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SUPERADDITIVITY AND SUBADDITIVITY OF SOME FUNCTIONALS WITH APPLICATIONS TO INEQUALITIES

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Abstract. The superadditivity and subadditivity properties of certain composite functionals are investigated. Applications in refining Jensen's, Hölder's, Minkowski's and Schwarz's inequalities are given.

1. Introduction

Let X be a linear space. A subset $C \subseteq X$ is called a *convex cone* in X provided the following conditions hold:

(i) $x, y \in C$ imply $x + y \in C$;
(ii) $x \in C$, $\alpha \geq 0$ imply $\alpha x \in C$.

A functional $h : C \to \mathbb{R}$ is called *superadditive (subadditive)* on C if

(iii) $h(x + y) \geq (\leq)\, h(x) + h(y)$ for any $x, y \in C$

and *nonnegative (strictly positive)* on C if, obviously, it satisfies

(iv) $h(x) \geq (>)\, 0$ for each $x \in C$.

The functional h is $s-$*positive homogeneous* on C, for a given $s > 0$, if

(v) $h(\alpha x) = \alpha^s h(x)$ for any $\alpha \geq 0$ and $x \in C$.

In [4] we obtained the following results concerning the quasilinearity of some composite functionals:

Theorem 1. *Let C be a convex cone in the linear space X and $v : C \to (0, \infty)$ an additive functional on C. If $h : C \to [0, \infty)$ is a superadditive (subadditive) functional on C and $p, q \geq 1$ $(0 < p, q < 1)$ then the functional*

$$\Psi_{p,q} : C \to [0, \infty),\, \Psi_{p,q}(x) = h^q(x)\, v^{q\left(1-\frac{1}{p}\right)}(x) \tag{1.1}$$

is superadditive (subadditive) on C.

Theorem 2. *Let C be a convex cone in the linear space X and $v : C \to (0, \infty)$ an additive functional on C. If $h : C \to [0, \infty)$ is a superadditive functional on C and $0 < p, q < 1$ then the functional*

$$\Phi_{p,q} : C \to [0, \infty),\, \Phi_{p,q}(x) = \frac{v^{q\left(1-\frac{1}{p}\right)}(x)}{h^q(x)} \tag{1.2}$$

is subadditive on C.

In [3], the following result was obtained as well:

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Theorem 3. *Let $x, y \in C$ and $h : C \to \mathbb{R}$ a nonnegative, superadditive and $s-$positive homogeneous functional on C. If $M \geq m \geq 0$ are such that $x - my$ and $My - x \in C$, then*

$$M^s h(y) \geq h(x) \geq m^s h(y). \tag{1.3}$$

Now, consider $v : C \to \mathbb{R}$ an *additive* and *strictly positive* functional on C which is also *positive homogeneous* on C, i.e.,

(vi) $v(\alpha x) = \alpha v(x)$ for any $\alpha > 0$ and $x \in C$.

Another result of this type was also obtained in [4], namely

Theorem 4. *Let $x, y \in C$, $h : C \to \mathbb{R}$ be a nonnegative, superadditive and $s-$positive homogeneous functional on C and v an additive, strictly positive and positive homogeneous functional on C. If $p, q \geq 1$ and $M \geq m \geq 0$ are such that $x - my$, $My - x \in C$, then*

$$M^{sq+q\left(1-\frac{1}{p}\right)} \Psi_{p,q}(y) \geq \Psi_{p,q}(x) \geq m^{sq+q\left(1-\frac{1}{p}\right)} \Psi_{p,q}(y) \tag{1.4}$$

where $\Psi_{p,q}$ is defined by (1.1).

As shown in [3] and [4], the above results can be applied to obtain refinements of the Jensen, Hölder, Minkowski and Schwarz inequalities for weights satisfying certain conditions.

The main aim of the present paper is to study quasilinearity properties of other composite functionals and to apply the obtained results in improving some classical inequalities as those mentioned above.

2. General Results

The following result holds.

Theorem 5. *Let C be a convex cone in the linear space X. If $h : C \to [0, \infty)$ is superadditive (subadditive) and $v : C \to (0, \infty)$ is additive, then for any $q \in (0, 1)$ $(q > 1)$ the composite mapping $\Lambda_q : C \to [0, \infty)$, $\Lambda_q(x) = v^{1-q}(x) h^q(x)$ is superadditive (subadditive) as well on C.*

Proof. By the properties of h and v we have

$$\frac{h(x+y)}{v(x+y)} \geq (\leq) \frac{h(x) + h(y)}{v(x+y)} = \frac{h(x) + h(y)}{v(x) + v(y)} = \frac{v(x) \frac{h(x)}{v(x)} + v(y) \frac{h(y)}{v(y)}}{v(x) + v(y)} \tag{2.1}$$

for any $x, y \in C$.

Taking the power q in (2.1) we get

$$\frac{h^q(x+y)}{v^q(x+y)} \geq (\leq) \left[\frac{v(x) \frac{h(x)}{v(x)} + v(y) \frac{h(y)}{v(y)}}{v(x) + v(y)} \right]^q \tag{2.2}$$

for any $x, y \in C$.

By the concavity (convexity) of the function $g(t) = t^q$, $q \in (0,1)$ $(q \geq 1)$ we also have

$$\left[\frac{v(x)\frac{h(x)}{v(x)} + v(y)\frac{h(y)}{v(y)}}{v(x) + v(y)}\right]^q \geq (\leq) \frac{v(x)\left[\frac{h(x)}{v(x)}\right]^q + v(y)\left[\frac{h(y)}{v(y)}\right]^q}{v(x) + v(y)} = \frac{v^{1-q}(x)\,h^q(x) + v^{1-q}(y)\,h^q(x)}{v(x) + v(y)} \tag{2.3}$$

for any $x, y \in C$.

Combining (2.2) with (2.3) we deduce

$$\frac{h^q(x+y)}{v^q(x+y)} \geq (\leq) \frac{v^{1-q}(x)\,h^q(x) + v^{1-q}(y)\,h^q(x)}{v(x) + v(y)}$$

which is clearly equivalent with the desired result, since, obviously $v(x) + v(y) = v(x+y)$ for any $x, y \in C$. □

The following corollary is of interest in order to provide upper and lower bounds for some functionals associated with certain inequalities.

Corollary 1. *Let $x, y \in C$, $h : C \to \mathbb{R}$ be a nonnegative, superadditive and $s-$positive homogeneous functional on C and v an additive, strictly positive and positive homogeneous functional on C. If $q \in (0,1)$ and $M \geq m \geq 0$ are such that $x - my$, $My - x \in C$, then*

$$M^{(s-1)q+1}\Lambda_q(y) \geq \Lambda_q(x) \geq m^{(s-1)q+1}\Lambda_q(y). \tag{2.4}$$

Proof. Follows by Theorem 3 applied for the functional Λ_q which is $(s-1)q+1$-positive homogeneous on C. □

The following result may be stated as well:

Theorem 6. *Let C be a convex cone in the linear space X. If $h : C \to [0,\infty)$ is superadditive and $v : C \to (0,\infty)$ is subadditive, then the composite mapping $\delta : C \to [0,\infty)$, $\delta(x) = \frac{v^2(x)}{h(x)}$ is subadditive as well on C.*

Proof. Since h is superadditive and v is subadditive on C, we also have (compare with (2.1)) that

$$\frac{h(x+y)}{v(x+y)} \geq \frac{h(x) + h(y)}{v(x+y)} \geq \frac{h(x) + h(y)}{v(x) + v(y)} = \frac{v(x)\frac{h(x)}{v(x)} + v(y)\frac{h(y)}{v(y)}}{v(x) + v(y)} \tag{2.5}$$

for any $x, y \in C$.

Utilising the elementary inequality between the weighted arithmetic mean and the weighted harmonic mean, i.e.,

$$\frac{\alpha a + \beta b}{\alpha + \beta} \geq \frac{\alpha + \beta}{\frac{\alpha}{a} + \frac{\beta}{b}}, \alpha, \beta, a, b > 0$$

for the choices $a = \frac{h(x)}{v(x)}, b = \frac{h(y)}{v(y)}, \alpha = v(x)$ and $\beta = v(y)$, we have

$$\frac{v(x)\frac{h(x)}{v(x)} + v(y)\frac{h(y)}{v(y)}}{v(x) + v(y)} \geq \frac{v(x) + v(y)}{\frac{v^2(x)}{h(x)} + \frac{v^2(y)}{h(y)}} \geq \frac{v(x+y)}{\frac{v^2(x)}{h(x)} + \frac{v^2(y)}{h(y)}} \tag{2.6}$$

for any $x, y \in C$, where for the last inequality we have used the subadditivity of the functional v.

Combining (2.5) with (2.6) we get

$$\frac{h(x+y)}{v^2(x+y)} \geq \frac{1}{\frac{v^2(x)}{h(x)} + \frac{v^2(y)}{h(y)}}$$

that is equivalent with

$$\frac{v^2(x)}{h(x)} + \frac{v^2(y)}{h(y)} \geq \frac{v^2(x+y)}{h(x+y)}$$

for any $x, y \in C$, which proves that δ is subadditive on C. □

3. Applications for Jensen's Inequality

Let C be a convex subset of the real linear space X and let $f : C \to \mathbb{R}$ be a convex mapping. Here we consider the following well-known form of *Jensen's discrete inequality*:

$$f\left(\frac{1}{P_I}\sum_{i\in I} p_i x_i\right) \leq \frac{1}{P_I}\sum_{i\in I} p_i f(x_i), \tag{3.1}$$

where I denotes a finite subset of the set $\mathbb{N}$ of natural numbers, $x_i \in C$, $p_i \geq 0$ for $i \in I$ and $P_I := \sum_{i\in I} p_i > 0$.

Let us fix $I \in \mathcal{P}_f(\mathbb{N})$ (the class of finite parts of $\mathbb{N}$) and $x_i \in C$ $(i \in I)$. Now consider the functional $J : S_+(I) \to \mathbb{R}$ given by

$$J_I(\mathbf{p}) := \sum_{i\in I} p_i f(x_i) - P_I f\left(\frac{1}{P_I}\sum_{i\in I} p_i x_i\right) \geq 0 \tag{3.2}$$

where $S_+(I) := \left\{\mathbf{p} = (p_i)_{i\in I} \middle| p_i \geq 0,\ i \in I \text{ and } P_I > 0\right\}$ and f is convex on C.

We observe that $S_+(I)$ is a convex cone and the functional J_I is nonnegative and positive homogeneous on $S_+(I)$.

Lemma 1 ([6])**.** *The functional $J_I(\cdot)$ is a superadditive functional on $S_+(I)$.*

Define the following functional

$$J_{q,I}(\mathbf{p}) := P_I^{1-q} J_I^q(\mathbf{p}) = \left[P_I^{\frac{1}{q}-1}\sum_{i\in I} p_i f(x_i) - P_I^{\frac{1}{q}} f\left(\frac{1}{P_I}\sum_{i\in I} p_i x_i\right)\right]^q \tag{3.3}$$

for $q \in (0,1)$.

The following proposition can be stated:

Proposition 1. *The functional $J_{q,I}(\cdot)$ is superadditive on $S_+(I)$ for any $q \in (0,1)$.*

Proof. Define $v(\mathbf{p}) = P_I$ and $h(\mathbf{p}) = J_I(\mathbf{p})$. Then for $q \in (0,1)$, we have

$$\Lambda_q(x) = v^{1-q}(x)\, h^q(x) = P_I^{1-q} J_I^q(\mathbf{p}) = J_{q,I}(\mathbf{p})$$

for any $\mathbf{p} \in S_+(I)$.

Since $v(\cdot)$ is additive and $J_I(\cdot)$ is superadditive on $S_+(I)$ on applying Theorem 5 we conclude that $J_{q,I}(\cdot)$ is also superadditive on $S_+(I)$. □

Remark 1. *We observe that, in particular, the following functional*

$$\breve{J}_I(\mathbf{p}) := \left[P_I \sum_{i\in I} p_i f(x_i) - P_I^{\frac{1}{2}} f\left(\frac{1}{P_I}\sum_{i\in I} p_i x_i\right)\right]^{\frac{1}{2}}$$

is superadditive on $S_+(I)$.

We can state the following result that provides a refinement and a reverse for the Jensen's inequality when bounds for the weights are known:

Proposition 2. *If* $\mathbf{p},\mathbf{q}\in S_+(I)$ *and* $M\geq m\geq 0$ *are such that* $M\mathbf{p}\geq\mathbf{q}\geq m\mathbf{p}$, *i.e.,* $Mp_i\geq q_i\geq mp_i$ *for each* $i\in I$, *then:*

$$M^{\frac{1}{q}}\left(\frac{P_I}{Q_I}\right)^{\frac{1}{q}}\left[\frac{1}{P_I}\sum_{i\in I}p_if(x_i)-f\left(\frac{1}{P_I}\sum_{i\in I}p_ix_i\right)\right] \tag{3.4}$$
$$\geq\frac{1}{Q_I}\sum_{i\in I}q_if(x_i)-f\left(\frac{1}{Q_I}\sum_{i\in I}q_ix_i\right)$$
$$\geq m^{\frac{1}{q}}\left(\frac{P_I}{Q_I}\right)^{\frac{1}{q}}\left[\frac{1}{P_I}\sum_{i\in I}p_if(x_i)-f\left(\frac{1}{P_I}\sum_{i\in I}p_ix_i\right)\right],$$

for any $q\in(0,1)$.

Proof. Applying Corollary 1 for the functional $J_{q,I}(\mathbf{p})$ and $s=1$ we have

$$M\left[P_I^{\frac{1}{q}-1}\sum_{i\in I}p_if(x_i)-P_I^{\frac{1}{q}}f\left(\frac{1}{P_I}\sum_{i\in I}p_ix_i\right)\right]^q \tag{3.5}$$
$$\geq\left[Q_I^{\frac{1}{q}-1}\sum_{i\in I}q_if(x_i)-Q_I^{\frac{1}{q}}f\left(\frac{1}{Q_I}\sum_{i\in I}q_ix_i\right)\right]^q$$
$$\geq m\left[P_I^{\frac{1}{q}-1}\sum_{i\in I}p_if(x_i)-P_I^{\frac{1}{q}}f\left(\frac{1}{P_I}\sum_{i\in I}p_ix_i\right)\right]^q.$$

Taking the power $\frac{1}{q}>0$ in the inequality (3.5) we deduce the desired result (3.4). □

The above Proposition 2 can be utilised to obtain various inequalities generated by the appropriate choices of the convex function f.

(1) If $f:X\to\mathbb{R}$, $f(x)=\|x\|^r$, $r\geq 1, q\in(0,1)$ where $(X,\|\cdot\|)$ is a normed linear space, then we can state the inequality:

$$M^{\frac{1}{q}}\left(\frac{P_I}{Q_I}\right)^{\frac{1}{q}}\left[\frac{1}{P_I}\sum_{i\in I}p_i\|x_i\|^r-\left\|\frac{1}{P_I}\sum_{i\in I}p_ix_i\right\|^r\right] \tag{3.6}$$
$$\geq\frac{1}{Q_I}\sum_{i\in I}q_i\|x_i\|^r-\left\|\frac{1}{Q_I}\sum_{i\in I}q_ix_i\right\|^r$$
$$\geq m^{\frac{1}{q}}\left(\frac{P_I}{Q_I}\right)^{\frac{1}{q}}\left[\frac{1}{P_I}\sum_{i\in I}p_i\|x_i\|^r-\left\|\frac{1}{P_I}\sum_{i\in I}p_ix_i\right\|^r\right]$$

and, in particular,

$$M^{\frac{1}{q}}\left(\frac{P_I}{Q_I}\right)^{\frac{1}{q}}\left[\frac{1}{P_I}\sum_{i\in I}p_i\left\|x_i\right\|-\left\|\frac{1}{P_I}\sum_{i\in I}p_ix_i\right\|\right] \tag{3.7}$$
$$\geq\frac{1}{Q_I}\sum_{i\in I}q_i\left\|x_i\right\|-\left\|\frac{1}{Q_I}\sum_{i\in I}q_ix_i\right\|$$
$$\geq m^{\frac{1}{q}}\left(\frac{P_I}{Q_I}\right)^{\frac{1}{q}}\left[\frac{1}{P_I}\sum_{i\in I}p_i\left\|x_i\right\|-\left\|\frac{1}{P_I}\sum_{i\in I}p_ix_i\right\|\right]$$

for $I\in\mathcal{P}_f(\mathbb{N})$ and $\mathbf{p},\mathbf{q}\in S_+(I)$ with $M\mathbf{p}\geq\mathbf{q}\geq m\mathbf{p}$ and $M\geq m>0$ and for any vectors $x_i\in X$, $i\in I$.

(2) For $x_i>0$ and $p_i\geq0$, $(i\in\mathbb{N})$ so that $P_I>0$, let us denote

$$A(I,\mathbf{p},x):=\frac{1}{P_I}\sum_{i\in I}p_ix_i,\qquad G(I,\mathbf{p},x):=\left(\prod_{i\in I}(x_i)^{p_i}\right)^{\frac{1}{P_I}},$$

the *weighted arithmetic* and *geometric means* respectively.

Applying the above Proposition 2 for the convex function $f(x)=-\ln x$, $x\in(0,\infty)$, we can state the following inequality:

$$\left[\frac{A(I,\mathbf{p},x)}{G(I,\mathbf{p},x)}\right]^{M^{\frac{1}{q}}\left(\frac{P_I}{Q_I}\right)^{\frac{1}{q}}}\geq\frac{A(I,\mathbf{q},x)}{G(I,\mathbf{q},x)}\geq\left[\frac{A(I,\mathbf{p},x)}{G(I,\mathbf{p},x)}\right]^{m^{\frac{1}{q}}\left(\frac{P_I}{Q_I}\right)^{\frac{1}{q}}} \tag{3.8}$$

for $q\in(0,1)$, $I\in\mathcal{P}_f(\mathbb{N})$, $\mathbf{p},\mathbf{q}\in S_+(I)$ with $M\mathbf{p}\geq\mathbf{q}\geq m\mathbf{p}$ and $M\geq m>0$ and for any $x_i>0$, $i\in I$.

4. Applications for Hölder's Inequality

Let $(X,\|\cdot\|)$ be a normed space and $I\in\mathcal{P}_f(\mathbb{N})$. We define

$$E(I):=\left\{x=(x_j)_{j\in I}\,\middle|\,x_j\in X,\ j\in I\right\}$$

and

$$\mathbb{K}(I):=\left\{\lambda=(\lambda_j)_{j\in I}\,\middle|\,\lambda_j\in\mathbb{K},\ j\in I\right\}.$$

We consider for $\alpha,\beta>1$, $\frac{1}{\alpha}+\frac{1}{\beta}=1$ the functional

$$H_I(\mathbf{p},\lambda,x;\alpha,\beta):=\left(\sum_{j\in I}p_j|\lambda_j|^{\alpha}\right)^{\frac{1}{\alpha}}\left(\sum_{j\in I}p_j\|x_j\|^{\beta}\right)^{\frac{1}{\beta}}-\left\|\sum_{j\in I}p_j\lambda_jx_j\right\|.$$

The following result has been proved in [3]:

Lemma 2. *For any* $\mathbf{p},\mathbf{q}\in S_+(I)$ *we have*

$$H_I(\mathbf{p}+\mathbf{q},\lambda,x;\alpha,\beta)\geq H_I(\mathbf{p},\lambda,x;\alpha,\beta)+H_I(\mathbf{q},\lambda,x;\alpha,\beta), \tag{4.1}$$

where $x\in E(I)$, $\lambda\in\mathbb{K}(I)$ *and* $\alpha,\beta>1$ *with* $\frac{1}{\alpha}+\frac{1}{\beta}=1$.

Remark 2. *The same result can be stated if* $(B,\|\cdot\|)$ *is a normed algebra and the functional* H *is defined by*

$$H_I(\mathbf{p},\lambda,x;\alpha,\beta):=\left(\sum_{i\in I}p_i\|x_i\|^\alpha\right)^{\frac{1}{\alpha}}\left(\sum_{i\in I}p_i\|y_i\|^\beta\right)^{\frac{1}{\beta}}-\left\|\sum_{i\in I}p_ix_iy_i\right\|,$$

where $x=(x_i)_{i\in I}, y=(y_i)_{i\in I}\subset B$, $\mathbf{p}\in S_+(I)$ *and* $\alpha,\beta>1$ *with* $\frac{1}{\alpha}+\frac{1}{\beta}=1$.

Define the following functional on $S_+(I)$:

$$H_{q,I}(\mathbf{p},\lambda,x;\alpha,\beta):=P_I^{1-q}H_I^q(\mathbf{p},\lambda,x;\alpha,\beta). \tag{4.2}$$

Since, $H(\cdot,\lambda,x;\alpha,\beta)$ is positive homogeneous, on utilising Corollary 1, we can state the following result.

Proposition 3. *If* $\mathbf{p},\mathbf{q}\in S_+(I)$ *and* $M\geq m\geq 0$ *with* $M\mathbf{p}\geq\mathbf{q}\geq m\mathbf{p}$, *then we have:*

$$M^{\frac{1}{q}}\left(\frac{P_I}{Q_I}\right)^{\frac{1}{q}-1}\left[\left(\sum_{i\in I}p_i|\lambda_i|^\alpha\right)^{\frac{1}{\alpha}}\left(\sum_{i\in I}p_i\|x_i\|^\beta\right)^{\frac{1}{\beta}}-\left\|\sum_{i\in I}p_i\lambda_ix_i\right\|\right] \tag{4.3}$$
$$\geq\left(\sum_{i\in I}q_i|\lambda_i|^\alpha\right)^{\frac{1}{\alpha}}\left(\sum_{i\in I}q_i\|x_i\|^\beta\right)^{\frac{1}{\beta}}-\left\|\sum_{i\in I}q_i\lambda_ix_i\right\|$$
$$\geq m^{\frac{1}{q}}\left(\frac{P_I}{Q_I}\right)^{\frac{1}{q}-1}\left[\left(\sum_{i\in I}p_i|\lambda_i|^\alpha\right)^{\frac{1}{\alpha}}\left(\sum_{i\in I}p_i\|x_i\|^\beta\right)^{\frac{1}{\beta}}-\left\|\sum_{i\in I}p_i\lambda_ix_i\right\|\right]$$

for $x\in E(I)$, $\lambda\in\mathbb{K}(I)$, $q\in(0,1)$ *and* $\alpha,\beta>1$, $\frac{1}{\alpha}+\frac{1}{\beta}=1$.

Proof. By Corollary 1 applied for the functional $H_{q,I}(\mathbf{p},\lambda,x;\alpha,\beta)$ we have

$$MP_I^{1-q}\left[\left(\sum_{i\in I}p_i|\lambda_i|^\alpha\right)^{\frac{1}{\alpha}}\left(\sum_{i\in I}p_i\|x_i\|^\beta\right)^{\frac{1}{\beta}}-\left\|\sum_{i\in I}p_i\lambda_ix_i\right\|\right]^q \tag{4.4}$$
$$\geq Q_I^{1-q}\left[\left(\sum_{i\in I}q_i|\lambda_i|^\alpha\right)^{\frac{1}{\alpha}}\left(\sum_{i\in I}q_i\|x_i\|^\beta\right)^{\frac{1}{\beta}}-\left\|\sum_{i\in I}q_i\lambda_ix_i\right\|\right]^q$$
$$\geq mP_I^{1-q}\left[\left(\sum_{i\in I}p_i|\lambda_i|^\alpha\right)^{\frac{1}{\alpha}}\left(\sum_{i\in I}p_i\|x_i\|^\beta\right)^{\frac{1}{\beta}}-\left\|\sum_{i\in I}p_i\lambda_ix_i\right\|\right]^q$$

Taking the power $\frac{1}{q}>0$ in the inequality (4.4) we deduce the desired result (4.3). □

5. Applications for Minkowski's Inequality

Let $(X,\|\cdot\|)$ be a normed space and $I\in\mathcal{P}_f(\mathbb{N})$. We define the functional:

$$M_I(\mathbf{p},x,y;\alpha)=\left[\left(\sum_{i\in I}p_i\|x_i\|^\alpha\right)^{\frac{1}{\alpha}}+\left(\sum_{i\in I}p_i\|y_i\|^\alpha\right)^{\frac{1}{\alpha}}\right]^\alpha-\sum_{i\in I}p_i\|x_i+y_i\|^\alpha, \tag{5.1}$$

where $\mathbf{p}\in S_{+}(I)$, $\alpha\geq 1$ and $x,y\in E(I)$.

The following result concerning the superadditivity of the functional $M_{I}(\cdot,x,y;\alpha)$ holds [3]:

Lemma 3. *For any* $\mathbf{p},\mathbf{q}\in S_{+}(I)$, *we have*

$$M_{I}(\mathbf{p}+\mathbf{q},x,y;\alpha)\geq M_{I}(\mathbf{p},x,y;\alpha)+M_{I}(\mathbf{q},x,y;\alpha),$$

where $x,y\in E(I)$ *and* $\alpha\geq 1$.

Since the functional $M_{I}(\cdot,x,y;\alpha)$ is positive homogeneous on $S_{+}(I)$, then on utilising Corollary 1, we can state the following proposition.

Proposition 4. *If* $\mathbf{p},\mathbf{q}\in S_{+}(I)$ *and* $M\geq m\geq 0$ *with* $M\mathbf{p}\geq\mathbf{q}\geq m\mathbf{p}$, *then we have:*

$$\begin{aligned}(5.2)\quad & M^{\frac{1}{q}}\left(\frac{P_I}{Q_I}\right)^{\frac{1}{q}-1}\\ &\times\left\{\left[\left(\sum_{i\in I}p_i\|x_i\|^{\alpha}\right)^{\frac{1}{\alpha}}+\left(\sum_{i\in I}p_i\|y_i\|^{\alpha}\right)^{\frac{1}{\alpha}}\right]^{\alpha}-\sum_{i\in I}p_i\|x_i+y_i\|^{\alpha}\right\}\\ &\geq\left[\left(\sum_{i\in I}q_i\|x_i\|^{\alpha}\right)^{\frac{1}{\alpha}}+\left(\sum_{i\in I}q_i\|y_i\|^{\alpha}\right)^{\frac{1}{\alpha}}\right]^{\alpha}-\sum_{i\in I}q_i\|x_i+y_i\|^{\alpha}\\ &\geq m^{\frac{1}{q}}\left(\frac{P_I}{Q_I}\right)^{\frac{1}{q}-1}\\ &\times\left\{\left[\left(\sum_{i\in I}p_i\|x_i\|^{\alpha}\right)^{\frac{1}{\alpha}}+\left(\sum_{i\in I}p_i\|y_i\|^{\alpha}\right)^{\frac{1}{\alpha}}\right]^{\alpha}-\sum_{i\in I}p_i\|x_i+y_i\|^{\alpha}\right\},\end{aligned}$$

where $x,y\in E(I)$, $q\in(0,1)$ *and* $\alpha\geq 1$.

6. Applications for Schwarz's Inequality

Let X be a linear space over the real or complex number field $\mathbb{K}$ and let us denote by $\mathcal{H}(X)$ the class of all positive semi-definite Hermitian forms on X, or, for simplicity, nonnegative forms on X, i.e., the mapping $\langle\cdot,\cdot\rangle : X\times X\to\mathbb{K}$ belongs to $\mathcal{H}(X)$ if it satisfies the conditions

(i) $\langle x,x\rangle\geq 0$ for all $x\in X$;
(ii) $\langle\alpha x+\beta y,z\rangle=\alpha\langle x,z\rangle+\beta\langle y,z\rangle$ for all $x,y\in X$ and $\alpha,\beta\in\mathbb{K}$;
(iii) $\langle y,x\rangle=\overline{\langle x,y\rangle}$ for all $x,y\in X$.

If $\langle\cdot,\cdot\rangle\in\mathcal{H}(X)$, then the functional $\|\cdot\|=\langle\cdot,\cdot\rangle^{1/2}$ is a semi-norm on X and the following version of Schwarz's inequality holds:

$$\|x\|\,\|y\|\geq|\langle x,y\rangle| \tag{6.1}$$

for each $x,y\in H$.

Now, let us observe that $\mathcal{H}(X)$ is a *convex cone* in the linear space of all mappings defined on X^2 with values in $\mathbb{K}$. Also, we can introduce on $\mathcal{H}(X)$ the following *binary relation* [5]

$$\langle\cdot,\cdot\rangle_2\geq\langle\cdot,\cdot\rangle_1\quad\text{if and only if}\quad\|x\|_2\geq\|x\|_1\quad\text{for any}\quad x\in H. \tag{6.2}$$

This is an *order relation* on $\mathcal{H}(X)$.

Consider the following functional [5]:

$$\sigma : \mathcal{H}(X) \times X^2 \to \mathbb{R}_+, \qquad \sigma(\langle \cdot, \cdot \rangle ; x, y) := \|x\| \|y\| - |\langle x, y \rangle|,$$

which is closely related to the Schwarz inequality in (6.1).

Lemma 4 ([5])**.** *The functional $\sigma(\cdot; x, y)$ is nonnegative, superadditive and positive homogeneous on $\mathcal{H}(X)$.*

The following proposition can be stated.

Proposition 5. *Let $M \geq m > 0$ and $\langle \cdot, \cdot \rangle_1, \langle \cdot, \cdot \rangle_2$ be two inner products on X such that $M \|x\|_1 \geq \|x\|_2 \geq m \|x\|_1$ for each $x \in X$. If $e \in X, e \neq 0$, $q \in (0,1)$ then*

$$\begin{aligned} (6.3) \qquad & M^{\frac{2}{q}} \left(\frac{\|e\|_1}{\|e\|_2} \right)^{2\left(\frac{1}{q}-1\right)} \left(\|x\|_1 \|y\|_1 - |\langle x, y \rangle_1| \right) \\ & \geq \|x\|_2 \|y\|_2 - |\langle x, y \rangle_2| \\ & \geq m^{\frac{2}{q}} \left(\frac{\|e\|_1}{\|e\|_2} \right)^{2\left(\frac{1}{q}-1\right)} \left(\|x\|_1 \|y\|_1 - |\langle x, y \rangle_1| \right), \end{aligned}$$

for any $x, y \in H$.

Proof. On applying Corollary 1 for the functional

$$\begin{aligned} \Lambda_{q,e}(\langle \cdot, \cdot \rangle) &= \langle e, e \rangle^{1-q} \sigma^q (\langle \cdot, \cdot \rangle ; x, y) \\ &= \|e\|^{2(1-q)} \left[\|x\| \|y\| - |\langle x, y \rangle| \right]^q \end{aligned}$$

we deduce

$$\begin{aligned} (6.4) \qquad & M^2 \|e\|_1^{2(1-q)} \left[\|x\|_1 \|y\|_1 - |\langle x, y \rangle_1| \right]^q \\ & \geq \|e\|_2^{2(1-q)} \left[\|x\|_2 \|y\|_2 - |\langle x, y \rangle_2| \right]^q \\ & \geq m^2 \|e\|_1^{2(1-q)} \left[\|x\|_1 \|y\|_1 - |\langle x, y \rangle_1| \right]^q \end{aligned}$$

for x, y fixed in X.

Taking the power $\frac{1}{q} > 0$ in the inequality (6.4) we deduce the desired result (6.3). □

The above result can be used to obtain some inequalities for positive definite operators as follows:

Corollary 2. *Assume that $A : H \to H$ is a self-adjoint linear operator on the Hilbert space $(H, \langle \cdot, \cdot \rangle)$ satisfying the property that there exist $\Gamma \geq \gamma > 0$ such that $\Gamma I \geq A \geq \gamma I$ in the operation order (i.e., $\Gamma \|x\|^2 \geq \langle Ax, x \rangle \geq \gamma \|x\|^2$ for any $x \in H$), then for $q \in (0,1)$ we have the inequality:*

$$\begin{aligned} (6.5) \qquad & \frac{\Gamma^{\frac{1}{q}}}{\langle Ae, e \rangle^{\frac{1}{q}-1}} \left(\|x\| \|y\| - |\langle x, y \rangle| \right) \\ & \geq \langle Ax, x \rangle^{1/2} \langle Ay, y \rangle^{1/2} - |\langle Ax, y \rangle| \\ & \geq \frac{\gamma^{\frac{1}{q}}}{\langle Ae, e \rangle^{\frac{1}{q}-1}} \left(\|x\| \|y\| - |\langle x, y \rangle| \right), \end{aligned}$$

for any $x, y \in H$ and $e \in H$ with $\|e\| = 1$.

Remark 3. *Similar results can be stated if one uses the following nonnegative, superadditive and positive homogeneous functionals on $\mathcal{H}(X)$ (see* [2, pp. 8-15]*):*

$$\sigma_r(\langle\cdot,\cdot\rangle;x,y) := \|x\|\,\|y\| - \operatorname{Re}\langle x,y\rangle;$$
$$\delta(\langle\cdot,\cdot\rangle;x,y) := \|x\|^2\,\|y\|^2 - |\langle x,y\rangle|^2;$$
$$\delta_r(\langle\cdot,\cdot\rangle;x,y) := \|x\|^2\,\|y\|^2 - (\operatorname{Re}\langle x,y\rangle)^2;$$
$$\gamma(\langle\cdot,\cdot\rangle;x,y) := \frac{\|x\|^2\,\|y\|^2 - |\langle x,y\rangle|^2}{\|y\|^2};$$

where in the definition of γ, $\langle\cdot,\cdot\rangle$ is an inner product and y is not zero, and

$$\beta(\langle\cdot,\cdot\rangle;x,y) := \left(\|x\|^2\,\|y\|^2 - |\langle x,y\rangle|^2\right)^{\frac{1}{2}},$$

for each $x,y \in X$.

The details are left to the interested reader.

Remark 4. *For other examples of superadditive (subadditive) functionals that can provide interesting inequalities similar to the ones outlined above, we refer to* [1], [7]*,* [8] [9] *and* [10]*.*

7. More on Jensen's Inequality for Positive Functions

Let C be a convex subset of the real linear space X and let $f : C \to (0,\infty)$ be a convex mapping on C with positive values. Let us fix $I \in \mathcal{P}_f(\mathbb{N})$ (the class of finite parts of $\mathbb{N}$) and $x_i \in C$ $(i \in I)$. Now consider the functional $L : S_+(I) \to (0,\infty)$ given by

$$L_I(\mathbf{p}) := P_I f\left(\frac{1}{P_I}\sum_{i\in I} p_i x_i\right) \tag{7.1}$$

where $S_+(I) := \left\{\mathbf{p} = (p_i)_{i\in I} \,\middle|\, p_i \geq 0,\ i \in I \text{ and } P_I > 0\right\}$.

We observe that $S_+(I)$ is a convex cone and the functional L_I is *positive homogeneous* on $S_+(I)$.

Lemma 5. *The functional L_I is subadditive on $S_+(I)$.*

Proof. Indeed, by the convexity of f, we have that

$$\begin{aligned} &L_I(\mathbf{p}+\mathbf{q}) \\ &= (P_I+Q_I)\, f\left(\frac{1}{P_I+Q_I}\sum_{i\in I}(p_i+q_i)\,x_i\right) \\ &= (P_I+Q_I)\, f\left(\frac{P_I}{P_I+Q_I}\cdot\frac{1}{P_I}\sum_{i\in I}p_i x_i + \frac{Q_I}{P_I+Q_I}\cdot\frac{1}{Q_I}\sum_{i\in I}q_i x_i\right) \\ &\leq (P_I+Q_I)\left(\frac{P_I}{P_I+Q_I}\cdot f\left(\frac{1}{P_I}\sum_{i\in I}p_i x_i\right) + \frac{Q_I}{P_I+Q_I}\cdot f\left(\frac{1}{Q_I}\sum_{i\in I}q_i x_i\right)\right) \\ &= L_I(\mathbf{p}) + L_I(\mathbf{q}) \end{aligned} \tag{7.2}$$

since for any $\mathbf{p},\mathbf{q} \in S_+(I)$, and the statement is proven. □

It is well known that if f is strictly convex and not all $x_i \in C$ $(i \in I)$ are equal between them, then we have a strict inequality in Jensen's inequality, i.e.,

$$f\left(\frac{1}{P_I}\sum_{i\in I}p_i x_i\right) < \frac{1}{P_I}\sum_{i\in I}p_i f\left(x_i\right),$$

for any $\mathbf{p} \in S_+(I)$ and then we can consider the functional $\varkappa : S_+(I) \to (0,\infty)$,

$$\varkappa_{I,f}\left(\mathbf{p}\right) = \frac{P_I f\left(\frac{1}{P_I}\sum_{i\in I}p_i x_i\right)}{\frac{\sum_{i\in I}p_i f(x_i)}{P_I f\left(\frac{1}{P_I}\sum_{i\in I}p_i x_i\right)} - 1}. \tag{7.3}$$

Proposition 6. *If $f : C \to (0,\infty)$ be a strictly convex mapping on the convex set C and $x_i \in C$ $(i \in I)$ such that not all are equal between them, then the mapping $\varkappa$ defined by (7.3) is subadditive on $S_+(I)$.*

Proof. Observe that

$$\varkappa_{I,f}\left(\mathbf{p}\right) = \frac{L_I^2\left(\mathbf{p}\right)}{J_I\left(\mathbf{p}\right)}, \mathbf{p} \in S_+(I).$$

Then, by applying Theorem 6 for the functionals $v = L_I$ and $h = J_I$ we deduce the desired result. □

The above Proposition 6 provides some results of interest for various choices of strictly convex functions as follows:

1. If $f : (0,\infty) \to \mathbb{R}$ is defined by $f(x) = -\ln x$, then f is strictly convex and $\varkappa_{I,f}(\mathbf{p})$ becomes

$$\varkappa_{I,-\ln}\left(\mathbf{p}\right) = \frac{P_I\left[\ln A\left(I,\mathbf{p},x\right)\right]^2}{\ln\left[\frac{A(I,\mathbf{p},x)}{G(I,\mathbf{p},x)}\right]},$$

where $A(I,\mathbf{p},x)$ and $G(I,\mathbf{p},x)$ are the weighted arithmetic and geometric means, respectively.

Now, if we assume that $x_i > 0$ $(i \in I)$ such that not all are equal between them, then the functional $\varkappa_{I,-\ln}(\cdot)$ is *subadditive* on $S_+(I)$.

2. If we consider the power function $f : (0,\infty) \to \mathbb{R}$ defined by $f(x) = x^\alpha, \alpha \in (-\infty,0) \cup (1,\infty)$, then f is strictly convex and $\varkappa_{I,f}(\mathbf{p})$ becomes

$$\varkappa_{I,(\cdot)^\alpha}\left(\mathbf{p}\right) = \frac{P_I\left[A\left(I,\mathbf{p},x\right)\right]^{2\alpha}}{A\left(I,\mathbf{p},x^\alpha\right) - \left[A\left(I,\mathbf{p},x\right)\right]^\alpha},$$

where $A(I,\mathbf{p},x^\alpha) := \frac{1}{P_I}\sum_{i\in I}p_i x_i^\alpha$ and $x_i > 0$ $(i \in I)$.

Now, if we assume that $x_i > 0$ $(i \in I)$ such that not all are equal between them, then the functional $\varkappa_{I,(\cdot)^\alpha}(\cdot)$ is *subadditive* on $S_+(I)$.

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