15}$$

for any $x,y\in H$, with $\|x\|=\|y\|=1$.

Now, since

$$\left\langle Ax-\langle Ax,x\rangle x, B^*y-\frac{\bar\gamma+\bar\delta}{2}y\right\rangle = \langle BAx,y\rangle-\langle Ax,x\rangle\langle Bx,y\rangle-\frac{\gamma+\delta}{2}\langle Ax-\langle Ax,x\rangle x,y\rangle,$$

on taking the modulus in this equality we have

$$\begin{aligned}&\left|\left\langle Ax-\langle Ax,x\rangle x, B^*y-\frac{\bar\gamma+\bar\delta}{2}y\right\rangle\right| \\ &\qquad\ge|\langle BAx,y\rangle|-|\langle Ax,x\rangle\langle Bx,y\rangle|-\left|\frac{\gamma+\delta}{2}\right||\langle Ax-\langle Ax,x\rangle x,y\rangle|,\end{aligned} \tag{2.16}$$

for any $x,y\in H$, with $\|x\|=\|y\|=1$.

On making use of (2.15) and (2.16) we get

$$\begin{aligned}
&|\langle BAx,y\rangle| \\
&\le |\langle Ax,x\rangle\langle Bx,y\rangle| + \left|\frac{\gamma+\delta}{2}\right| |\langle Ax-\langle Ax,x\rangle x,y\rangle| + \frac{1}{4}|\beta-\alpha||\gamma-\delta| \\
&\le |\langle Ax,x\rangle\langle Bx,y\rangle| + \left|\frac{\gamma+\delta}{2}\right| \left\|Ax-\frac{\alpha+\beta}{2}x\right\| + \frac{1}{4}|\beta-\alpha||\gamma-\delta| \\
&\le |\langle Ax,x\rangle\langle Bx,y\rangle| + \frac{1}{4}|\beta-\alpha|(|\gamma+\delta|+|\gamma-\delta|),
\end{aligned} \tag{2.17}$$

for any $x,y\in H$, with $\|x\|=\|y\|=1$.

Taking the supremum over $\|x\|=\|y\|=1$ in (2.17) we deduce the desired inequality (2.14). □

In a similar manner we can state the following results as well:

Theorem 16 (Dragomir, 2008, [21])**.** *With the assumptions from Theorem 14 we have the inequality*

$$\|BA\| \le w(A)\|B\| + \frac{1}{2}|\gamma+\delta|(w(A)+\|A\|) + \frac{1}{4}|\beta-\alpha||\gamma-\delta|. \tag{2.18}$$

Indeed, we observe that

$$\begin{aligned}
&\left\langle Ax-\langle Ax,x\rangle x, B^*y-\frac{\bar\gamma+\bar\delta}{2}y\right\rangle \\
&\qquad = \langle BAx,y\rangle - \langle Ax,x\rangle\langle Bx,y\rangle - \frac{\gamma+\delta}{2}\langle Ax,y\rangle + \frac{\gamma+\delta}{2}\langle Ax,x\rangle\langle x,y\rangle
\end{aligned}$$

which produces the inequality

$$\begin{aligned}
&\left|\left\langle Ax-\langle Ax,x\rangle x, B^*y-\frac{\bar\gamma+\bar\delta}{2}y\right\rangle\right| + |\langle Ax,x\rangle\langle Bx,y\rangle| \\
&\qquad + \left|\frac{\gamma+\delta}{2}\right||\langle Ax,y\rangle| + \left|\frac{\gamma+\delta}{2}\right||\langle Ax,x\rangle||\langle x,y\rangle| \ge |\langle BAx,y\rangle|,
\end{aligned}$$

for any $x,y\in H$, with $\|x\|=\|y\|=1$.

On utilizing the same argument as in the proof of the above theorem, we get the desired result (2.18). The details are omitted.

2.3. **Related Results.** The following result concerning an upper bound for the norm of the operator product may be stated.

Theorem 17 (Dragomir, 2008, [21])**.** *With the assumptions from Theorem 14 we have the inequality*

$$\begin{aligned}
\|BA\| &\le \frac{1}{4}|\beta-\alpha||\gamma-\delta| + \left\|\frac{\alpha+\beta}{2}\cdot B + \frac{\gamma+\delta}{2}\cdot A \right. \\
&\qquad \left. - \frac{\alpha+\beta}{2}\cdot\frac{\gamma+\delta}{2}\cdot I\right\| \\
&\le \frac{1}{4}|\beta-\alpha||\gamma-\delta| + \min\left\{\left|\frac{\alpha+\beta}{2}\right|\left(\|B\|+\frac{1}{2}|\beta-\alpha|\right),\right. \\
&\qquad \left.\left|\frac{\gamma+\delta}{2}\right|\left(\|A\|+\frac{1}{2}|\gamma-\delta|\right)\right\}.
\end{aligned} \tag{2.19}$$

Proof. By the Schwarz inequality and utilizing the assumptions about A and B we have

$$\left|\left\langle Ax-\frac{\alpha+\beta}{2}x,B^{*}y-\frac{\bar{\gamma}+\bar{\delta}}{2}y\right\rangle\right| \leq\left\|Ax-\frac{\alpha+\beta}{2}x\right\|\left\|B^{*}y-\frac{\bar{\gamma}+\bar{\delta}}{2}y\right\| \leq\frac{1}{4}\left|\beta-\alpha\right|\left|\gamma-\delta\right|, \tag{2.20}$$

for any $x,y\in H$, with $\|x\|=\|y\|=1$.

Also, the following identity is of interest in itself

$$\left\langle Ax-\frac{\alpha+\beta}{2}x,B^{*}y-\frac{\bar{\gamma}+\bar{\delta}}{2}y\right\rangle =\left\langle BAx,y\right\rangle+\frac{\alpha+\beta}{2}\cdot\frac{\gamma+\delta}{2}\left\langle x,y\right\rangle-\frac{\alpha+\beta}{2}\left\langle Bx,y\right\rangle-\frac{\gamma+\delta}{2}\left\langle Ax,y\right\rangle, \tag{2.21}$$

for any $x,y\in H$, with $\|x\|=\|y\|=1$.

This identity gives

$$\left\langle Ax-\frac{\alpha+\beta}{2}x,B^{*}y-\frac{\bar{\gamma}+\bar{\delta}}{2}y\right\rangle +\left\langle\frac{\alpha+\beta}{2}\cdot Bx+\frac{\gamma+\delta}{2}\cdot Ax-\frac{\alpha+\beta}{2}\cdot\frac{\gamma+\delta}{2}x,y\right\rangle=\left\langle BAx,y\right\rangle,$$

for any $x,y\in H$, with $\|x\|=\|y\|=1$.

Taking the modulus and utilizing (2.20) we get

$$\begin{aligned}\left|\left\langle BAx,y\right\rangle\right| &\leq\left|\left\langle Ax-\frac{\alpha+\beta}{2}x,B^{*}y-\frac{\bar{\gamma}+\bar{\delta}}{2}y\right\rangle\right| \\ &\quad+\left|\left\langle\frac{\alpha+\beta}{2}\cdot Bx+\frac{\gamma+\delta}{2}\cdot Ax-\frac{\alpha+\beta}{2}\cdot\frac{\gamma+\delta}{2}x,y\right\rangle\right| \\ &\leq\frac{1}{4}\left|\beta-\alpha\right|\left|\gamma-\delta\right| \\ &\quad+\left\|\frac{\alpha+\beta}{2}\cdot Bx+\frac{\gamma+\delta}{2}\cdot Ax-\frac{\alpha+\beta}{2}\cdot\frac{\gamma+\delta}{2}x\right\|,\end{aligned}$$

for any $x,y\in H$, with $\|x\|=\|y\|=1$.

Finally, taking the supremum over $\|x\|=\|y\|=1$ we deduce the first part of the desired inequality (2.19). The second part is obvious by the triangle inequality and by the assumptions on A and B. □

The following particular case also holds

Corollary 2. *Let $A\in B(H)$ and $\alpha,\beta\in\mathbb{K}$ be such that the transforms $C_{\alpha,\beta}(A)$ is accretive. Then*

$$\left\|A^{2}\right\|\leq\frac{1}{4}\left|\beta-\alpha\right|^{2}+\left|\frac{\alpha+\beta}{2}\right|\left\|2\cdot A-\frac{\alpha+\beta}{2}\cdot I\right\| \left(\leq\frac{1}{4}\left|\beta-\alpha\right|^{2}+\left|\frac{\alpha+\beta}{2}\right|\left(\left\|A\right\|+\frac{1}{2}\left|\beta-\alpha\right|\right)\right) \tag{2.22}$$

and

$$\|A\|^2 \leq \frac{1}{4}|\beta-\alpha|^2 + \left\|\frac{\bar{\alpha}+\bar{\beta}}{2}\cdot A^* + \frac{\alpha+\beta}{2}\cdot A - \left|\frac{\alpha+\beta}{2}\right|^2\cdot I\right\| \left(\leq \frac{1}{4}|\beta-\alpha|^2 + \left|\frac{\alpha+\beta}{2}\right|\left(\|A\| + \frac{1}{2}|\beta-\alpha|\right)\right), \tag{2.23}$$

respectively.

The following result provides an approximation for the operator product in terms of some simpler quantities:

Theorem 18 (Dragomir, 2008, [21])**.** *With the assumptions from Theorem 14 we have the inequality*

$$\left\|BA - \frac{\alpha+\beta}{2}\cdot B - \frac{\gamma+\delta}{2}\cdot A + \frac{\alpha+\beta}{2}\cdot\frac{\gamma+\delta}{2}\cdot I\right\| \leq \frac{1}{4}|\beta-\alpha||\gamma-\delta|. \tag{2.24}$$

Proof. The identity (2.21) can written in an equivalent form as

$$\left\langle Ax - \frac{\alpha+\beta}{2}x, B^*y - \frac{\bar{\gamma}+\bar{\delta}}{2}y\right\rangle = \left\langle\left(BA - \frac{\alpha+\beta}{2}\cdot B - \frac{\gamma+\delta}{2}\cdot A + \frac{\alpha+\beta}{2}\cdot\frac{\gamma+\delta}{2}\cdot I\right)x, y\right\rangle, \tag{2.25}$$

for any $x, y \in H$, with $\|x\| = \|y\| = 1$.

Taking the modulus and making use of the inequality (2.20) we get

$$\left|\left\langle\left(BA - \frac{\alpha+\beta}{2}\cdot B - \frac{\gamma+\delta}{2}\cdot A + \frac{\alpha+\beta}{2}\cdot\frac{\gamma+\delta}{2}\cdot I\right)x, y\right\rangle\right| \leq \frac{1}{4}|\beta-\alpha||\gamma-\delta|,$$

for any $x, y \in H$, with $\|x\| = \|y\| = 1$, which implies the desired result (2.24). □

Corollary 3. *Let $A \in B(H)$ and $\alpha, \beta \in \mathbb{K}$ be such that the transform $C_{\alpha,\beta}(A)$ is accretive, then*

$$\left\|A^2 - (\alpha+\beta)\cdot A + \left(\frac{\alpha+\beta}{2}\right)^2\cdot I\right\| \leq \frac{1}{4}|\beta-\alpha|^2 \tag{2.26}$$

and

$$\left\|A^*A - \frac{\alpha+\beta}{2}\cdot A^* - \frac{\bar{\alpha}+\bar{\beta}}{2}\cdot A + \left|\frac{\alpha+\beta}{2}\right|^2\cdot I\right\| \leq \frac{1}{4}|\beta-\alpha|^2, \tag{2.27}$$

respectively.

Remark 13. *It is an open problem whether or not the constant $\frac{1}{4}$ is best possible in either of the inequalities (2.24), (2.26) or (2.27) above.*

The following theorem provides an approximation for the operator $\frac{1}{2}(U^*U+UU^*)$ when some information about the real or imaginary part of the operator U are given.

We recall that $U=\operatorname{Re}(U)+i\operatorname{Im}(U)$, i.e., $\operatorname{Re}(U)=\frac{1}{2}(U+U^*)$ and $\operatorname{Im}(U)=\frac{1}{2i}(U-U^*)$. For simplicity, we denote by A the real part of U and by B its imaginary part.

Theorem 19 (Dragomir, 2008, [21])**.** *Suppose that $a,b,c,d\in\mathbb{R}$ are such that $C_{a,c}(A)$ and $C_{b,d}(B)$ are accretive. Denote $\alpha:=a+ib$ and $\beta:=c+id\in\mathbb{C}$, then*

$$\left\|\frac{1}{2}(U^*U+UU^*)-\frac{\bar{\alpha}+\bar{\beta}}{2}\cdot U-\frac{\alpha+\beta}{2}\cdot U^*+\left|\frac{\alpha+\beta}{2}\right|^2\cdot I\right\| \leq \frac{1}{4}|\alpha-\beta|^2. \tag{2.28}$$

Proof. It is well known that for any operator T with the Cartesian decomposition $T=C+iD$ we have

$$\frac{1}{2}(T^*T+TT^*)=C^2+D^2. \tag{2.29}$$

For any $z\in\mathbb{C}$ we also have the identity

$$\frac{1}{2}\left[(U-zI)(U^*-\bar{z}I)+(U^*-\bar{z}I)(U-zI)\right] = \frac{1}{2}(U^*U+UU^*)-\bar{z}\cdot U-z\cdot U^*+|z|^2\cdot I. \tag{2.30}$$

For $z=\frac{\alpha+\beta}{2}$ we observe that

$$\operatorname{Re}(U-zI)=A-\frac{a+c}{2}\cdot I \qquad \text{and} \qquad \operatorname{Im}(U-zI)=B-\frac{b+d}{2}\cdot I$$

and utilizing the identities (2.29) and (2.30) we deduce

$$\begin{aligned}
&\left\|\frac{1}{2}(U^*U+UU^*)-\bar{z}\cdot U-z\cdot U^*+|z|^2\cdot I\right\| \\
&=\left\|\left(A-\frac{a+c}{2}\cdot I\right)^2+\left(B-\frac{b+d}{2}\cdot I\right)^2\right\| \\
&\leq\left\|A-\frac{a+c}{2}\cdot I\right\|^2+\left\|B-\frac{b+d}{2}\cdot I\right\|^2 \\
&\leq\frac{1}{4}\left[(c-a)^2+(d-b)^2\right]=\frac{1}{4}|\alpha-\beta|^2,
\end{aligned}$$

where for the last inequality we have used the fact that $C_{a,c}(A)$ and $C_{b,d}(B)$ are accretive. □

Remark 14. *It is an open problem whether or not the constant $\frac{1}{4}$ is best possible in (2.28).*

3. Power Inequalities for the Numerical Radius of a Product

3.1. **Inequalities for a Product of Two Operators.** The following result for powers of operators holds

Theorem 20 (Dragomir, 2009 [22])**.** *For any $A, B \in B(H)$ and $r \geq 1$, we have the inequality:*

$$w^r(B^*A) \leq \frac{1}{2}\left\|(A^*A)^r + (B^*B)^r\right\|. \tag{3.1}$$

The constant $\frac{1}{2}$ is best possible.

Proof. By the Schwarz inequality in the Hilbert space $(H; \langle\cdot,\cdot\rangle)$ we have:

$$\begin{aligned} |\langle B^*Ax, x\rangle| &= |\langle Ax, Bx\rangle| \leq \|Ax\| \cdot \|Bx\| \\ &= \langle A^*Ax, x\rangle^{\frac{1}{2}} \cdot \langle B^*Bx, x\rangle^{\frac{1}{2}}, \qquad x \in H. \end{aligned} \tag{3.2}$$

Utilising the arithmetic mean - geometric mean inequality and then the convexity of the function $f(t) = t^r$, $r \geq 1$, we have successively,

$$\begin{aligned} \langle A^*Ax, x\rangle^{\frac{1}{2}} \cdot \langle B^*Bx, x\rangle^{\frac{1}{2}} &\leq \frac{\langle A^*Ax, x\rangle + \langle B^*Bx, x\rangle}{2} \\ &\leq \left(\frac{\langle A^*Ax, x\rangle^r + \langle B^*Bx, x\rangle^r}{2}\right)^{\frac{1}{r}} \end{aligned} \tag{3.3}$$

for any $x \in H$.

It is known that if P is a positive operator then for any $r \geq 1$ and $x \in H$ with $\|x\| = 1$ we have the inequality (see for instance [31])

$$\langle Px, x\rangle^r \leq \langle P^r x, x\rangle. \tag{3.4}$$

Applying this property to the positive operator A^*A and B^*B, we deduce that

$$\begin{aligned} \left(\frac{\langle A^*Ax, x\rangle^r + \langle B^*Bx, x\rangle^r}{2}\right)^{\frac{1}{r}} &\leq \left(\frac{\langle (A^*A)^r x, x\rangle + \langle (B^*B)^r x, x\rangle}{2}\right)^{\frac{1}{r}} \\ &= \left(\frac{\langle [(A^*A)^r + (B^*B)^r] x, x\rangle}{2}\right)^{\frac{1}{r}} \end{aligned} \tag{3.5}$$

for any $x \in H$, $\|x\| = 1$.

Now, on making use of the inequalities (3.2), (3.3) and (3.5), we get the inequality:

$$|\langle (B^*A)^r x, x\rangle|^r \leq \frac{1}{2}\langle [(A^*A)^r + (B^*B)^r] x, x\rangle \tag{3.6}$$

for any $x \in H$, $\|x\| = 1$.

Taking the supremum over $x \in H$, $\|x\| = 1$ in (3.6) and since the operator $[(A^*A)^r + (B^*B)^r]$ is self-adjoint, we deduce the desired inequality (3.1).

For $r = 1$ and $B = A$, we get on both sides of (3.1) the same quantity $\|A\|^2$ which shows that the constant $\frac{1}{2}$ is best possible in general in the inequality (3.1). $\square$

Corollary 4. *For any $A \in B(H)$ and $r \geq 1$ we have the inequalities:*

$$w^r(A) \leq \frac{1}{2}\left\|(A^*A)^r + I\right\| \tag{3.7}$$

and

$$w^r\left(A^2\right) \leq \frac{1}{2}\left\|(A^*A)^r + (AA^*)^r\right\|, \tag{3.8}$$

respectively.

A different approach is considered in the following result:

Theorem 21 (Dragomir, 2009 [22])**.** *For any $A,B\in B(H)$ and any $\alpha\in(0,1)$ and $r\geq 1$, we have the inequality:*

$$w^{2r}(B^{*}A)\leq\left\|\alpha(A^{*}A)^{\frac{r}{\alpha}}+(1-\alpha)(B^{*}B)^{\frac{r}{1-\alpha}}\right\|. \tag{3.9}$$

Proof. By Schwarz's inequality, we have:

$$\begin{aligned}|\langle(B^{*}A)x,x\rangle|^{2}&\leq\langle(A^{*}A)x,x\rangle\cdot\langle(B^{*}B)x,x\rangle \\ &=\left\langle\left[(A^{*}A)^{\frac{1}{\alpha}}\right]^{\alpha}x,x\right\rangle\cdot\left\langle\left[(B^{*}B)^{\frac{1}{1-\alpha}}\right]^{1-\alpha}x,x\right\rangle,\end{aligned} \tag{3.10}$$

for any $x\in H$.

It is well known that (see for instance [31]) if P is a positive operator and $q\in(0,1]$ then for any $u\in H$, $\|u\|=1$, we have

$$\langle P^{q}u,u\rangle\leq\langle Pu,u\rangle^{q}. \tag{3.11}$$

Applying this property to the positive operators $(A^{*}A)^{\frac{1}{\alpha}}$ and $(B^{*}B)^{\frac{1}{1-\alpha}}$ $(\alpha\in(0,1))$, we have

$$\begin{aligned}&\left\langle\left[(A^{*}A)^{\frac{1}{\alpha}}\right]^{\alpha}x,x\right\rangle\cdot\left\langle\left[(B^{*}B)^{\frac{1}{1-\alpha}}\right]^{1-\alpha}x,x\right\rangle \\ &\qquad\leq\left\langle(A^{*}A)^{\frac{1}{\alpha}}x,x\right\rangle^{\alpha}\cdot\left\langle(B^{*}B)^{\frac{1}{1-\alpha}}x,x\right\rangle^{1-\alpha},\end{aligned} \tag{3.12}$$

for any $x\in H$, $\|x\|=1$.

Now, utilising the weighted arithmetic mean - geometric mean inequality, i.e., $a^{\alpha}b^{1-\alpha}\leq\alpha a+(1-\alpha)b$, $\alpha\in(0,1)$, $a,b\geq 0$, we get

$$\begin{aligned}&\left\langle(A^{*}A)^{\frac{1}{\alpha}}x,x\right\rangle^{\alpha}\cdot\left\langle(B^{*}B)^{\frac{1}{1-\alpha}}x,x\right\rangle^{1-\alpha} \\ &\qquad\leq\alpha\left\langle(A^{*}A)^{\frac{1}{\alpha}}x,x\right\rangle+(1-\alpha)\left\langle(B^{*}B)^{\frac{1}{1-\alpha}}x,x\right\rangle\end{aligned} \tag{3.13}$$

for any $x\in H$, $\|x\|=1$.

Moreover, by the elementary inequality following from the convexity of the function $f(t)=t^{r}$, $r\geq 1$, namely

$$\alpha a+(1-\alpha)b\leq(\alpha a^{r}+(1-\alpha)b^{r})^{\frac{1}{r}},\qquad\alpha\in(0,1),\ a,b\geq 0,$$

we deduce that

$$\begin{aligned}&\alpha\left\langle(A^{*}A)^{\frac{1}{\alpha}}x,x\right\rangle+(1-\alpha)\left\langle(B^{*}B)^{\frac{1}{1-\alpha}}x,x\right\rangle \\ &\leq\left[\alpha\left\langle(A^{*}A)^{\frac{1}{\alpha}}x,x\right\rangle^{r}+(1-\alpha)\left\langle(B^{*}B)^{\frac{1}{1-\alpha}}x,x\right\rangle^{r}\right]^{\frac{1}{r}} \\ &\leq\left[\alpha\left\langle(A^{*}A)^{\frac{r}{\alpha}}x,x\right\rangle+(1-\alpha)\left\langle(B^{*}B)^{\frac{r}{1-\alpha}}x,x\right\rangle\right]^{\frac{1}{r}},\end{aligned} \tag{3.14}$$

for any $x\in H$, $\|x\|=1$, where, for the last inequality we used the inequality (3.4) for the positive operators $(A^{*}A)^{\frac{1}{\alpha}}$ and $(B^{*}B)^{\frac{1}{1-\alpha}}$.

Now, on making use of the inequalities (3.10), (3.12), (3.13) and (3.14), we get

$$|\langle(B^{*}A)x,x\rangle|^{2r}\leq\left\langle\left[\alpha(A^{*}A)^{\frac{r}{\alpha}}+(1-\alpha)(B^{*}B)^{\frac{r}{1-\alpha}}\right]x,x\right\rangle \tag{3.15}$$

for any $x\in H$, $\|x\|=1$. Taking the supremum over $x\in H$, $\|x\|=1$ in (3.15) produces the desired inequality (3.9). □

Remark 15. *The particular case* $\alpha = \frac{1}{2}$ *produces the inequality*

$$w^{2r}\left(B^{*}A\right) \leq \frac{1}{2}\left\|\left(A^{*}A\right)^{2r} + \left(B^{*}B\right)^{2r}\right\|, \tag{3.16}$$

for $r \geq 1$. *Notice that* $\frac{1}{2}$ *is best possible in (3.16) since for* $r = 1$ *and* $B = A$ *we get on both sides of (3.16) the same quantity* $\|A\|^{4}$.

Corollary 5. *For any* $A \in B(H)$ *and* $\alpha \in (0,1)$, $r \geq 1$, *we have the inequalities*

$$w^{2r}(A) \leq \left\|\alpha\left(A^{*}A\right)^{\frac{r}{\alpha}} + (1-\alpha) I\right\| \tag{3.17}$$

and

$$w^{2r}\left(A^{2}\right) \leq \left\|\alpha\left(A^{*}A\right)^{\frac{r}{\alpha}} + (1-\alpha)\left(AA^{*}\right)^{\frac{r}{1-\alpha}}\right\|, \tag{3.18}$$

respectively.

Moreover, we have

$$\|A\|^{4r} \leq \left\|\alpha\left(A^{*}A\right)^{\frac{r}{\alpha}} + (1-\alpha)\left(A^{*}A\right)^{\frac{r}{1-\alpha}}\right\|. \tag{3.19}$$

3.2. **Inequalities for the Sum of Two Products.** The following result may be stated:

Theorem 22 (Dragomir, 2009 [22])**.** *For any* $A, B, C, D \in B(H)$ *and* $r, s \geq 1$ *we have:*

$$\left\|\frac{B^{*}A + D^{*}C}{2}\right\|^{2} \leq \left\|\frac{\left(A^{*}A\right)^{r} + \left(C^{*}C\right)^{r}}{2}\right\|^{\frac{1}{r}} \cdot \left\|\frac{\left(B^{*}B\right)^{s} + \left(D^{*}D\right)^{s}}{2}\right\|^{\frac{1}{s}}. \tag{3.20}$$

Proof. By the Schwarz inequality in the Hilbert space $(H; \langle\cdot,\cdot\rangle)$ we have:

$$\begin{aligned} &\left|\left\langle\left(B^{*}A + D^{*}C\right)x, y\right\rangle\right|^{2} \\ &= \left|\left\langle B^{*}Ax, y\right\rangle + \left\langle D^{*}Cx, y\right\rangle\right|^{2} \\ &\leq \left[\left|\left\langle B^{*}Ax, y\right\rangle\right| + \left|\left\langle D^{*}Cx, y\right\rangle\right|\right]^{2} \\ &\leq \left[\left\langle A^{*}Ax, x\right\rangle^{\frac{1}{2}} \cdot \left\langle B^{*}By, y\right\rangle^{\frac{1}{2}} + \left\langle C^{*}Cx, x\right\rangle^{\frac{1}{2}} \cdot \left\langle D^{*}Dy, y\right\rangle^{\frac{1}{2}}\right]^{2}, \end{aligned} \tag{3.21}$$

for any $x, y \in H$.

Now, on utilising the elementary inequality:

$$(ab + cd)^{2} \leq \left(a^{2} + c^{2}\right)\left(b^{2} + d^{2}\right), \qquad a, b, c, d \in \mathbb{R},$$

we then conclude that:

$$\begin{aligned} &\left[\left\langle A^{*}Ax, x\right\rangle^{\frac{1}{2}} \cdot \left\langle B^{*}By, y\right\rangle^{\frac{1}{2}} + \left\langle C^{*}Cx, x\right\rangle^{\frac{1}{2}} \cdot \left\langle D^{*}Dy, y\right\rangle^{\frac{1}{2}}\right]^{2} \\ &\qquad \leq \left(\left\langle A^{*}Ax, x\right\rangle + \left\langle C^{*}Cx, x\right\rangle\right) \cdot \left(\left\langle B^{*}By, y\right\rangle + \left\langle D^{*}Dy, y\right\rangle\right), \end{aligned} \tag{3.22}$$

for any $x, y \in H$.

Now, on making use of a similar argument to the one in the proof of Theorem 20, we have for $r, s \geq 1$ that

$$\begin{aligned} &\left(\left\langle A^{*}Ax, x\right\rangle + \left\langle C^{*}Cx, x\right\rangle\right) \cdot \left(\left\langle B^{*}By, y\right\rangle + \left\langle D^{*}Dy, y\right\rangle\right) \\ &\qquad \leq 4\left\langle\left[\frac{\left(A^{*}A\right)^{r} + \left(C^{*}C\right)^{r}}{2}\right]x, x\right\rangle^{\frac{1}{r}} \cdot \left\langle\left[\frac{\left(B^{*}B\right)^{s} + \left(D^{*}D\right)^{s}}{2}\right]y, y\right\rangle^{\frac{1}{s}} \end{aligned} \tag{3.23}$$

for any $x, y \in H$, $\|x\| = \|y\| = 1$.

Consequently, by (3.21) – (3.23) we have:

$$\left|\left\langle\left[\frac{B^*A+D^*C}{2}\right]x,y\right\rangle\right|^2 \le \left\langle\left[\frac{(A^*A)^r+(C^*C)^r}{2}\right]x,x\right\rangle^{\frac{1}{r}}\cdot\left\langle\left[\frac{(B^*B)^s+(D^*D)^s}{2}\right]y,y\right\rangle^{\frac{1}{s}} \tag{3.24}$$

for any $x,y\in H$, $\|x\|=\|y\|=1$.

Taking the supremum over $x,y\in H$, $\|x\|=\|y\|=1$ we deduce the desired inequality (3.20). $\square$

Remark 16. *If $s=r$, then the inequality (3.20) is equivalent with:*

$$\left\|\frac{B^*A+D^*C}{2}\right\|^{2r}\le\left\|\frac{(A^*A)^r+(C^*C)^r}{2}\right\|\cdot\left\|\frac{(B^*B)^r+(D^*D)^r}{2}\right\|. \tag{3.25}$$

Corollary 6. *For any $A,C\in B(H)$ we have:*

$$\left\|\frac{A+C}{2}\right\|^{2r}\le\left\|\frac{(A^*A)^r+(C^*C)^r}{2}\right\|, \tag{3.26}$$

where $r\ge 1$. Also, we have

$$\left\|\frac{A^2+C^2}{2}\right\|^{2}\le\left\|\frac{(A^*A)^r+(C^*C)^r}{2}\right\|^{\frac{1}{r}}\cdot\left\|\frac{(AA^*)^s+(CC^*)^s}{2}\right\|^{\frac{1}{s}} \tag{3.27}$$

for all $r,s\ge 1$, and in particular

$$\left\|\frac{A^2+C^2}{2}\right\|^{2r}\le\left\|\frac{(A^*A)^r+(C^*C)^r}{2}\right\|\cdot\left\|\frac{(AA^*)^r+(CC^*)^r}{2}\right\| \tag{3.28}$$

for $r\ge 1$.

The inequality (3.26) follows from (3.20) for $B=D=I$, while the inequality (3.27) is obtained from the same inequality (3.20) for $B=A^*$ and $D=C^*$.

Another particular result of interest is the following one:

Corollary 7. *For any $A,B\in B(H)$ we have:*

$$\left\|\frac{B^*A+A^*B}{2}\right\|^{2}\le\left\|\frac{(A^*A)^r+(B^*B)^r}{2}\right\|^{\frac{1}{r}}\cdot\left\|\frac{(A^*A)^s+(B^*B)^s}{2}\right\|^{\frac{1}{s}} \tag{3.29}$$

for $r,s\ge 1$ and, in particular,

$$\left\|\frac{B^*A+A^*B}{2}\right\|^{r}\le\left\|\frac{(A^*A)^r+(B^*B)^r}{2}\right\| \tag{3.30}$$

for any $r\ge 1$.

The inequality (3.28) follows from (3.20) for $D=A$ and $C=B$.

Another particular case that might be of interest is the following one.

Corollary 8. *For any $A,D\in B(H)$ we have:*

$$\left\|\frac{A+D}{2}\right\|^{2}\le\left\|\frac{(A^*A)^r+I}{2}\right\|^{\frac{1}{r}}\cdot\left\|\frac{(DD^*)^s+I}{2}\right\|^{\frac{1}{s}}, \tag{3.31}$$

where $r,s\geq 1$. In particular

$$\|A\|^{2}\leq\left\|\frac{(A^{*}A)^{r}+I}{2}\right\|^{\frac{1}{r}}\cdot\left\|\frac{(AA^{*})^{s}+I}{2}\right\|^{\frac{1}{s}}. \tag{3.32}$$

Moreover, for any $r\geq 1$ we have

$$\|A\|^{2r}\leq\left\|\frac{(A^{*}A)^{r}+I}{2}\right\|\cdot\left\|\frac{(AA^{*})^{r}+I}{2}\right\|.$$

The proof is obvious by the inequality (3.20) on choosing $B=I$, $C=I$ and writing the inequality for D^{*} instead of D.

Remark 17. *If $T\in B(H)$ and $T=A+iC$, i.e., A and C are its Cartesian decomposition, then we get from (3.26) that*

$$\|T\|^{2r}\leq 2^{2r-1}\left\|A^{2r}+C^{2r}\right\|,$$

for any $r\geq 1$.

Also, since $A=\operatorname{Re}(T)=\frac{T+T^{*}}{2}$ and $C=\operatorname{Im}(T)=\frac{T-T^{*}}{2i}$, then from (3.26) we get the following inequalities as well:

$$\|\operatorname{Re}(T)\|^{2r}\leq\left\|\frac{(T^{*}T)^{r}+(TT^{*})^{r}}{2}\right\|$$

and

$$\|\operatorname{Im}(T)\|^{2r}\leq\left\|\frac{(T^{*}T)^{r}+(TT^{*})^{r}}{2}\right\|$$

for any $r\geq 1$.

In terms of the *Euclidean radius* of two operators $w_{e}(\cdot,\cdot)$, where, as in [9],

$$w_{e}(T,U):=\sup_{\|x\|=1}\left(|\langle Tx,x\rangle|^{2}+|\langle Ux,x\rangle|^{2}\right)^{\frac{1}{2}},$$

we have the following result as well.

Theorem 23 (Dragomir, 2009 [22])**.** *For any $A,B,C,D\in B(H)$ and $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, we have the inequality:*

$$w_{e}^{2}(B^{*}A,D^{*}C)\leq\|(A^{*}A)^{p}+(C^{*}C)^{p}\|^{1/p}\cdot\|(B^{*}B)^{q}+(D^{*}D)^{q}\|^{1/q}. \tag{3.33}$$

Proof. For any $x\in H$, $\|x\|=1$ we have the inequalities

$$\begin{aligned}
&|\langle B^{*}Ax,x\rangle|^{2}+|\langle D^{*}Cx,x\rangle|^{2}\\
&\leq\langle A^{*}Ax,x\rangle\cdot\langle B^{*}Bx,x\rangle+\langle C^{*}Cx,x\rangle\cdot\langle D^{*}Dx,x\rangle\\
&\leq\left(\langle A^{*}Ax,x\rangle^{p}+\langle C^{*}Cx,x\rangle^{p}\right)^{1/p}\cdot\left(\langle B^{*}Bx,x\rangle^{q}+\langle D^{*}Dx,x\rangle^{q}\right)^{1/q}\\
&\leq\left(\langle (A^{*}A)^{p}x,x\rangle+\langle (C^{*}C)^{p}x,x\rangle\right)^{1/p}\cdot\left(\langle (B^{*}B)^{q}x,x\rangle+\langle (D^{*}D)^{q}x,x\rangle\right)^{1/q}\\
&\leq\langle [(A^{*}A)^{p}+(C^{*}C)^{p}]x,x\rangle^{1/p}\cdot\langle [(B^{*}B)^{q}+(D^{*}D)^{q}]x,x\rangle^{1/q}.
\end{aligned}$$

Taking the supremum over $x\in H$, $\|x\|=1$ and noticing that the operators $(A^{*}A)^{p}+(C^{*}C)^{p}$ and $(B^{*}B)^{q}+(D^{*}D)^{q}$ are self-adjoint, we deduce the desired inequality (3.33). □

The following particular case is of interest.

Corollary 9. *For any* $A, C \in B(H)$ *and* $p, q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$, *we have:*

$$w_e^2(A, C) \leq 2^{\frac{1}{q}} \|(A^*A)^p + (C^*C)^p\|^{\frac{1}{p}}.$$

The proof follows from (3.33) for $B = D = I$.

Corollary 10. *For any* $A, D \in B(H)$ *and* $p, q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$, *we have:*

$$w_e^2(A, D) \leq \|(A^*A)^p + I\|^{\frac{1}{p}} \cdot \|(D^*D)^q + I\|^{\frac{1}{q}}.$$

3.3. **Vector Inequalities for the Commutator.** The commutator of two bounded linear operators T and U is the operator $TU - UT$. For the usual norm $\|\cdot\|$ and for any two operators T and U, by using the triangle inequality and the submultiplicity of the norm, we can state the following inequality:

$$\|TU - UT\| \leq 2\|U\|\|T\|. \tag{3.34}$$

In [17], the following result has been obtained as well

$$\|TU - UT\| \leq 2\min\{\|T\|, \|U\|\}\min\{\|T - U\|, \|T + U\|\}. \tag{3.35}$$

By utilising Theorem 22 we can state the following result for the numerical radius of the commutator.

Proposition 1 (Dragomir, 2009 [22])**.** *For any* $T, U \in B(H)$ *and* $r, s \geq 1$ *we have*

$$\|TU - UT\|^2 \leq 2^{2-\frac{1}{r}-\frac{1}{s}} \|(T^*T)^r + (U^*U)^r\|^{\frac{1}{r}} \cdot \|(TT^*)^s + (UU^*)^s\|^{\frac{1}{s}}. \tag{3.36}$$

Proof. Follows by Theorem 22 on choosing $B = T^*$, $A = U$, $D = -U^*$ and $C = T$. □

Remark 18. *In particular, for* $r = s$ *we get from (3.36) that*

$$\|TU - UT\|^{2r} \leq 2^{2r-2} \|(T^*T)^r + (U^*U)^r\| \cdot \|(TT^*)^r + (UU^*)^r\| \tag{3.37}$$

and for $r = 1$ *we get*

$$\|TU - UT\|^2 \leq \|T^*T + U^*U\| \cdot \|TT^* + UU^*\|. \tag{3.38}$$

For a bounded linear operator $T \in B(H)$, the self-commutator is the operator $T^*T - TT^*$. Observe that the operator $V := -i(T^*T - TT^*)$ is self-adjoint and $w(V) = \|V\|$, i.e.,

$$w(T^*T - TT^*) = \|T^*T - TT^*\|.$$

Now, utilising (3.36) for $U = T^*$ we can state the following corollary.

Corollary 11. *For any* $T \in B(H)$ *we have the inequality:*

$$\|T^*T - TT^*\|^2 \leq 2^{2-\frac{1}{r}-\frac{1}{s}} \|(T^*T)^r + (TT^*)^r\|^{\frac{1}{r}} \cdot \|(T^*T)^s + (TT^*)^s\|^{\frac{1}{s}}. \tag{3.39}$$

In particular, we have

$$\|T^*T - TT^*\|^r \leq 2^{r-1} \|(T^*T)^r + (TT^*)^r\|, \tag{3.40}$$

for any $r \geq 1$.

Moreover, for $r = 1$ *we have*

$$\|T^*T - TT^*\| \leq \|T^*T + TT^*\|. \tag{3.41}$$

4. A Functional Associated with Two Bounded Linear Operators

4.1. **Some Preliminary Facts.** For two bounded linear operators A, B in the Hilbert space $(H, \langle \cdot, \cdot \rangle)$, we define the functional

$$\mu(A,B) := \sup_{\|x\|=1} \{\|Ax\| \|Bx\|\} (\geq 0). \tag{4.1}$$

It is obvious that μ is symmetric and sub-additive in each variable, $\mu(A,A) = \|A\|^2$, $\mu(A,I) = \|A\|$, where I is the identity operator, $\mu(\alpha A, \beta B) = |\alpha\beta| \mu(A,B)$ and $\mu(A,B) \leq \|A\| \|B\|$. We also have the following inequalities

$$\mu(A,B) \geq w(B^* A) \tag{4.2}$$

and

$$\mu(A,B) \|A\| \|B\| \geq \mu(AB, BA). \tag{4.3}$$

The inequality (4.2) follows by the Schwarz inequality $\|Ax\| \|Bx\| \geq |\langle Ax, Bx \rangle|$, $x \in H$, while (4.3) can be obtained by multiplying the inequalities $\|ABx\| \leq \|A\| \|Bx\|$ and $\|BAx\| \leq \|B\| \|Ax\|$.

From (4.2) we also get

$$\|A\|^2 \geq \mu(A, A^*) \geq w(A^2) \tag{4.4}$$

for any A.

Motivated by the above results we establish in this section several inequalities for the functional $\mu(\cdot,\cdot)$ under various assumptions for the operators involved, including operators satisfying the uniform (α, β) −property and operators for which the transform $C_{\alpha,\beta}(\cdot,\cdot)$ is accretive.

4.2. **General Inequalities.** The following result concerning some general power operator inequalities may be stated:

Theorem 24 (Dragomir, 2008 [20]). *For any $A, B \in B(H)$ and $r \geq 1$ we have the inequality*

$$\mu^r(A,B) \leq \frac{1}{2} \|(A^*A)^r + (B^*B)^r\|. \tag{4.5}$$

The constant $\frac{1}{2}$ is best possible.

Proof. Utilising the arithmetic mean - geometric mean inequality and the convexity of the function $f(t) = t^r$ for $r \geq 1$ and $t \geq 0$ we have successively

$$\begin{aligned} \|Ax\| \|Bx\| &\leq \frac{1}{2} [\langle A^*Ax, x \rangle + \langle B^*Bx, x \rangle] \\ &\leq \left[\frac{\langle A^*Ax, x \rangle^r + \langle B^*Bx, x \rangle^r}{2} \right]^{\frac{1}{r}} \end{aligned} \tag{4.6}$$

for any $x \in H$.

It is well known that if P is a positive operator, then for any $r \geq 1$ and $x \in H$ with $\|x\| = 1$ we have the inequality (see for instance [30])

$$\langle Px, x \rangle^r \leq \langle P^r x, x \rangle. \tag{4.7}$$

Applying this inequality to the positive operators A^*A and B^*B we deduce that

$$\left[\frac{\langle A^*Ax, x \rangle^r + \langle B^*Bx, x \rangle^r}{2} \right]^{\frac{1}{r}} \leq \left\langle \frac{[(A^*A)^r + (B^*B)^r] x}{2}, x \right\rangle^{\frac{1}{r}} \tag{4.8}$$

for any $x \in H$ with $\|x\| = 1$.

Now, on making use of the inequalities (4.6) and (4.8) we get

$$\|Ax\| \|Bx\| \leq \left\langle \frac{[(A^*A)^r + (B^*B)^r] x}{2}, x \right\rangle^{\frac{1}{r}} \tag{4.9}$$

for any $x \in H$ with $\|x\| = 1$. Taking the supremum over $x \in H$ with $\|x\| = 1$ we obtain the desired result (4.5).

For $r = 1$ and $B = A$ we get on both sides of (4.5) the same quantity $\|A\|^2$ which shows that the constant $\frac{1}{2}$ is best possible in general in the inequality (4.5). □

Corollary 12. *For any $A \in B(H)$ and $r \geq 1$ we have the inequalities*

$$\mu^r(A, A^*) \leq \frac{1}{2} \|(A^*A)^r + (AA^*)^r\| \tag{4.10}$$

and

$$\|A\|^r \leq \frac{1}{2} \|(A^*A)^r + I\|, \tag{4.11}$$

respectively.

The following similar result for powers of operators can be stated as well:

Theorem 25 (Dragomir, 2008 [20])**.** *For any $A, B \in B(H)$, any $\alpha \in (0,1)$ and $r \geq 1$ we have the inequality*

$$\mu^{2r}(A, B) \leq \left\| \alpha \cdot (A^*A)^{\frac{r}{\alpha}} + (1-\alpha) \cdot (B^*B)^{\frac{r}{1-\alpha}} \right\|. \tag{4.12}$$

The inequality is sharp.

Proof. Observe that, for any $\alpha \in (0,1)$ we have

$$\begin{aligned} \|Ax\|^2 \|Bx\|^2 &= \langle (A^*A) x, x\rangle \langle (B^*B) x, x\rangle \\ &= \left\langle \left[(A^*A)^{\frac{1}{\alpha}}\right]^{\alpha} x, x \right\rangle \left\langle \left[(B^*B)^{\frac{1}{1-\alpha}}\right]^{1-\alpha} x, x \right\rangle, \end{aligned} \tag{4.13}$$

where $x \in H$.

It is well known that (see for instance [30]), if P is a positive operator and $q \in (0,1)$, then

$$\langle P^q x, x\rangle \leq \langle Px, x\rangle^q. \tag{4.14}$$

Applying this property to the positive operators $(A^*A)^{1/\alpha}$ and $(B^*B)^{1/(1-\alpha)}$, where $\alpha \in (0,1)$, we have

$$\begin{aligned} &\left\langle \left[(A^*A)^{\frac{1}{\alpha}}\right]^{\alpha} x, x \right\rangle \left\langle \left[(B^*B)^{\frac{1}{1-\alpha}}\right]^{1-\alpha} x, x \right\rangle \\ &\qquad \leq \left\langle (A^*A)^{\frac{1}{\alpha}} x, x \right\rangle^{\alpha} \left\langle (B^*B)^{\frac{1}{1-\alpha}} x, x \right\rangle^{1-\alpha} \end{aligned} \tag{4.15}$$

for any $x \in H$ with $\|x\| = 1$.

Now, on utilising the weighted arithmetic mean-geometric mean inequality, i.e.,

$$a^{\alpha} b^{1-\alpha} \leq \alpha a + (1-\alpha) b, \quad \text{where } \alpha \in (0,1) \text{ and } a, b \geq 0,$$

we get

$$\left\langle (A^*A)^{\frac{1}{\alpha}} x, x \right\rangle^{\alpha} \left\langle (B^*B)^{\frac{1}{1-\alpha}} x, x \right\rangle^{1-\alpha} \leq \alpha \cdot \left\langle (A^*A)^{\frac{1}{\alpha}} x, x \right\rangle + (1-\alpha) \cdot \left\langle (B^*B)^{\frac{1}{1-\alpha}} x, x \right\rangle \tag{4.16}$$

for any $x \in H$ with $\|x\| = 1$.

Moreover, by the elementary inequality

$$\alpha a + (1-\alpha) b \leq (\alpha a^r + (1-\alpha) b^r)^{\frac{1}{r}}, \quad \text{where } \alpha \in (0,1) \text{ and } a, b \geq 0;$$

we have successively

$$\begin{aligned} &\alpha \cdot \left\langle (A^*A)^{\frac{1}{\alpha}} x, x \right\rangle + (1-\alpha) \cdot \left\langle (B^*B)^{\frac{1}{1-\alpha}} x, x \right\rangle \\ &\leq \left[\alpha \cdot \left\langle (A^*A)^{\frac{1}{\alpha}} x, x \right\rangle^r + (1-\alpha) \cdot \left\langle (B^*B)^{\frac{1}{1-\alpha}} x, x \right\rangle^r \right]^{\frac{1}{r}} \\ &\leq \left[\alpha \cdot \left\langle (A^*A)^{\frac{r}{\alpha}} x, x \right\rangle + (1-\alpha) \cdot \left\langle (B^*B)^{\frac{r}{1-\alpha}} x, x \right\rangle \right]^{\frac{1}{r}}, \end{aligned} \tag{4.17}$$

for any $x \in H$ with $\|x\| = 1$, where for the last inequality we have used the property (4.7) for the positive operators $(A^*A)^{1/\alpha}$ and $(B^*B)^{1/(1-\alpha)}$.

Now, on making use of the identity (4.13) and the inequalities (4.15)-(4.17) we get

$$\|Ax\|^2 \|Bx\|^2 \leq \left[\left\langle \left[\alpha \cdot (A^*A)^{\frac{r}{\alpha}} + (1-\alpha) \cdot (B^*B)^{\frac{r}{1-\alpha}} \right] x, x \right\rangle \right]^{\frac{1}{r}}$$

for any $x \in H$ with $\|x\| = 1$. Taking the supremum over $x \in H$ with $\|x\| = 1$ we deduce the desired result (4.12).

Notice that the inequality is sharp since for $r = 1$ and $B = A$ we get on both sides of (4.12) the same quantity $\|A\|^4$. □

Corollary 13. *For any $A \in B(H)$, any $\alpha \in (0,1)$ and $r \geq 1$, we have the inequalities*

$$\mu^{2r}(A, A^*) \leq \left\| \alpha \cdot (A^*A)^{\frac{r}{\alpha}} + (1-\alpha) \cdot (AA^*)^{\frac{r}{1-\alpha}} \right\|,$$

$$\|A\|^{2r} \leq \left\| \alpha \cdot (A^*A)^{\frac{r}{\alpha}} + (1-\alpha) \cdot I \right\|$$

and

$$\|A\|^{4r} \leq \left\| \alpha \cdot (A^*A)^{\frac{r}{\alpha}} + (1-\alpha) \cdot (A^*A)^{\frac{r}{1-\alpha}} \right\|,$$

respectively.

The following reverse of the inequality (4.2) may be stated as well:

Theorem 26 (Dragomir, 2008 [20])**.** *For any $A, B \in B(H)$ we have the inequalities*

$$(0 \leq)\, \mu(A, B) - w(B^*A) \leq \frac{1}{2} \|A - B\|^2 \tag{4.18}$$

and

$$\mu\left(\frac{A+B}{2}, \frac{A-B}{2}\right) \leq \frac{1}{2} w(B^*A) + \frac{1}{4} \|A - B\|^2, \tag{4.19}$$

respectively.

Proof. We have

$$\|Ax-Bx\|^2 = \|Ax\|^2 + \|Bx\|^2 - 2\operatorname{Re}\langle B^*Ax, x\rangle \geq 2\|Ax\|\|Bx\| - 2|\langle B^*Ax, x\rangle|, \tag{4.20}$$

for any $x \in H, \|x\| = 1$, which gives the inequality

$$\|Ax\|\|Bx\| \leq |\langle B^*Ax, x\rangle| + \frac{1}{2}\|Ax-Bx\|^2,$$

for any $x \in H, \|x\| = 1$.

Taking the supremum over $\|x\| = 1$ we deduce the desired result (4.18).

By the parallelogram identity in the Hilbert space H we also have

$$\|Ax\|^2 + \|Bx\|^2 = \frac{1}{2}\left(\|Ax+Bx\|^2 + \|Ax-Bx\|^2\right) \geq \|Ax+Bx\|\|Ax-Bx\|,$$

for any $x \in H$.

Combining this inequality with the first part of (4.20) we get

$$\|Ax+Bx\|\|Ax-Bx\| \leq \|Ax-Bx\|^2 + 2|\langle B^*Ax, x\rangle|,$$

for any $x \in H$. Taking the supremum in this inequality over $\|x\| = 1$ we deduce the desired result (4.19). □

Corollary 14. *Let $A \in B(H)$. If $\operatorname{Re}(A) := \frac{A+A^*}{2}$ and $\operatorname{Im}(A) := \frac{A-A^*}{2i}$ are the real and imaginary parts of A, then we have the inequalities*

$$(0 \leq)\, \mu(A, A^*) - w(A^2) \leq 2 \cdot \|\operatorname{Im}(A)\|^2$$

and

$$\mu(\operatorname{Re}(A), \operatorname{Im}(A)) \leq \frac{1}{2}w(A^2) + \|\operatorname{Im}(A)\|^2,$$

respectively.

Moreover, we have

$$(0 \leq)\, \mu(\operatorname{Re}(A), \operatorname{Im}(A)) - w(\operatorname{Re}(A)\operatorname{Im}(A)) \leq \frac{1}{2}\|A\|^2.$$

Corollary 15. *For any $A \in B(H)$ and $\lambda \in \mathbb{C}$ with $\lambda \neq 0$ we have the inequality (see also* [7]*)*

$$(0 \leq)\, \|A\| - w(A) \leq \frac{1}{2|\lambda|}\|A - \lambda I\|^2. \tag{4.21}$$

For a bounded linear operator T consider the quantity $\ell(T) := \inf_{\|x\|=1}\|Tx\|$. We can state the following result as well.

Theorem 27 (Dragomir, 2008 [20])**.** *For any $A, B \in B(H)$ with $A \neq B$ and such that $\ell(B) \geq \|A - B\|$ we have*

$$(0 \leq)\, \mu^2(A, B) - w^2(B^*A) \leq \|A\|^2\|A-B\|^2. \tag{4.22}$$

Proof. Denote $r := \|A - B\| > 0$. Then for any $x \in H$ with $\|x\| = 1$ we have $\|Bx\| \geq r$ and by the first part of (4.20) we can write that

$$\|Ax\|^2 + \left(\sqrt{\|Bx\|^2 - r^2}\right)^2 \leq 2|\langle B^*Ax, x\rangle| \tag{4.23}$$

for any $x \in H$ with $\|x\| = 1$.

On the other hand we have

$$\|Ax\|^2 + \left(\sqrt{\|Bx\|^2 - r^2}\right)^2 \geq 2 \cdot \|Ax\| \sqrt{\|Bx\|^2 - r^2} \tag{4.24}$$

for any $x \in H$ with $\|x\| = 1$.

Combining (4.23) with (4.24) we deduce

$$\|Ax\| \sqrt{\|Bx\|^2 - r^2} \leq |\langle B^* Ax, x\rangle|$$

which is clearly equivalent to

$$\|Ax\|^2 \|Bx\|^2 \leq |\langle B^* Ax, x\rangle|^2 + \|Ax\|^2 \|A - B\|^2 \tag{4.25}$$

for any $x \in H$ with $\|x\| = 1$. Taking the supremum in (4.25) over $x \in H$ with $\|x\| = 1$, we deduce the desired inequality (4.22). □

Corollary 16. *For any $A \in B(H)$ a non-self-adjoint operator in $B(H)$ and such that $\ell(A^*) \geq \|\mathrm{Im}(A)\|$ we have*

$$(0 \leq)\, \mu^2(A, A^*) - w^2\left(A^2\right) \leq 4 \cdot \|A\|^2 \|\mathrm{Im}(A)\|^2 . \tag{4.26}$$

Corollary 17. *For any $A \in B(H)$ and $\lambda \in \mathbb{C}$ with $\lambda \neq 0$ and $|\lambda| \geq \|A - \lambda I\|$ we have the inequality (see also* [7]*)*

$$(0 \leq)\, \|A\|^2 - w^2(A) \leq \frac{1}{|\lambda|^2} \cdot \|A\|^2 \|A - \lambda I\|^2$$

or, equivalently,

$$(0 \leq) \sqrt{1 - \frac{\|A - \lambda I\|^2}{|\lambda|^2}} \leq \frac{w(A)}{\|A\|}\, (\leq 1) .$$

4.3. **Inequalities for Operators Satisfying the Uniform (α, β)-property.** The following result that may be of interest in itself holds:

Lemma 2. *Let $T \in B(H)$ and $\alpha, \beta \in \mathbb{C}$ with $\alpha \neq \beta$. The following statements are equivalent:*

(i) *We have*

$$\mathrm{Re}\,\langle \beta y - Tx, Tx - \alpha y\rangle \geq 0 \tag{4.27}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$;

(ii) *We have*

$$\left\| Tx - \frac{\alpha + \beta}{2} \cdot y \right\| \leq \frac{1}{2} |\alpha - \beta| \tag{4.28}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

Proof. This follows by the following identity

$$\mathrm{Re}\,\langle \beta y - Tx, Tx - \alpha y\rangle = \frac{1}{4} |\alpha - \beta|^2 - \left\| Tx - \frac{\alpha + \beta}{2} \cdot y \right\|^2 ,$$

that holds for any $x, y \in H$ with $\|x\| = \|y\| = 1$. □

Remark 19. *For any operator $T \in B(H)$ if we choose $\alpha = a\|T\|(1+2i)$ and $\beta = a\|T\|(1-2i)$ with $a \geq 1$, then*

$$\frac{\alpha+\beta}{2} = a\|T\| \quad \text{and} \quad \frac{|\alpha-\beta|}{2} = 2a\|T\|$$

showing that

$$\begin{aligned}\left\|Tx - \frac{\alpha+\beta}{2}\cdot y\right\| &\leq \|Tx\| + \left|\frac{\alpha+\beta}{2}\right| \leq \|T\| + a\|T\| \\ &\leq 2a\|T\| = \frac{1}{2}\cdot|\alpha-\beta|,\end{aligned}$$

that holds for any $x, y \in H$ with $\|x\| = \|y\| = 1$, i.e., T satisfies the condition (4.27) with the scalars α and β given above.

Definition 1. *For given $\alpha, \beta \in \mathbb{C}$ with $\alpha \neq \beta$ and $y \in H$ with $\|y\| = 1$, we say that the operator $T \in B(H)$ has the (α, β, y)-property if either (4.27) or, equivalently, (4.28) holds true for any $x \in H$ with $\|x\| = 1$. Moreover, if T has the (α, β, y)-property for any $y \in H$ with $\|y\| = 1$, then we say that this operator has the uniform (α, β)-property.*

Remark 20. *The above Remark 19 shows that any bounded linear operator has the uniform (α, β)-property for infinitely many (α, β) appropriately chosen. For a given operator satisfying an (α, β)-property, it is an open problem to find the lower bound for the nonzero quantity $|\alpha - \beta|$.*

The following results may be stated:

Theorem 28 (Dragomir, 2008 [20])**.** *Let $A, B \in B(H)$ and $\alpha, \beta, \gamma, \delta \in \mathbb{K}$ with $\alpha \neq \beta$ and $\gamma \neq \delta$. For $y \in H$ with $\|y\| = 1$ assume that A^* has the (α, β, y)-property while B^* has the (γ, δ, y)-property. Then*

$$\left|\|Ay\|\|By\| - \|BA^*\|\right| \leq \frac{1}{4}|\beta-\alpha||\gamma-\delta|. \tag{4.29}$$

Moreover, if A^* has the uniform (α, β)-property and B^* has the uniform (γ, δ)-property, then

$$\left|\mu(A, B) - \|BA^*\|\right| \leq \frac{1}{4}|\beta-\alpha||\gamma-\delta|. \tag{4.30}$$

Proof. A^* has the (α, β, y)-property while B^* has the (γ, δ, y)-property, then on making use of Lemma 2 we have that

$$\left\|A^*x - \frac{\alpha+\beta}{2}\cdot y\right\| \leq \frac{1}{2}|\beta-\alpha|$$

and

$$\left\|B^*z - \frac{\gamma+\delta}{2}\cdot y\right\| \leq \frac{1}{2}|\gamma-\delta|$$

for any $x, z \in H$ with $\|x\| = \|z\| = 1$.

Now, we make use of the following Grüss type inequality for vectors in inner product spaces obtained by the author in [1] (see also [2] or [6, p. 43]):

Let $(H, \langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$, $u, v, e \in H$, $\|e\| = 1$, and $\alpha, \beta, \gamma, \delta \in \mathbb{K}$ such that

$$\operatorname{Re}\langle\beta e - u, u - \alpha e\rangle \geq 0, \qquad \operatorname{Re}\langle\delta e - v, v - \gamma e\rangle \geq 0 \tag{4.31}$$

or, equivalently,

$$\left\| u - \frac{\alpha+\beta}{2} e \right\| \le \frac{1}{2} |\beta - \alpha|, \qquad \left\| v - \frac{\gamma+\delta}{2} e \right\| \le \frac{1}{2} |\delta - \gamma|. \tag{4.32}$$

Then

$$|\langle u, v\rangle - \langle u, e\rangle \langle e, v\rangle| \le \frac{1}{4} |\beta - \alpha| |\delta - \gamma|. \tag{4.33}$$

Applying (4.33) for $u = A^*x$, $v = B^*z$ and $e = y$ we deduce

$$|\langle BA^*x, z\rangle - \langle x, Ay\rangle \langle By, z\rangle| \le \frac{1}{4} |\beta - \alpha| |\delta - \gamma|, \tag{4.34}$$

for any $x, z \in H$, $\|x\| = \|z\| = 1$, which is an inequality of interest in itself.

Observing that

$$||\langle BA^*x, z\rangle| - |\langle x, Ay\rangle \langle z, By\rangle|| \le |\langle BA^*x, z\rangle - \langle x, Ay\rangle \langle By, z\rangle|,$$

then by (4.34) we deduce the inequality

$$||\langle BA^*x, z\rangle| - |\langle x, Ay\rangle \langle z, By\rangle|| \le \frac{1}{4} |\beta - \alpha| |\delta - \gamma|$$

for any $x, z \in H$, $\|x\| = \|z\| = 1$. This is equivalent to the following two inequalities

$$|\langle BA^*x, z\rangle| \le |\langle x, Ay\rangle \langle z, By\rangle| + \frac{1}{4} |\beta - \alpha| |\delta - \gamma| \tag{4.35}$$

and

$$|\langle x, Ay\rangle \langle z, By\rangle| \le |\langle BA^*x, z\rangle| + \frac{1}{4} |\beta - \alpha| |\delta - \gamma| \tag{4.36}$$

for any $x, z \in H$, $\|x\| = \|z\| = 1$.

Taking the supremum over $x, z \in H$, $\|x\| = \|z\| = 1$, in (4.35) and (4.36) we get the inequalities

$$\|BA^*\| \le \|Ay\| \|By\| + \frac{1}{4} |\beta - \alpha| |\delta - \gamma| \tag{4.37}$$

and

$$\|Ay\| \|By\| \le \|BA^*\| + \frac{1}{4} |\beta - \alpha| |\delta - \gamma|, \tag{4.38}$$

which are clearly equivalent to (4.29).

Now, if A^* has the uniform (α, β)-property and B^* has the uniform (γ, δ)-property, then the inequalities (4.37) and (4.38) hold for any $y \in H$ with $\|y\| = 1$. Taking the supremum over $y \in H$ with $\|y\| = 1$ in these inequalities we deduce

$$\|BA^*\| \le \mu(A, B) + \frac{1}{4} |\beta - \alpha| |\delta - \gamma|$$

and

$$\mu(A, B) \le \|BA^*\| + \frac{1}{4} |\beta - \alpha| |\delta - \gamma|,$$

which are equivalent to (4.30). □

Corollary 18. *Let $A \in B(H)$ and $\alpha, \beta, \gamma, \delta \in \mathbb{K}$ with $\alpha \ne \beta$ and $\gamma \ne \delta$. For $y \in H$ with $\|y\| = 1$ assume that A has the (α, β, y)-property while A^* has the (γ, δ, y)-property. Then*

$$\left| \|A^*y\| \|Ay\| - \left\|A^2\right\| \right| \le \frac{1}{4} |\beta - \alpha| |\gamma - \delta|.$$

Moreover, if A has the uniform (α,β)-property and A^* has the uniform (γ,δ)-property, then

$$\left|\mu(A,A^*)-\left\|A^2\right\|\right|\le\frac{1}{4}\left|\beta-\alpha\right|\left|\gamma-\delta\right|.$$

The following results may be stated as well:

Theorem 29 (Dragomir, 2008 [20])**.** *Let $A,B\in B(H)$ and $\alpha,\beta,\gamma,\delta\in\mathbb{K}$ with $\alpha+\beta\ne 0$ and $\gamma+\delta\ne 0$. For $y\in H$ with $\|y\|=1$ assume that A^* has the (α,β,y)-property while B^* has the (γ,δ,y)-property. Then*

$$\begin{aligned}(4.39)\quad &\left|\|Ay\|\,\|By\|-\|BA^*\|\right|\\ &\le\frac{1}{4}\cdot\frac{|\beta-\alpha|\,|\delta-\gamma|}{\sqrt{|\beta+\alpha|\,|\delta+\gamma|}}\sqrt{(\|A\|+\|Ay\|)(\|B\|+\|By\|)}.\end{aligned}$$

Moreover, if A^* has the uniform (α,β)-property and B^* has the uniform (γ,δ)-property, then

$$|\mu(A,B)-\|BA^*\||\le\frac{1}{2}\cdot\frac{|\beta-\alpha|\,|\delta-\gamma|}{\sqrt{|\beta+\alpha|\,|\delta+\gamma|}}\sqrt{\|A\|\,\|B\|}. \tag{4.40}$$

Proof. We make use of the following inequality obtained by the author in [5] (see also [6, p. 65]):

Let $(H,\langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$, $u,v,e\in H$, $\|e\|=1$, and $\alpha,\beta,\gamma,\delta\in\mathbb{K}$ with $\alpha+\beta\ne0$ and $\gamma+\delta\ne0$ such that

$$\operatorname{Re}\langle\beta e-u,u-\alpha e\rangle\ge0,\qquad \operatorname{Re}\langle\delta e-v,v-\gamma e\rangle\ge0$$

or, equivalently,

$$\left\|u-\frac{\alpha+\beta}{2}e\right\|\le\frac{1}{2}|\beta-\alpha|,\qquad\left\|v-\frac{\gamma+\delta}{2}e\right\|\le\frac{1}{2}|\delta-\gamma|.$$

Then

$$\begin{aligned}(4.41)\quad &|\langle u,v\rangle-\langle u,e\rangle\langle e,v\rangle|\\ &\le\frac{1}{4}\cdot\frac{|\beta-\alpha|\,|\delta-\gamma|}{\sqrt{|\beta+\alpha|\,|\delta+\gamma|}}\sqrt{(\|u\|+|\langle u,e\rangle|)(\|v\|+|\langle v,e\rangle|)}.\end{aligned}$$

Applying (4.41) for $u=A^*x$, $v=B^*z$ and $e=y$ we deduce

$$\begin{aligned}&|\langle BA^*x,z\rangle-\langle x,Ay\rangle\langle By,z\rangle|\\ &\le\frac{1}{4}\cdot\frac{|\beta-\alpha|\,|\delta-\gamma|}{\sqrt{|\beta+\alpha|\,|\delta+\gamma|}}\sqrt{(\|A^*x\|+|\langle x,Ay\rangle|)(\|B^*z\|+|\langle z,By\rangle|)}\end{aligned}$$

for any $x,y,z\in H$, $\|x\|=\|y\|=\|z\|=1$.

Now, on making use of a similar argument to the one from the proof of Theorem 28, we deduce the desired results (4.39) and (4.40). The details are omitted. □

Corollary 19. *Let $A\in B(H)$ and $\alpha,\beta,\gamma,\delta\in\mathbb{K}$ with $\alpha+\beta\ne0$ and $\gamma+\delta\ne0$. For $y\in H$ with $\|y\|=1$ assume that A has the (α,β,y)-property while A^* has the (γ,δ,y)-property. Then*

$$\left|\|A^*y\|\,\|Ay\|-\left\|A^2\right\|\right|\le\frac{1}{4}\cdot\frac{|\beta-\alpha|\,|\delta-\gamma|}{\sqrt{|\beta+\alpha|\,|\delta+\gamma|}}\sqrt{(\|A\|+\|A^*y\|)(\|A\|+\|Ay\|)}.$$

Moreover, if A has the uniform (α,β)-property and A^* has the uniform (γ,δ)-property, then

$$\left|\mu\left(A,A^*\right)-\left\|A^2\right\|\right|\le\frac{1}{2}\cdot\frac{|\beta-\alpha|\,|\delta-\gamma|}{\sqrt{|\beta+\alpha|\,|\delta+\gamma|}}\left\|A\right\|.$$

4.4. **The Transform $C_{\alpha,\beta}\left(\cdot,\cdot\right)$ and Other Inequalities.** For two given operators $T,U\in B\left(H\right)$ and two given scalars $\alpha,\beta\in\mathbb{C}$ consider the transform

$$C_{\alpha,\beta}\left(T,U\right)=\left(T^*-\bar{\alpha}U^*\right)\left(\beta U-T\right).$$

This transform generalizes the transform $C_{\alpha,\beta}\left(T\right):=\left(T^*-\bar{\alpha}I\right)\left(\beta I-T\right)=C_{\alpha,\beta}\left(T,I\right),$ where I is the identity operator, which has been introduced in [16] in order to provide some generalizations of the well known Kantorovich inequality for operators in Hilbert spaces.

We recall that a bounded linear operator T on the complex Hilbert space $(H,\langle\cdot,\cdot\rangle)$ is called *accretive* if $\operatorname{Re}\langle Ty,y\rangle\ge 0$ for any $y\in H$.

Utilizing the following identity

$$\begin{aligned}\operatorname{Re}\left\langle C_{\alpha,\beta}\left(T,U\right)x,x\right\rangle&=\operatorname{Re}\left\langle C_{\beta,\alpha}\left(T,U\right)x,x\right\rangle\\&=\frac{1}{4}\left|\beta-\alpha\right|^2\left\|Ux\right\|^2-\left\|Tx-\frac{\alpha+\beta}{2}\cdot Ux\right\|^2,\end{aligned}\tag{4.42}$$

that holds for any scalars α,β and any vector $x\in H$, we can give a simple characterization result that is useful in the following:

Lemma 3. *For $\alpha,\beta\in\mathbb{C}$ and $T,U\in B(H)$ the following statements are equivalent:*

(i) *The transform $C_{\alpha,\beta}\left(T,U\right)$ (or, equivalently, $C_{\beta,\alpha}\left(T,U\right)$) is accretive;*
(ii) *We have the norm inequality*

$$\left\|Tx-\frac{\alpha+\beta}{2}\cdot Ux\right\|\le\frac{1}{2}\left|\beta-\alpha\right|\left\|Ux\right\|\tag{4.43}$$

for any $x\in H$.

As a consequence of the above lemma we can state

Corollary 20. *Let $\alpha,\beta\in\mathbb{C}$ and $T,U\in B(H)$. If $C_{\alpha,\beta}\left(T,U\right)$ is accretive, then*

$$\left\|T-\frac{\alpha+\beta}{2}\cdot U\right\|\le\frac{1}{2}\left|\beta-\alpha\right|\left\|U\right\|.\tag{4.44}$$

Remark 21. *In order to give examples of linear operators $T,U\in B(H)$ and numbers $\alpha,\beta\in\mathbb{C}$ such that the transform $C_{\alpha,\beta}\left(T,U\right)$ is accretive, it suffices to select two bounded linear operators S and V and the complex numbers z,w $(w\neq 0)$ with the property that $\|Sx-zVx\|\le|w|\,\|Vx\|$ for any $x\in H$, and, by choosing $T=S$, $U=V$, $\alpha=\frac{1}{2}(z+w)$ and $\beta=\frac{1}{2}(z-w)$ we observe that T and U satisfy (4.43), i.e., $C_{\alpha,\beta}\left(T,U\right)$ is accretive.*

We are able now to give the following result concerning other reverse inequalities for the case when the involved operators satisfy the accretivity property described above.

Theorem 30 (Dragomir, 2008 [20])**.** *Let $\alpha,\beta\in\mathbb{C}$ and $A,B\in B(H)$. If $C_{\alpha,\beta}\left(A,B\right)$ is accretive, then*

$$(0\le)\,\mu^2\left(A,B\right)-w^2\left(B^*A\right)\le\frac{1}{4}\cdot\left|\beta-\alpha\right|^2\left\|B\right\|^4.\tag{4.45}$$

Moreover, if $\alpha+\beta\neq 0$*, then*

$$(0\leq)\,\mu(A,B)-w(B^{*}A)\leq\frac{1}{4}\cdot\frac{|\beta-\alpha|^{2}}{|\beta+\alpha|}\|B\|^{2}. \tag{4.46}$$

In addition, if $\operatorname{Re}\left(\alpha\bar{\beta}\right)>0$ *and* $B^{*}A\neq 0$*, then also*

$$(1\leq)\,\frac{\mu(A,B)}{w(B^{*}A)}\leq\frac{1}{2}\cdot\frac{|\beta+\alpha|}{\sqrt{\operatorname{Re}\left(\alpha\bar{\beta}\right)}} \tag{4.47}$$

and

$$(0\leq)\,\mu^{2}(A,B)-w^{2}(B^{*}A)\leq\left(|\beta+\alpha|-2\cdot\sqrt{\operatorname{Re}\left(\alpha\bar{\beta}\right)}\right)w(B^{*}A)\|B\|^{2}, \tag{4.48}$$

respectively.

Proof. By Lemma 3, since $C_{\alpha,\beta}(A,B)$ is accretive, then

$$\left\|Ax-\frac{\alpha+\beta}{2}\cdot Bx\right\|\leq\frac{1}{2}|\beta-\alpha|\,\|Bx\| \tag{4.49}$$

for any $x\in H$.

We utilize the following reverse of the Schwarz inequality in inner product spaces obtained by the author in [3] (see also [6, p. 4]):

If $\gamma,\Gamma\in\mathbb{K}$ ($\mathbb{K}=\mathbb{C},\mathbb{R}$) and $u,v\in H$ are such that

$$\operatorname{Re}\langle\Gamma v-u,u-\gamma v\rangle\geq 0 \tag{4.50}$$

or, equivalently,

$$\left\|u-\frac{\gamma+\Gamma}{2}\cdot v\right\|\leq\frac{1}{2}|\Gamma-\gamma|\,\|v\|, \tag{4.51}$$

then

$$0\leq\|u\|^{2}\|v\|^{2}-|\langle u,v\rangle|^{2}\leq\frac{1}{4}|\Gamma-\gamma|^{2}\|v\|^{4}. \tag{4.52}$$

Now, on making use of (4.52) for $u=Ax$, $v=Bx$, $x\in H$, $\|x\|=1$ and $\gamma=\alpha$, $\Gamma=\beta$ we can write the inequality

$$\|Ax\|^{2}\|Bx\|^{2}\leq|\langle B^{*}Ax,x\rangle|^{2}+\frac{1}{4}|\beta-\alpha|^{2}\|Bx\|^{4},$$

for any $x\in H$, $\|x\|=1$. Taking the supremum over $\|x\|=1$ in this inequality produces the desired result (4.45).

Now, by utilizing the result from [5] (see also [6, p. 29]) namely:

If $\gamma,\Gamma\in\mathbb{K}$ with $\gamma+\Gamma\neq 0$ and $u,v\in H$ are such that either (4.50) or, equivalently, (4.51) holds true, then

$$0\leq\|u\|\|v\|-|\langle u,v\rangle|\leq\frac{1}{4}\cdot\frac{|\Gamma-\gamma|^{2}}{|\Gamma+\gamma|}\|v\|^{2}. \tag{4.53}$$

Now, on making use of (4.53) for $u=Ax$, $v=Bx$, $x\in H$, $\|x\|=1$ and $\gamma=\alpha,\Gamma=\beta$ and using the same procedure outlined above, we deduce the second inequality (4.46).

The inequality (4.47) follows from the result presented below obtained in [4] (see also [6, p. 21]):

If $\gamma, \Gamma \in \mathbb{K}$ with $\operatorname{Re}(\Gamma\bar{\gamma}) > 0$ and $u, v \in H$ are such that either (4.50) or, equivalently, (4.51) holds true, then

$$\|u\| \|v\| \leq \frac{1}{2} \cdot \frac{|\Gamma + \gamma|}{\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})}} |\langle u, v\rangle|, \tag{4.54}$$

by choosing $u = Ax$, $v = Bx$, $x \in H$, $\|x\| = 1$ and $\gamma = \alpha, \Gamma = \beta$ and taking the supremum over $\|x\| = 1$.

Finally, on making use of the inequality (see [7])

$$\|u\|^2 \|v\|^2 - |\langle u, v\rangle|^2 \leq \left(|\Gamma + \gamma| - 2\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})}\right) |\langle u, v\rangle| \|v\|^2 \tag{4.55}$$

that is valid provided $\gamma, \Gamma \in \mathbb{K}$ with $\operatorname{Re}(\Gamma\bar{\gamma}) > 0$ and $u, v \in H$ are such that either (4.50) or, equivalently, (4.51) holds true, we obtain the last inequality (4.48). The details are omitted. □

Remark 22. *Let $M, m > 0$ and $A, B \in B(H)$. If $C_{m,M}(A, B)$ is accretive, then*

$$(0 \leq)\, \mu^2(A, B) - w^2(B^*A) \leq \frac{1}{4} \cdot (M - m)^2 \|B\|^4,$$

$$(0 \leq)\, \mu(A, B) - w(B^*A) \leq \frac{1}{4} \cdot \frac{(M - m)^2}{m + M} \|B\|^2,$$

$$(1 \leq)\, \frac{\mu(A, B)}{w(B^*A)} \leq \frac{1}{2} \cdot \frac{m + M}{\sqrt{mM}}$$

and

$$(0 \leq)\, \mu^2(A, B) - w^2(B^*A) \leq \left(\sqrt{M} - \sqrt{m}\right)^2 w(B^*A) \|B\|^2,$$

respectively.

Corollary 21. *Let $\alpha, \beta \in \mathbb{C}$ and $A \in B(H)$. If $C_{\alpha,\beta}(A, A^*)$ is accretive, then*

$$(0 \leq)\, \mu^2(A, A^*) - w^2(A^2) \leq \frac{1}{4} \cdot |\beta - \alpha|^2 \|A\|^4.$$

Moreover, if $\alpha + \beta \neq 0$, then

$$(0 \leq)\, \mu(A, A^*) - w(A^2) \leq \frac{1}{4} \cdot \frac{|\beta - \alpha|^2}{|\beta + \alpha|} \|A\|^2.$$

In addition, if $\operatorname{Re}(\alpha\bar{\beta}) > 0$ and $A^2 \neq 0$, then also

$$(1 \leq)\, \frac{\mu(A, A^*)}{w(A^2)} \leq \frac{1}{2} \cdot \frac{|\beta + \alpha|}{\sqrt{\operatorname{Re}(\alpha\bar{\beta})}}$$

and

$$(0 \leq)\, \mu^2(A, A^*) - w^2(A^2) \leq \left(|\beta + \alpha| - 2 \cdot \sqrt{\operatorname{Re}(\alpha\bar{\beta})}\right) w(A^2) \|A\|^2,$$

respectively.

Remark 23. *In a similar manner, if $N, n > 0$, $A \in B(H)$ and $C_{n,N}(A, A^*)$ is accretive, then*

$$(0 \leq)\, \mu^2(A, A^*) - w^2(A^2) \leq \frac{1}{4} \cdot (N - n)^2 \|A\|^4,$$

$$(0 \leq)\, \mu(A, A^*) - w(A^2) \leq \frac{1}{4} \cdot \frac{(N - n)^2}{n + N} \|A\|^2,$$

$$(1 \leq) \frac{\mu(A, A^*)}{w(A^2)} \leq \frac{1}{2} \cdot \frac{n+N}{\sqrt{nN}} \qquad \textit{(for } A^2 \neq 0\textit{)}$$

and

$$(0 \leq) \mu^2(A, A^*) - w^2(A^2) \leq \left(\sqrt{N} - \sqrt{n}\right)^2 w(A^2) \|A\|^2,$$

respectively.

5. Some Inequalities of the Grüss Type for the Numerical Radius

5.1. **Introduction.** Motivated by the natural questions that arise, in order to compare the quantity $w(AB)$ with other expressions comprising the norm or the numerical radius of the involved operators A and B (or certain expressions constructed with these operators), we establish in this section some natural inequalities of the form

$$w(BA) \leq w(A) w(B) + K_1 \quad \text{(additive Grüss' type inequality)}$$

or

$$\frac{w(BA)}{w(A) w(B)} \leq K_2 \quad \text{(multiplicative Grüss' type inequality)}$$

where K_1 and K_2 are specified and desirably simple constants (depending on the given operators A and B).

Applications in providing upper bounds for the non negative quantities

$$\|A\|^2 - w^2(A) \quad \text{and} \quad w^2(A) - w(A^2)$$

and the *super unitary* quantities

$$\frac{\|A\|^2}{w^2(A)} \quad \text{and} \quad \frac{w^2(A)}{w(A^2)}$$

are also given.

5.2. **Numerical Radius Inequalities of Grüss Type.** For the complex numbers α, β and the bounded linear operator T we define the following transform

$$C_{\alpha,\beta}(T) := (T^* - \overline{\alpha} I)(\beta I - T), \tag{5.1}$$

where by T^* we denote the adjoint of T.

We list some properties of the transform $C_{\alpha,\beta}(\cdot)$ that are useful in the following:

(i) For any $\alpha, \beta \in \mathbb{C}$ and $T \in B(H)$ we have:

$$C_{\alpha,\beta}(I) = (1 - \overline{\alpha})(\beta - 1) I, \qquad C_{\alpha,\alpha}(T) = -(\alpha I - T)^* (\alpha I - T), \tag{5.2}$$

$$C_{\alpha,\beta}(\gamma T) = |\gamma|^2 C_{\frac{\alpha}{\gamma}, \frac{\beta}{\gamma}}(T) \quad \text{for each } \gamma \in \mathbb{C} \setminus \{0\}, \tag{5.3}$$

$$[C_{\alpha,\beta}(T)]^* = C_{\beta,\alpha}(T) \tag{5.4}$$

and

$$C_{\overline{\beta},\overline{\alpha}}(T^*) - C_{\alpha,\beta}(T) = T^*T - TT^*. \tag{5.5}$$

(ii) The operator $T \in B(H)$ is normal if and only if $C_{\overline{\beta},\overline{\alpha}}(T^*) = C_{\alpha,\beta}(T)$ for each $\alpha, \beta \in \mathbb{C}$.

We recall that a bounded linear operator T on the complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$ is called *accretive* if $\operatorname{Re} \langle Ty, y \rangle \geq 0$ for any $y \in H$.

Utilizing the following identity

$$\operatorname{Re} \langle C_{\alpha,\beta}(T)x, x\rangle = \operatorname{Re} \langle C_{\beta,\alpha}(T)x, x\rangle = \frac{1}{4}|\beta-\alpha|^2 - \left\|\left(T - \frac{\alpha+\beta}{2}I\right)x\right\|^2 \tag{5.6}$$

that holds for any scalars α, β and any vector $x \in H$ with $\|x\| = 1$ we can give a simple characterization result that is useful in the following:

Lemma 4. *For $\alpha, \beta \in \mathbb{C}$ and $T \in B(H)$ the following statements are equivalent:*

(i) *The transform $C_{\alpha,\beta}(T)$ (or, equivalently $C_{\beta,\alpha}(T)$) is accretive;*

(ii) *The transform $C_{\overline{\alpha},\overline{\beta}}(T^*)$ $\left(\text{or, equivalently } C_{\overline{\beta},\bar{\alpha}}(T^*)\right)$ is accretive;*

(iii) *We have the norm inequality*

$$\left\|T - \frac{\alpha+\beta}{2} \cdot I\right\| \leq \frac{1}{2}|\beta-\alpha| \tag{5.7}$$

or, equivalently,

$$\left\|T^* - \frac{\bar{\alpha}+\bar{\beta}}{2} \cdot I\right\| \leq \frac{1}{2}|\beta-\alpha|. \tag{5.8}$$

Remark 24. *In order to give examples of operators $T \in B(H)$ and numbers $\alpha, \beta \in \mathbb{C}$ such that the transform $C_{\alpha,\beta}(T)$ is accretive, it suffices to select a bounded linear operator S and the complex numbers z, w with the property that $\|S - zI\| \leq |w|$ and, by choosing $T = S$, $\alpha = \frac{1}{2}(z+w)$ and $\beta = \frac{1}{2}(z-w)$. We observe that T satisfies (5.7), i.e., $C_{\alpha,\beta}(T)$ is accretive.*

The following results compare the quantities $w(AB)$ and $w(A)w(B)$ provided that some information about the transforms $C_{\alpha,\beta}(A)$ and $C_{\gamma,\delta}(B)$ are available, where $\alpha, \beta, \gamma, \delta \in \mathbb{K}$.

Theorem 31 (Dragomir, 2008 [18])**.** *Let $A, B \in B(H)$ and $\alpha, \beta, \gamma, \delta \in \mathbb{K}$ be such that the transforms $C_{\alpha,\beta}(A)$ and $C_{\gamma,\delta}(B)$ are accretive, then*

$$w(BA) \leq w(A)w(B) + \frac{1}{4}|\beta-\alpha||\gamma-\delta|. \tag{5.9}$$

Proof. Since $C_{\alpha,\beta}(A)$ and $C_{\gamma,\delta}(B)$ are accretive, then, on making use of Lemma 4 we have that

$$\left\|Ax - \frac{\alpha+\beta}{2}x\right\| \leq \frac{1}{2}|\beta-\alpha|$$

and

$$\left\|B^*x - \frac{\bar{\gamma}+\bar{\delta}}{2}x\right\| \leq \frac{1}{2}\left|\bar{\gamma}-\bar{\delta}\right|$$

for any $x \in H, \|x\| = 1$.

Now, we make use of the following Grüss type inequality for vectors in inner product spaces obtained by the author in [1] (see also [2] or [6, p. 43]):

Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$, $u, v, e \in H$, $\|e\| = 1$, and $\alpha, \beta, \gamma, \delta \in \mathbb{K}$ such that

$$\operatorname{Re}\langle \beta e - u, u - \alpha e\rangle \geq 0, \quad \operatorname{Re}\langle \delta e - v, v - \gamma e\rangle \geq 0 \tag{5.10}$$

or equivalently,

$$\left\|u-\frac{\alpha+\beta}{2}e\right\|\leq\frac{1}{2}\left|\beta-\alpha\right|,\quad\left\|v-\frac{\gamma+\delta}{2}e\right\|\leq\frac{1}{2}\left|\delta-\gamma\right|,\tag{5.11}$$

then

$$\left|\langle u,v\rangle-\langle u,e\rangle\langle e,v\rangle\right|\leq\frac{1}{4}\left|\beta-\alpha\right|\left|\delta-\gamma\right|.\tag{5.12}$$

Applying (5.12) for $u=Ax,v=B^{*}x$ and $e=x$ we deduce

$$\left|\langle BAx,x\rangle-\langle Ax,x\rangle\langle Bx,x\rangle\right|\leq\frac{1}{4}\left|\beta-\alpha\right|\left|\delta-\gamma\right|,\tag{5.13}$$

for any $x\in H,\|x\|=1$, which is an inequality of interest in itself.

Observing that

$$\left|\langle BAx,x\rangle\right|-\left|\langle Ax,x\rangle\langle Bx,x\rangle\right|\leq\left|\langle BAx,x\rangle-\langle Ax,x\rangle\langle Bx,x\rangle\right|,$$

then by (5.12) we deduce the inequality

$$\left|\langle BAx,x\rangle\right|\leq\left|\langle Ax,x\rangle\langle Bx,x\rangle\right|+\frac{1}{4}\left|\beta-\alpha\right|\left|\delta-\gamma\right|,\tag{5.14}$$

for any $x\in H,\|x\|=1$. On taking the supremum over $\|x\|=1$ in (5.14) we deduce the desired result (5.9). □

The following particular case provides a upper bound for the nonnegative quantity $\|A\|^{2}-w(A)^{2}$ when some information about the operator A is available:

Corollary 22. *Let $A\in B(H)$ and $\alpha,\beta\in\mathbb{K}$ be such that the transform $C_{\alpha,\beta}(A)$ is accretive, then*

$$(0\leq)\left\|A\right\|^{2}-w^{2}(A)\leq\frac{1}{4}\left|\beta-\alpha\right|^{2}.\tag{5.15}$$

Proof. Follows on applying Theorem 31 above for the choice $B=A^{*}$, taking into account that $C_{\alpha,\beta}(A)$ is accretive implies that $C_{\overline{\alpha},\overline{\beta}}(A^{*})$ is the same and $w(A^{*}A)=\|A\|^{2}$. □

Remark 25. *Let $A\in B(H)$ and $M>m>0$ are such that the transform $C_{m,M}(A)=(A^{*}-mI)(MI-A)$ is accretive. Then*

$$(0\leq)\left\|A\right\|^{2}-w^{2}(A)\leq\frac{1}{4}(M-m)^{2}.\tag{5.16}$$

A sufficient simple condition for $C_{m,M}(A)$ to be accretive is that A is a selfadjoint operator on H and such that $MI\geq A\geq mI$ in the partial operator order of $B(H)$.

The following result may be stated as well:

Theorem 32 (Dragomir, 2008 [18])**.** *Let $A,B\in B(H)$ and $\alpha,\beta,\gamma,\delta\in\mathbb{K}$ be such that* $\operatorname{Re}(\beta\overline{\alpha})>0,\operatorname{Re}(\delta\overline{\gamma})>0$ *and the transforms $C_{\alpha,\beta}(A),C_{\gamma,\delta}(B)$ are accretive, then*

$$\frac{w(BA)}{w(A)w(B)}\leq1+\frac{1}{4}\cdot\frac{\left|\beta-\alpha\right|\left|\delta-\gamma\right|}{\left[\operatorname{Re}(\beta\overline{\alpha})\operatorname{Re}(\delta\overline{\gamma})\right]^{\frac{1}{2}}}\tag{5.17}$$

and

$$w(BA) \le w(A)\,w(B) + \left[\left(|\alpha+\beta| - 2\left[\operatorname{Re}(\beta\overline{\alpha})\right]^{\frac{1}{2}}\right) \times \left(|\delta+\gamma| - 2\left[\operatorname{Re}(\delta\overline{\gamma})\right]^{\frac{1}{2}}\right)\right]^{\frac{1}{2}} \times \left[w(A)\,w(B)\right]^{\frac{1}{2}} \tag{5.18}$$

respectively.

Proof. With the assumptions (5.10) (or, equivalently, (5.11) in the proof of Theorem 31) and if $\operatorname{Re}(\beta\overline{\alpha}) > 0, \operatorname{Re}(\delta\overline{\gamma}) > 0$ then

$$|\langle u,v\rangle - \langle u,e\rangle\langle e,v\rangle| \le \begin{cases} \frac{1}{4}\frac{|\beta-\alpha||\delta-\gamma|}{[\operatorname{Re}(\beta\overline{\alpha})\operatorname{Re}(\delta\overline{\gamma})]^{\frac{1}{2}}}\,|\langle u,e\rangle\langle e,v\rangle|, \\ \left[\left(|\alpha+\beta| - 2\left[\operatorname{Re}(\beta\overline{\alpha})\right]^{\frac{1}{2}}\right)\left(|\delta+\gamma| - 2\left[\operatorname{Re}(\delta\overline{\gamma})\right]^{\frac{1}{2}}\right)\right]^{\frac{1}{2}} \\ \qquad \times \left[|\langle u,e\rangle\langle e,v\rangle|\right]^{\frac{1}{2}}. \end{cases} \tag{5.19}$$

The first inequality has been established in [4] (see [6, p. 62]) while the second one can be obtained in a canonical manner from the reverse of the Schwarz inequality given in [7]. The details are omitted.

Applying (5.12) for $u = Ax, v = B^*x$ and $e = x$ we deduce

$$|\langle BAx,x\rangle - \langle Ax,x\rangle\langle Bx,x\rangle| \le \begin{cases} \frac{1}{4}\frac{|\beta-\alpha||\delta-\gamma|}{[\operatorname{Re}(\beta\overline{\alpha})\operatorname{Re}(\delta\overline{\gamma})]^{\frac{1}{2}}}\,|\langle A,x\rangle\langle Bx,x\rangle|, \\ \left[\left(|\alpha+\beta| - 2\left[\operatorname{Re}(\beta\overline{\alpha})\right]^{\frac{1}{2}}\right)\left(|\delta+\gamma| - 2\left[\operatorname{Re}(\delta\overline{\gamma})\right]^{\frac{1}{2}}\right)\right]^{\frac{1}{2}} \\ \qquad \times \left[|\langle A,x\rangle\langle Bx,x\rangle|\right]^{\frac{1}{2}} \end{cases} \tag{5.20}$$

for any $x \in H, \|x\| = 1$, which are of interest in themselves.

A similar argument to that in the proof of Theorem 31 yields the desired inequalities (5.17) and (5.18). The details are omitted. □

Corollary 23. *Let $A \in B(H)$ and $\alpha, \beta \in \mathbb{K}$ be such that $\operatorname{Re}(\beta\overline{\alpha}) > 0$ and the transform $C_{\alpha,\beta}(A)$ is accretive, then*

$$(1 \le)\,\frac{\|A\|^2}{w^2(A)} \le 1 + \frac{1}{4}\cdot\frac{|\beta-\alpha|^2}{\operatorname{Re}(\beta\overline{\alpha})} \tag{5.21}$$

and

$$(0 \le)\,\|A\|^2 - w^2(A) \le \left(|\alpha+\beta| - 2\left[\operatorname{Re}(\beta\overline{\alpha})\right]^{\frac{1}{2}}\right) w(A) \tag{5.22}$$

respectively.

The proof is obvious from Theorem 32 on choosing $B = A^*$ and the details are omitted.

Remark 26. *Let $A \in B(H)$ and $M > m > 0$ are such that the transform $C_{m,M}(A) = (A^* - mI)(MI - A)$ is accretive. Then, on making use of Corollary 23, we may state the following simpler results*

$$(1 \le)\,\frac{\|A\|}{w(A)} \le \frac{1}{2}\cdot\frac{M+m}{\sqrt{Mm}} \tag{5.23}$$

and

$$(0 \le) \|A\|^2 - w^2(A) \le \left(\sqrt{M} - \sqrt{m}\right)^2 w(A) \tag{5.24}$$

respectively. These two inequalities were obtained earlier by the author using a different approach, see [8].

Problem 1. *Find general examples of bounded linear operators realizing the equality case in each of the inequalities (5.9), (5.17) and (5.18), respectively.*

5.3. **Some Particular Cases of Interest.** The following result is well known in the literature (see for instance [33]):

$$w(A^n) \le w^n(A),$$

for each positive integer n and any operator $A \in B(H)$.

The following reverse inequalities for $n = 2$, can be stated:

Proposition 2 (Dragomir, 2008 [18])**.** *Let $A \in B(H)$ and $\alpha, \beta \in \mathbb{K}$ be such that the transform $C_{\alpha,\beta}(A)$ is accretive, then*

$$(0 \le) w^2(A) - w(A^2) \le \frac{1}{4} |\beta - \alpha|^2 . \tag{5.25}$$

Proof. On applying the inequality (5.13) from Theorem 31 for the choice $B = A$, we get the following inequality of interest in itself:

$$\left| \langle Ax, x\rangle^2 - \langle A^2 x, x\rangle \right| \le \frac{1}{4} |\beta - \alpha|^2 , \tag{5.26}$$

for any $x \in H, \|x\| = 1$. Since obviously,

$$|\langle Ax, x\rangle|^2 - \left|\langle A^2 x, x\rangle\right| \le \left| \langle Ax, x\rangle^2 - \langle A^2 x, x\rangle \right| ,$$

then by (5.26) we get

$$|\langle Ax, x\rangle|^2 \le \left|\langle A^2 x, x\rangle\right| + \frac{1}{4} |\beta - \alpha|^2 , \tag{5.27}$$

for any $x \in H, \|x\| = 1$. Taking the supremum over $\|x\| = 1$ in (5.27) we deduce the desired result (5.25). □

Remark 27. *Let $A \in B(H)$ and $M > m > 0$ are such that the transform $C_{m,M}(A) = (A^* - mI)(MI - A)$ is accretive. Then*

$$(0 \le) w^2(A) - w(A^2) \le \frac{1}{4} (M - m)^2 . \tag{5.28}$$

If $MI \ge A \ge mI$ in the partial operator order of $B(H)$, then (5.28) is valid.

Finally, we also have

Proposition 3 (Dragomir, 2008 [18])**.** *Let $A \in B(H)$ and $\alpha, \beta \in \mathbb{K}$ be such that $\operatorname{Re}(\beta\overline{\alpha}) > 0$ and the transform $C_{\alpha,\beta}(A)$ is accretive, then*

$$(1 \le) \frac{w^2(A)}{w(A^2)} \le 1 + \frac{1}{4} \cdot \frac{|\beta - \alpha|^2}{\operatorname{Re}(\beta\overline{\alpha})} \tag{5.29}$$

and

$$(0 \le) w^2(A) - w(A^2) \le \left(|\alpha + \beta| - 2 [\operatorname{Re}(\beta\overline{\alpha})]^{\frac{1}{2}} \right) w(A) \tag{5.30}$$

respectively.

Proof. On applying the inequality (5.20) from Theorem 32 for the choice $B = A$, we get the following inequality of interest in itself:

$$\left|\langle Ax, x\rangle^2 - \langle A^2x, x\rangle\right| \leq \begin{cases} \frac{1}{4}\cdot\frac{|\beta-\alpha|^2}{\operatorname{Re}(\beta\overline{\alpha})}\left|\langle A, x\rangle\right|^2, \\ \left(|\alpha+\beta| - 2\left[\operatorname{Re}(\beta\overline{\alpha})\right]^{\frac{1}{2}}\right)\left|\langle A, x\rangle\right|. \end{cases} \tag{5.31}$$

for any $x \in H, \|x\| = 1$.

Now, on making use of a similar argument to the one in the proof of Proposition 2 we deduce the desired results (5.29) and (5.30). The details are omitted. □

Remark 28. *Let $A \in B(H)$ and $M > m > 0$ are such that the transform $C_{m,M}(A) = (A^* - mI)(MI - A)$ is accretive. Then, on making use of Proposition 3 we may state the following simpler results*

$$(1 \leq)\, \frac{w^2(A)}{w(A^2)} \leq \frac{1}{4}\cdot\frac{(M+m)^2}{Mm} \tag{5.32}$$

and

$$(0 \leq)\, w^2(A) - w\left(A^2\right) \leq \left(\sqrt{M} - \sqrt{m}\right)^2 w(A) \tag{5.33}$$

respectively.

6. Some Inequalities for the Euclidean Operator Radius

6.1. **Some Preliminary Facts.** Let $B(H)$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space H with inner product $\langle\cdot,\cdot\rangle$. For $A \in B(H)$, let $w(A)$ and $\|A\|$ denote the numerical radius and the usual operator norm of A, respectively. It is well known that $w(\cdot)$ defines a norm on $B(H)$, and for every $A \in B(H)$,

$$\frac{1}{2}\|A\| \leq w(A) \leq \|A\|. \tag{6.1}$$

For other results concerning the numerical range and radius of bounded linear operators on a Hilbert space, see [26] and [28].

In [32], F. Kittaneh has improved (6.1) in the following manner:

$$\frac{1}{4}\|A^*A + AA^*\| \leq w^2(A) \leq \frac{1}{2}\|A^*A + AA^*\|, \tag{6.2}$$

with the constants $\frac{1}{4}$ and $\frac{1}{2}$ as best possible.

Following Popescu's work [34], we consider the *Euclidean operator radius* of a pair (C, D) of bounded linear operators defined on a Hilbert space $(H; \langle\cdot,\cdot\rangle)$. Note that in [34], the author has introduced the concept for an $n-$tuple of operators and pointed out its main properties.

Let (C, D) be a pair of bounded linear operators on H. The *Euclidean operator radius* is defined by:

$$w_e(C, D) := \sup_{\|x\|=1}\left(\left|\langle Cx, x\rangle\right|^2 + \left|\langle Dx, x\rangle\right|^2\right)^{\frac{1}{2}}. \tag{6.3}$$

As pointed out in [34], $w_e : B^2(H) \to [0,\infty)$ is a norm and the following inequality holds:

$$\frac{\sqrt{2}}{4}\left\|C^*C+D^*D\right\|^{\frac{1}{2}} \le w_e(C,D) \le \left\|C^*C+D^*D\right\|^{\frac{1}{2}}, \tag{6.4}$$

where the constants $\frac{\sqrt{2}}{4}$ and 1 are best possible in (6.4).

We observe that, if C and D are self-adjoint operators, then (6.4) becomes

$$\frac{\sqrt{2}}{4}\left\|C^2+D^2\right\|^{\frac{1}{2}} \le w_e(C,D) \le \left\|C^2+D^2\right\|^{\frac{1}{2}}. \tag{6.5}$$

We observe also that if $A \in B(H)$ and $A = B + iC$ is the Cartesian decomposition of A, then

$$\begin{aligned} w_e^2(B,C) &= \sup_{\|x\|=1}\left[|\langle Bx,x\rangle|^2 + |\langle Cx,x\rangle|^2\right] \\ &= \sup_{\|x\|=1}|\langle Ax,x\rangle|^2 = w^2(A). \end{aligned}$$

By the inequality (6.5) and since (see [32])

$$A^*A + AA^* = 2\left(B^2+C^2\right), \tag{6.6}$$

then we have

$$\frac{1}{16}\left\|A^*A+AA^*\right\| \le w^2(A) \le \frac{1}{2}\left\|A^*A+AA^*\right\|. \tag{6.7}$$

We remark that the lower bound for $w^2(A)$ in (6.7) provided by Popescu's inequality (6.4) is not as good as the first inequality of Kittaneh from (6.2). However, the upper bounds for $w^2(A)$ are the same and have been proved using different arguments.

The main aim of this section is to extend Kittaneh's result to the Euclidean radius of two operators and investigate other particular instances of interest. Related results connecting the Euclidean operator radius, the usual numerical radius of a composite operator and the operator norm are also provided.

6.2. **Some Inequalities for the Euclidean Operator Radius.** The following result concerning a sharp lower bound for the Euclidean operator radius may be stated:

Theorem 33 (Dragomir, 2008 [9]). *Let $B,C : H \to H$ be two bounded linear operators on the Hilbert space $(H;\langle\cdot,\cdot\rangle)$. Then*

$$\frac{\sqrt{2}}{2}\left[w\left(B^2+C^2\right)\right]^{\frac{1}{2}} \le w_e(B,C)\left(\le \left\|B^*B+C^*C\right\|^{\frac{1}{2}}\right). \tag{6.8}$$

The constant $\frac{\sqrt{2}}{2}$ is best possible in the sense that it cannot be replaced by a larger constant.

Proof. We follow a similar argument to the one from [32].

For any $x \in H$, $\|x\| = 1$, we have

$$\begin{aligned} |\langle Bx,x\rangle|^2 + |\langle Cx,x\rangle|^2 &\ge \frac{1}{2}\left(|\langle Bx,x\rangle| + |\langle Cx,x\rangle|\right)^2 \\ &\ge \frac{1}{2}\left|\langle (B \pm C)x,x\rangle\right|^2. \end{aligned} \tag{6.9}$$

Taking the supremum in (6.9), we deduce

$$w_e^2(B,C) \geq \frac{1}{2} w^2(B \pm C). \tag{6.10}$$

Utilising the inequality (6.10) and the properties of the numerical radius, we have successively:

$$\begin{aligned} 2w_e^2(B,C) &\geq \frac{1}{2}\left[w^2(B+C)+w^2(B-C)\right] \\ &\geq \frac{1}{2}\left\{w\left[(B+C)^2\right]+w\left[(B-C)^2\right]\right\} \\ &\geq \frac{1}{2}\left\{w\left[(B+C)^2+(B-C)^2\right]\right\} \\ &= w\left(B^2+C^2\right), \end{aligned}$$

which gives the desired inequality (6.8).

The sharpness of the constant will be shown in a particular case, later on. □

Corollary 24. *For any two self-adjoint bounded linear operators B, C on H, we have*

$$\frac{\sqrt{2}}{2}\left\|B^2+C^2\right\|^{\frac{1}{2}} \leq w_e(B,C)\left(\leq \left\|B^2+C^2\right\|^{\frac{1}{2}}\right). \tag{6.11}$$

The constant $\frac{\sqrt{2}}{2}$ is sharp in (6.11).

Remark 29. *The inequality (6.11) is better than the first inequality in (6.5) which follows from Popescu's first inequality in (6.4). It also provides, for the case that B, C are the self-adjoint operators in the Cartesian decomposition of A, exactly the lower bound obtained by Kittaneh in (6.2) for the numerical radius $w(A)$. Moreover, since $\frac{1}{4}$ is a sharp constant in Kittaneh's inequality (6.2), it follows that $\frac{\sqrt{2}}{2}$ is also the best possible constant in (6.11) and (6.8), respectively.*

The following particular case may be of interest:

Corollary 25. *For any bounded linear operator $A : H \to H$ and $\alpha, \beta \in \mathbb{C}$ we have:*

$$\begin{aligned} \frac{1}{2} w\left[\alpha^2 A^2+\beta^2 (A^*)^2\right] &\leq \left(|\alpha|^2+|\beta|^2\right) w^2(A) \\ &\quad \left(\leq \left\||\alpha|^2 A^*A+|\beta|^2 AA^*\right\|\right). \end{aligned} \tag{6.12}$$

Proof. If we choose in Theorem 33, $B = \alpha A$ and $C = \beta A^*$, we get

$$w_e^2(B,C) = \left(|\alpha|^2+|\beta|^2\right) w^2(A)$$

and

$$w\left(B^2+C^2\right) = w\left[\alpha^2 A^2+\beta^2 (A^*)^2\right],$$

which, by (6.8) implies the desired result (6.12). □

Remark 30. *If we choose in (6.12) $\alpha = \beta \neq 0$, then we get the inequality*

$$\frac{1}{4}\left\|A^2+(A^*)^2\right\| \leq w^2(A)\left(\leq \frac{1}{2}\|A^*A+AA^*\|\right), \tag{6.13}$$

for any bounded linear operator $A \in B(H)$.

If we choose in (6.12), $\alpha = 1$, $\beta = i$, then we get

$$\frac{1}{4} w\left[A^2 - (A^*)^2\right] \le w^2(A), \tag{6.14}$$

for every bounded linear operator $A : H \to H$.

The following result may be stated as well.

Theorem 34 (Dragomir, 2008 [9])**.** *For any two bounded linear operators B, C on H we have:*

$$\begin{aligned} \frac{\sqrt{2}}{2} \max\{w(B+C), w(B-C)\} &\le w_e(B, C) \\ &\le \frac{\sqrt{2}}{2}\left[w^2(B+C) + w^2(B-C)\right]^{\frac{1}{2}}. \end{aligned} \tag{6.15}$$

The constant $\frac{\sqrt{2}}{2}$ is sharp in both inequalities.

Proof. The first inequality follows from (6.10).

For the second inequality, we observe that

$$|\langle Cx, x\rangle \pm \langle Bx, x\rangle|^2 \le w^2(C \pm B) \tag{6.16}$$

for any $x \in H$, $\|x\| = 1$.

The inequality (6.16) and the parallelogram identity for complex numbers give:

$$\begin{aligned} 2\left[|\langle Bx, x\rangle|^2 + |\langle Cx, x\rangle|^2\right] &= |\langle Bx, x\rangle - \langle Cx, x\rangle|^2 + |\langle Bx, x\rangle + \langle Cx, x\rangle|^2 \\ &\le w^2(B+C) + w^2(B-C), \end{aligned} \tag{6.17}$$

for any $x \in H$, $\|x\| = 1$.

Taking the supremum in (6.16) we deduce the desired result (6.15).

The fact that $\frac{\sqrt{2}}{2}$ is the best possible constant follows from the fact that for $B = C \ne 0$ one would obtain the same quantity $\sqrt{2}w(B)$ in all terms of (6.15). □

Corollary 26. *For any two self-adjoint operators B, C on H we have:*

$$\begin{aligned} &\frac{\sqrt{2}}{2} \max\{\|B+C\|, \|B-C\|\} \\ &\qquad \le w_e(B, C) \le \frac{\sqrt{2}}{2}\left[\|B+C\|^2 + \|B-C\|^2\right]^{\frac{1}{2}}. \end{aligned} \tag{6.18}$$

The constant $\frac{\sqrt{2}}{2}$ is best possible in both inequalities.

Corollary 27. *Let A be a bounded linear operator on H. Then*

$$\begin{aligned} &\frac{\sqrt{2}}{2} \max\left\{\left\|\frac{(1-i)A + (1+i)A^*}{2}\right\|, \left\|\frac{(1+i)A + (1-i)A^*}{2}\right\|\right\} \\ &\le w(A) \\ &\le \frac{\sqrt{2}}{2}\left[\left\|\frac{(1-i)A + (1+i)A^*}{2}\right\|^2 + \left\|\frac{(1+i)A + (1-i)A^*}{2}\right\|^2\right]^{\frac{1}{2}}. \end{aligned} \tag{6.19}$$

Proof. Follows from (6.18) applied for the Cartesian decomposition of A. □

The following result may be stated as well:

Corollary 28. *For any A a bounded linear operator on H and $\alpha, \beta \in \mathbb{C}$, we have:*

$$\frac{\sqrt{2}}{2} \max \{w(\alpha A + \beta A^*), w(\alpha A - \beta A^*)\} \leq \left(|\alpha|^2 + |\beta|^2\right)^{\frac{1}{2}} w(A) \leq \frac{\sqrt{2}}{2} \left[w^2(\alpha A + \beta A^*) + w^2(\alpha A - \beta A^*)\right]^{\frac{1}{2}}. \tag{6.20}$$

Remark 31. *The above inequality (6.20) contains some particular cases of interest. For instance, if $\alpha = \beta \neq 0$, then by (6.20) we get*

$$\frac{1}{2} \max \{\|A + A^*\|, \|A - A^*\|\} \leq w(A) \leq \frac{1}{2} \left[\|A + A^*\|^2 + \|A - A^*\|^2\right]^{\frac{1}{2}}, \tag{6.21}$$

since, obviously $w(A + A^*) = \|A + A^*\|$ and $w(A - A^*) = \|A - A^*\|$, $A - A^*$ being a normal operator.

Now, if we choose in (6.20), $\alpha = 1$ and $\beta = i$, and taking into account that $A + iA^*$ and $A - iA^*$ are normal operators, then we get

$$\frac{1}{2} \max \{\|A + iA^*\|, \|A - iA^*\|\} \leq w(A) \leq \frac{1}{2} \left[\|A + iA^*\|^2 + \|A - iA^*\|^2\right]^{\frac{1}{2}}. \tag{6.22}$$

The constant $\frac{1}{2}$ is best possible in both inequalities (6.21) and (6.22).

The following simple result may be stated as well.

Proposition 4 (Dragomir, 2008 [9])**.** *For any two bounded linear operators B and C on H, we have the inequality:*

$$w_e(B, C) \leq \left[w^2(C - B) + 2w(B)w(C)\right]^{\frac{1}{2}}. \tag{6.23}$$

Proof. For any $x \in H$, $\|x\| = 1$, we have

$$|\langle Cx, x\rangle|^2 - 2\operatorname{Re}\left[\langle Cx, x\rangle \overline{\langle Bx, x\rangle}\right] + |\langle Bx, x\rangle|^2 = |\langle Cx, x\rangle - \langle Bx, x\rangle|^2 \leq w^2(C - B),$$

giving

$$|\langle Cx, x\rangle|^2 + |\langle Bx, x\rangle|^2 \leq w^2(C - B) + 2\operatorname{Re}\left[\langle Cx, x\rangle \overline{\langle Bx, x\rangle}\right] \leq w^2(C - B) + 2|\langle Cx, x\rangle||\langle Bx, x\rangle| \tag{6.24}$$

for any $x \in H$, $\|x\| = 1$.

Taking the supremum in (6.24) over $\|x\| = 1$, we deduce the desired inequality (6.23). □

In particular, if B and C are self-adjoint operators, then

$$w_e(B, C) \leq \left(\|B - C\|^2 + 2\|B\|\|C\|\right)^{\frac{1}{2}}. \tag{6.25}$$

Now, if we apply the inequality (6.25) for $B = \frac{A+A^*}{2}$ and $C = \frac{A-A^*}{2i}$, where $A \in B(H)$, then we deduce:

$$w(A) \leq \left[\left\| \frac{(1+i)A + (1-i)A^*}{2} \right\|^2 + 2 \cdot \left\| \frac{A+A^*}{2} \right\| \left\| \frac{A-A^*}{2} \right\| \right]^{\frac{1}{2}}.$$

The following result provides a different upper bound for the Euclidean operator radius than (6.23).

Proposition 5 (Dragomir, 2008 [9]). *For any two bounded linear operators B and C on H, we have*

$$w_e(B,C) \leq \left[2 \min\left\{w^2(B), w^2(C)\right\} + w(B-C)\,w(B+C)\right]^{\frac{1}{2}}. \tag{6.26}$$

Proof. Utilising the parallelogram identity (6.17), we have, by taking the supremum over $x \in H, \|x\| = 1$, that

$$2w_e^2(B,C) = w_e^2(B-C, B+C). \tag{6.27}$$

Now, if we apply Proposition 4 for $B-C, B+C$ instead of B and C, then we can state

$$w_e^2(B-C, B+C) \leq 4w^2(C) + 2w(B-C)\,w(B+C)$$

giving

$$w_e^2(B,C) \leq 2w^2(C) + w(B-C)\,w(B+C). \tag{6.28}$$

Now, if in (6.28) we swap the C with B then we also have

$$w_e^2(B,C) \leq 2w^2(B) + w(B-C)\,w(B+C). \tag{6.29}$$

The conclusion follows now by (6.28) and (6.29). □

6.3. **Other Results.** A different upper bound for the Euclidean operator radius is incorporated in the following

Theorem 35 (Dragomir, 2008 [9]). *Let $(H; \langle\cdot,\cdot\rangle)$ be a Hilbert space and B, C two bounded linear operators on H. Then*

$$w_e^2(B,C) \leq \max\left\{\|B\|^2, \|C\|^2\right\} + w(C^*B). \tag{6.30}$$

The inequality (6.30) is sharp.

Proof. Firstly, let us observe that for any $y, u, v \in H$ we have successively

$$\begin{aligned} &\|\langle y,u\rangle u + \langle y,v\rangle v\|^2 \\ &= |\langle y,u\rangle|^2 \|u\|^2 + |\langle y,v\rangle|^2 \|v\|^2 + 2\operatorname{Re}\left[\langle y,u\rangle \overline{\langle y,v\rangle} \langle u,v\rangle\right] \\ &\leq |\langle y,u\rangle|^2 \|u\|^2 + |\langle y,v\rangle|^2 \|v\|^2 + 2|\langle y,u\rangle| |\langle y,v\rangle| |\langle u,v\rangle| \\ &\leq |\langle y,u\rangle|^2 \|u\|^2 + |\langle y,v\rangle|^2 \|v\|^2 + \left(|\langle y,u\rangle|^2 + |\langle y,v\rangle|^2\right) |\langle u,v\rangle| \\ &\leq \left(|\langle y,u\rangle|^2 + |\langle y,v\rangle|^2\right) \left(\max\left\{\|u\|^2, \|v\|^2\right\} + |\langle u,v\rangle|\right). \end{aligned} \tag{6.31}$$

On the other hand,

$$\left(|\langle y,u\rangle|^2+|\langle y,v\rangle|^2\right)^2 = [\langle y,u\rangle\langle u,y\rangle+\langle y,v\rangle\langle v,y\rangle]^2 = [\langle y,\langle y,u\rangle u+\langle y,v\rangle v\rangle]^2 \le \|y\|^2\|\langle y,u\rangle u+\langle y,v\rangle v\|^2 \tag{6.32}$$

for any $y,u,v\in H$.

Making use of (6.31) and (6.32) we deduce that

$$|\langle y,u\rangle|^2+|\langle y,v\rangle|^2 \le \|y\|^2\left[\max\left\{\|u\|^2,\|v\|^2\right\}+|\langle u,v\rangle|\right] \tag{6.33}$$

for any $y,u,v\in H$, which is a vector inequality of interest in itself.

Now, if we apply the inequality (6.33) for $y=x$, $u=Bx$, $v=Cx$, $x\in H$, $\|x\|=1$, then we can state that

$$|\langle Bx,x\rangle|^2+|\langle Cx,x\rangle|^2 \le \max\left\{\|Bx\|^2,\|Cx\|^2\right\}+|\langle Bx,Cx\rangle| \tag{6.34}$$

for any $x\in H$, $\|x\|=1$, which is of interest in itself.

Taking the supremum over $x\in H$, $\|x\|=1$, we deduce the desired result (6.30).

To prove the sharpness of the inequality (6.30) we choose $C=B$, B a self-adjoint operator on H. In this case, both sides of (6.30) become $2\|B\|^2$. □

If information about the sum and the difference of the operators B and C is available, then one may use the following result:

Corollary 29. *For any two operators $B,C\in B(H)$ we have*

$$w_e^2(B,C)\le\frac{1}{2}\left\{\max\left\{\|B-C\|^2,\|B+C\|^2\right\}+w\left[(B^*-C^*)(B+C)\right]\right\}. \tag{6.35}$$

The constant $\frac{1}{2}$ is best possible in (6.35).

Proof. Follows by the inequality (6.30) written for $B+C$ and $B-C$ instead of B and C and by utilising the identity (6.27).

The fact that $\frac{1}{2}$ is best possible in (6.35) follows by the fact that for $C=B$, B a self-adjoint operator, we get in both sides of the inequality (6.35) the quantity $2\|B\|^2$. □

Corollary 30. *Let $A:H\to H$ be a bounded linear operator on the Hilbert space H. Then:*

$$w^2(A)\le\frac{1}{4}\left[\max\left\{\|A+A^*\|^2,\|A-A^*\|^2\right\}+w\left[(A^*-A)(A+A^*)\right]\right]. \tag{6.36}$$

The constant $\frac{1}{4}$ is best possible.

Proof. If $B=\frac{A+A^*}{2}$, $C=\frac{A-A^*}{2i}$ is the Cartesian decomposition of A, then

$$w_e^2(B,C)=w^2(A)$$

and

$$w(C^*B)=\frac{1}{4}w\left[(A^*-A)(A+A^*)\right].$$

Utilising (6.30) we deduce (6.36). □

Remark 32. *If we choose in (6.30), $B = A$ and $C = A^*, A \in B(H)$ then we can state that*

$$w^2(A) \le \frac{1}{2}\left[\|A\|^2 + w\left(A^2\right)\right]. \tag{6.37}$$

The constant $\frac{1}{2}$ is best possible in (6.37).

Note that this inequality has been obtained in [12] *by the use of a different argument based on the Buzano's inequality.*

Finally, the following upper bound for the Euclidean radius involving different composite operators also holds:

Theorem 36 (Dragomir, 2008 [9])**.** *With the assumptions of Theorem 35, we have*

$$w_e^2(B,C) \le \frac{1}{2}\left[\|B^*B + C^*C\| + \|B^*B - C^*C\|\right] + w\left(C^*B\right). \tag{6.38}$$

The inequality (6.38) is sharp.

Proof. We use (6.34) to write that

$$\begin{aligned} &|\langle Bx, x\rangle|^2 + |\langle Cx, x\rangle|^2 \\ &\le \frac{1}{2}\left[\|Bx\|^2 + \|Cx\|^2 + \left|\|Bx\|^2 - \|Cx\|^2\right|\right] + |\langle Bx, Cx\rangle| \end{aligned} \tag{6.39}$$

for any $x \in H$, $\|x\| = 1$.

Since $\|Bx\|^2 = \langle B^*Bx, x\rangle$, $\|Cx\|^2 = \langle C^*Cx, x\rangle$, then (6.39) can be written as

$$\begin{aligned} &|\langle Bx, x\rangle|^2 + |\langle Cx, x\rangle|^2 \\ &\le \frac{1}{2}\left[\langle (B^*B + C^*C)x, x\rangle + |\langle (B^*B - C^*C)x, x\rangle|\right] + |\langle Bx, Cx\rangle| \end{aligned} \tag{6.40}$$

$x \in H$, $\|x\| = 1$.

Taking the supremum in (6.40) over $x \in H$, $\|x\| = 1$ and noting that the operators $B^*B \pm C^*C$ are self-adjoint, we deduce the desired result (6.38).

The sharpness of the constant will follow from that of (6.43) pointed out below. □

Corollary 31. *For any two operators $B, C \in B(H)$, we have*

$$\begin{aligned} &w_e^2(B,C) \\ &\le \frac{1}{2}\left\{\|B^*B + C^*C\| + \|B^*C + C^*B\| + w\left[(B^* - C^*)(B + C)\right]\right\}. \end{aligned} \tag{6.41}$$

The constant $\frac{1}{2}$ is best possible.

Proof. If we write (6.38) for $B+C, B-C$ instead of B, C and perform the required calculations then we get

$$\begin{aligned} &w_e^2(B+C, B-C) \\ &\le \frac{1}{2}\left[2\|B^*B + C^*C\| + 2\|B^*C + C^*B\|\right] + w\left[(B^* - C^*)(B + C)\right], \end{aligned}$$

which, by the identity (6.27) is clearly equivalent with (6.41).

Now, if we choose in (6.41) $B = C$, then we get the inequality $w(B) \le \|B\|$, which is a sharp inequality. □

Corollary 32. *If B, C are self-adjoint operators on H then*

$$w_e^2(B,C) \le \frac{1}{2}\left[\left\|B^2+C^2\right\|+\left\|B^2-C^2\right\|\right]+w(CB). \tag{6.42}$$

We observe that, if B and C are chosen to be the Cartesian decomposition for the bounded linear operator A, then we can get from (6.42) that

$$w^2(A) \le \frac{1}{4}\left\{\|A^*A+AA^*\|+\left\|A^2+(A^*)^2\right\|+w\left[(A^*-A)(A+A^*)\right]\right\}. \tag{6.43}$$

The constant $\frac{1}{4}$ is best possible. This follows by the fact that for A a self-adjoint operator, we obtain on both sides of (6.43) the same quantity $\|A\|^2$.

Now, if we choose in (6.38) $B = A$ and $C = A^*$, $A \in B(H)$, then we get

$$w^2(A) \le \frac{1}{4}\{\|A^*A+AA^*\|+\|A^*A-AA^*\|\}+\frac{1}{2}w\left(A^2\right). \tag{6.44}$$

This inequality is sharp. The equality holds if, for instance, we assume that A is normal, i.e., $A^*A = AA^*$. In this case we get on both sides of (6.44) the quantity $\|A\|^2$, since for normal operators, $w\left(A^2\right) = w^2(A) = \|A\|^2$.

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