Received 20/03/11

Quadrature Rules for the Riemann-Stieltjes Integral and Applications

S.S. Dragomir

This paper was presented at the 35th South African Symposium on Numerical and Applied Mathematics (SANUM 2011, 23-25 March, 2011) hosted by the Department of Mathematical Sciences, University of Stellenbosch, South Africa.

1. Introduction

The concept of **Riemann-Stieltjes integral**

$$\int_a^b f(t)\,du(t), \text{where } f \text{ is called } \textit{the integrand},$$
u is called *the integrator,*

plays an important role in Mathematics, for instance in the definition of complex integral, the representation of bounded linear functionals on the Banach space of all continuous functions on an interval $[a, b]$, in the spectral representation of selfadjoint operators on complex Hilbert spaces and other classes of operators such as the unitary operators etc...

However the *Numerical Analysis of this integral* is quite poor as pointed out by the seminal paper due to Michael Tortorella from 1990 [**23**], in which, I quote, for the first time in history

"*A generalization of the Newton–Cotes quadrature rules that provides a means for numerical computation of Stieltjes integrals without using derivatives is described. The methods find wide application in the numerical evaluation of many applied probability models. Numerical convolution of life distributions is discussed in this paper. Error analyses are provided.*"

Earlier results in this direction, however, with a modest influence in the literature were provided by Dubuc and Todor in their 1984 and 1987 papers [**18**] and [**19**], respectively.

For recent results see the work of Diethelm [**8**], Liu [**20**], Mercer [**21**], Munteanu [**22**], Mozyrska et al. [**24**] and the references therein.

In the following I will present some recent results obtained together with some members of the RGMIA, see [**1**], [**2**], [**3**], [**7**], [**5**], [**6**], [**12**] and [**13**]. A comprehensive list of preprints related to this subject may be found at http://rgmia.org

2. Trapezoidal Rules

We endeavour in the following to provide sharp bounds for the error in approximating the Riemann-Stieltjes integral by a *trapezoidal rule* under various assumptions for the integrand and the integrator for which the integral exists.

If we consider the error

$$E_T(f,u;a,b) := \frac{f(a)+f(b)}{2}\cdot[u(b)-u(a)] - \int_a^b f(t)\,du(t)$$

then we have the following results:

Theorem 1. *We have*

(1) *Let $f:[a,b]\to\mathbb{C}$ be a $p-H$-Hölder type function, that is, it satisfies the condition*

$$|f(x)-f(y)| \le H|x-y|^p \text{ for all } x,y\in[a,b], \tag{2.1}$$

where $H>0$ and $p\in(0,1]$ are given, and $u:[a,b]\to\mathbb{C}$ is a function of bounded variation on $[a,b]$. Then we have the inequality (see Dragomir [**9**]*):*

$$|E_T(f,u;a,b)| \le \frac{1}{2^p}H(b-a)^p\bigvee_a^b(u) \tag{2.2}$$

where $\bigvee_a^b(u)$ denotes the total variation of u on $[a,b]$. The constant $C=1$ on the right hand side of (2.2) *cannot be replaced by a smaller quantity.*

(2) *Let $f:[a,b]\to\mathbb{C}$ be a $p-H$-Hölder type mapping where $H>0$ and $p\in(0,1]$ are given, and $u:[a,b]\to\mathbb{C}$ is a Lipschitzian function on $[a,b]$, this means that*

$$|u(x)-u(y)| \le L|x-y| \text{ for all } x,y\in[a,b], \tag{2.3}$$

where $L>0$ is given. Then we have the inequality (see Dragomir [**11**]*):*

$$|E_T(f,u;a,b)| \le \frac{1}{p+1}HL(b-a)^{p+1}. \tag{2.4}$$

(3) *Let $f:[a,b]\to\mathbb{C}$ be a $p-H$-Hölder type mapping where $H>0$ and $p\in(0,1]$ are given, and $u:[a,b]\to\mathbb{R}$ a monotonic nondecreasing function on $[a,b]$. Then we have the inequality (see Dragomir* [**11**]*):*

$$\begin{aligned} &|E_T(f,u;a,b)| \\ &\le \frac{1}{2}H\Big\{(b-a)^p[u(b)-u(a)] \\ &-p\int_a^b\left[\frac{(b-t)^{1-p}-(t-a)^{1-p}}{(b-t)^{1-p}(t-a)^{1-p}}\right]u(t)\,dt\Big\} \\ &\le \frac{1}{2^p}H(b-a)^p[u(b)-u(a)]. \end{aligned} \tag{2.5}$$

The inequalities in (2.5) are sharp.

(4) *Let $f,\ u : [a,b] \to \mathbb{C}$ be of bounded variation on $[a,b]$. If one of them is continuous on $[a,b]$, then the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ exists and we have the inequality (see Dragomir* [**11**]*):*

$$|E_T(f,u;a,b)| \leq \frac{1}{2}\bigvee_a^b(f)\bigvee_a^b(u). \tag{2.6}$$

The constant $\frac{1}{2}$ is best possible in (2.6).

(5) *We consider the case when the function $f : [a,b] \to \mathbb{C}$ satisfies the end-point Lipschitzian conditions*

$$\begin{aligned} |f(t)-f(a)| &\leq L_a(t-a)^{\alpha} \text{ and} \\ |f(b)-f(t)| &\leq L_b(b-t)^{\beta} \end{aligned} \tag{2.7}$$

for any $t \in (a,b)$ where the constants $L_a, L_b > 0$ and $\alpha, \beta > -1$ are given. Assume that the function f satisfies the condition (2.7).

a) *If $u : [a,b] \to \mathbb{C}$ is Lipschitzian with the constant $K > 0$, then we have the inequality (see Dragomir* [**11**]*):*

$$\begin{aligned} &|E_T(f,u;a,b)| \\ &\leq \frac{1}{2}K\left[\frac{L_a}{\alpha+1}(b-a)^{\alpha+1} + \frac{L_\beta}{\beta+1}(b-a)^{\beta+1}\right]. \end{aligned} \tag{2.8}$$

b) *If $\alpha, \beta > 0$ and $u : [a,b] \to \mathbb{R}$ is monotonic nondecreasing on $[a,b]$, then (see Dragomir* [**11**]*):*

$$\begin{aligned} &|E_T(f,u;a,b)| \\ &\leq \frac{1}{2}L_a\left[(b-a)^{\alpha}u(b) - \alpha\int_a^b (t-a)^{\alpha-1}u(t)\,dt\right] \\ &+ \frac{1}{2}L_b\left[\beta\int_a^b (b-t)^{\beta-1}u(t)\,dt - (b-a)^{\beta}u(a)\right] \\ &\leq \frac{1}{2}\left[L_a(b-a)^{\alpha} + L_b(b-a)^{\beta}\right][u(b)-u(a)]. \end{aligned} \tag{2.9}$$

REMARK 1. *Since, integrating by parts in the Riemann-Stieltjes integral we have*

$$E_T(f,u;a,b) = -E_T(u,f;a,b)$$

then similar results can be stated by swapping the role of f with u in the above theorem. The details are omitted.

3. Applications for Functions of Selfadjoint Operators

Let U be a selfadjoint operator on the complex Hilbert space $(H, \langle .,. \rangle)$ with the spectrum $Sp(U)$ included in the interval $[m,M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its *spectral family.* Then for any continuous function $f : [m,M] \to \mathbb{R}$, it is well known that we have the following *spectral representation in terms of the Riemann-Stieltjes integral*:

$$\langle f(U)x, y \rangle = \int_{m-0}^{M} f(\lambda)\,d(\langle E_\lambda x, y \rangle), \tag{3.1}$$

for any $x, y \in H$. The function $g_{x,y}(\lambda) := \langle E_\lambda x, y\rangle$ is of *bounded variation* on the interval $[m, M]$ and

$$g_{x,y}(m-0) = 0 \text{ and } g_{x,y}(M) = \langle x, y\rangle$$

for any $x, y \in H$. It is also well known that $g_x(\lambda) := \langle E_\lambda x, x\rangle$ is *monotonic nondecreasing* and *right continuous* on $[m, M]$.

Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its *spectral family.*

Consider the partition $I_n : m = t_0 < t_1 < ... < t_{n-1} < t_n = M$ of the interval $[m, M]$, and define $h_i := t_{i+1} - t_i$ $(i = 0, ..., n-1)$, $\nu(h) := \max\{h_i | i = 0, ..., n-1\}$ and the *generalized trapezoidal quadrature rule* associated to the continuous function $f : [m, M] \to \mathbb{C}$, selfadjoint operator A and the vectors $x, y \in H$

$$\begin{aligned} &T_n(f, A, I_n; x, y) \\ &:= \sum_{i=0}^{n-1} \frac{f(t_i) + f(t_{i+1})}{2} \left\langle \left(E_{t_{i+1}} - E_{t_i}\right) x, y\right\rangle . \end{aligned} \tag{3.2}$$

Theorem 2. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family.*

a) *If $f : [m, M] \to \mathbb{C}$ is continuous and with bounded variation on $[m, M]$, then for any $x, y \in H$*

$$\begin{aligned} \langle f(A) x, y\rangle &= T_n(f, A, I_n; x, y) \\ &\quad + R_n(f, A, I_n; x, y) \end{aligned} \tag{3.3}$$

*and the remainder $R_n(f, A, I_n; x, y)$ satisfies the error bounds (see Dragomir [**11**]):*

$$\begin{aligned} &|R_n(f, A, I_n; x, y)| \\ &\leq \frac{1}{2} \max_{i \in \{0,...,n-1\}} \left\{ \bigvee_{t_i}^{t_{i+1}} (f) \right\} \bigvee_m^M \left(\langle E_{(\cdot)} x, y\rangle\right) \\ &\leq \frac{1}{2} \max_{i \in \{0,...,n-1\}} \left\{ \bigvee_{t_i}^{t_{i+1}} (f) \right\} \|x\| \|y\| . \end{aligned} \tag{3.4}$$

b) *Let $f : [m, M] \to \mathbb{C}$ be a $p - H$-Hölder continuous function on $[m, M]$, then for any $x, y \in H$ we have the equality (3.3) and the remainder $R_n(f, A, I_n; x, y)$ satisfies the error bounds (see Dragomir [**11**]):*

$$\begin{aligned} &|R_n(f, A, I_n; x, y)| \\ &\leq \frac{1}{2^p} H[\nu(h)]^p \bigvee_m^M \left(\langle E_{(\cdot)} x, y\rangle\right) \\ &\leq \frac{1}{2^p} H[\nu(h)]^p \|x\| \|y\| . \end{aligned} \tag{3.5}$$

4. Ostrowski Type Rules

Now define the error in approximating the Riemann-Stieltjes integral by the generalized mid-point rule as

$$E_O(f,u;a,b,x) := [u(b)-u(a)]\, f(x) - \int_a^b f(t)du(t)$$

where $x \in [a,b]$. Bounds for $E_O(f,u;a,b,x)$ in the case when $u(t) = t, t \in [a,b]$ are known in the literature as *Ostrowski type inequalities*.

THEOREM 3. *We have*

(1) *If $f : [a,b] \to \mathbb{R}$ is a function of bounded variation and $u : [a,b] \to \mathbb{R}$ is of r-H-Hölder type, then (see Dragomir* [**10**]*):*

$$\begin{aligned} &|E_O(f,u;a,b,x))| \\ &\leq H\left[(x-a)^r \bigvee_a^x (f) + (b-x)^r \bigvee_x^b (f)\right] \\ &\leq \begin{cases} H\left[(x-a)^r + (b-x)^r\right] \\ \times \left[\frac{1}{2}\bigvee_a^b(f) + \frac{1}{2}\left|\bigvee_a^x(f) - \bigvee_x^b(f)\right|\right]; \\ \\ H\left[(x-a)^{qr} + (b-x)^{qr}\right]^{\frac{1}{q}} \\ \times \left[\left(\bigvee_a^x(f)\right)^p + \left(\bigvee_x^b(f)\right)^p\right]^{\frac{1}{p}} \\ \qquad\qquad \text{if} \quad p>1, \frac{1}{p}+\frac{1}{q}=1 ; \\ \\ H\left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]^r \bigvee_a^b(f) ; \end{cases} \end{aligned} \tag{4.1}$$

(2) *Let $f : [a,b] \to \mathbb{R}$ be a function of r-H-Hölder type with $r \in (0,1]$ and $H > 0$, and $u : [a,b] \to \mathbb{R}$ be a monotonic nondecreasing function on $[a,b]$. Then (see Chung & Dragomir* [**4**]*):*

$$\begin{aligned} &|E_O(f,u;a,b,x))| \\ &\leq H\Bigg[(b-x)^r u(b) - (x-a)^r u(a) \\ &\qquad\qquad + r\left\{\int_a^x \frac{u(t)}{(x-t)^{1-r}}dt - \int_x^b \frac{u(t)}{(t-x)^{1-r}}dt\right\}\Bigg] \\ &\leq H\left\{(b-x)^r\left[(u(b)-u(x)\right] + (x-a)^r\left[u(x)-u(a)\right]\right\} \\ &\leq H\left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]^r \left[u(b)-u(a)\right] , \end{aligned} \tag{4.2}$$

for any $x \in [a,b]$.

(3) *Let* $f:[a,b]\to\mathbb{R}$ *be monotonic nondecreasing on* $[a,b]$ *and* $u:[a,b]\to\mathbb{R}$ *of* r*-*H*-Hölder type. Then (see Chung & Dragomir* [**4**]*)*

$$\begin{aligned}&|E_O(f,u;a,b,x))|\\&\leq H\left[\left[(x-a)^r-(b-a)^r\right]f(x)\right.\\&\left.+r\left\{\int_x^b\frac{f(t)dt}{(b-t)^{1-r}}-\int_a^x\frac{f(t)dt}{(t-a)^{1-r}}\right\}\right]\\&\leq H\left\{(b-x)^r\left[f(b)-f(x)\right]+(x-a)^r\left[f(x)-f(a)\right]\right\}\\&\leq H\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^r\left[f(b)-f(a)\right].\end{aligned}\tag{4.3}$$

(4) *Let* $f:[a,b]\to\mathbb{C}$ *be a* $r-H-$*Hölder continuous function on* $[a,b]$*, and* $u:[a,b]\to\mathbb{C}$ *is an* $L-$*Lipschitzian function on* $[a,b]$*, then (see Chung & Dragomir* [**4**]*) for any* $x\in[a,b]$ *we have:*

$$\begin{aligned}&|E_O(f,u;a,b,x))|\\&\leq\frac{LH}{r+1}\left[(x-a)^{r+1}+(b-x)^{r+1}\right],\end{aligned}\tag{4.4}$$

Remark 2. *The case* $x=\frac{a+b}{2}$ *provides the best error bounds in the above results and the corresponding mid-point quadrature rule is also the simplest to numerically implement for applications.*

5. Grüss Type Rules

In 1998, S.S. Dragomir and I. Fedotov [**16**], in order to approximate the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ with the simpler expression

$$\frac{1}{b-a}\left[u(b)-u(a)\right]\int_a^b f(t)\,dt$$

introduced the following error functional

$$\begin{aligned}&E_G(f,u;a,b)\\&:=\int_a^b f(t)\,du(t)-\frac{1}{b-a}\left[u(b)-u(a)\right]\int_a^b f(t)\,dt\end{aligned}\tag{5.1}$$

provided that both the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ and the Riemann integral $\int_a^b f(t)\,dt$ exist.

Theorem 4. *We have:*

(1) *If* u *is* $L-$*Lipschitzian on* $[a,b]$ *and* f *is Riemann integrable on* $[a,b]$*, then (see Dragomir & Fedotov)* [**16**]*:*

$$\begin{aligned}&|E_G(f,u;a,b)|\\&\leq L\int_a^b\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|dt.\end{aligned}\tag{5.2}$$

The inequality (5.2) is sharp. Moreover, if there exist the constants $m,M\in\mathbb{R}$ *such that*

$$m\leq f(t)\leq M\qquad\text{for any }\ t\in[a,b],\tag{5.3}$$

then:

$$|E_G(f,u;a,b)| \leq \frac{1}{2}L(M-m)(b-a). \tag{5.4}$$

The constant $\frac{1}{2}$ is sharp in (5.4).

(2) *Let $f,u:[a,b]\to\mathbb{R}$ be such that u is (l,L)-Lipschitzian on $[a,b]$, i.e., the function $u-\frac{l+L}{2}\cdot e$, where $e(t)=t$, $t\in[a,b]$ is $\frac{1}{2}(L-l)-$Lipschitzian.*

*(i) If f is of bounded variation, then (see Dragomir [**14**]):*

$$|E_G(f,u;a,b)| \leq \frac{1}{4}(L-l)(b-a)\bigvee_a^b(f). \tag{5.5}$$

The constant $\frac{1}{4}$ is best possible in (5.5).

*(ii) If f is $K-$Lipschitzian on $[a,b]$, then (see Dragomir [**14**]):*

$$|E_G(f,u;a,b)| \leq \frac{1}{6}K(L-l)(b-a)^2. \tag{5.6}$$

*(iii) If f is monotonic nondecreasing, then (see Dragomir [**14**]):*

$$|E_G(f,u;a,b)| \leq 2\cdot\frac{L-l}{b-a}\int_a^b\left(t-\frac{a+b}{2}\right)f(t)\,dt \leq \begin{cases} \frac{1}{2}(L-l)\max\{|f(a)|,|f(b)|\}(b-a); \\ \frac{1}{(q+1)^{\frac{1}{q}}}(L-l)\|f\|_p(b-a)^{\frac{1}{q}} \\ \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ (L-l)\|f\|_1, \end{cases} \tag{5.7}$$

where $\|f\|_p := \left(\int_a^b |f(t)|^p\,dt\right)^{\frac{1}{p}}$, $p\geq 1$ are the Lebesgue norms. The constants 2 and $\frac{1}{2}$ are best possible in (5.7).

The following results also holds:

Theorem 5. *We have*

(1) *If u is of bounded variation on $[a,b]$ and f is continuous on $[a,b]$, then:*

$$|E_G(f,u;a,b)| \leq \bigvee_a^b(u)\max_{t\in[a,b]}\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|. \tag{5.8}$$

*The inequality (5.8) is sharp. Moreover, if f is $K-$Lipschitzian, then (see Dragomir & Fedotov [**17**]):*

$$|E_G(f,u;a,b)| \leq \frac{1}{2}K(b-a)\bigvee_a^b(u). \tag{5.9}$$

The constant $\frac{1}{2}$ is best possible in (5.9).

(2) *Let $u:[a,b]\to\mathbb{R}$ be a continuous convex function on $[a,b]$.*

(i) If $f:[a,b]\to\mathbb{R}$ is a function of bounded variation on $[a,b]$, then (see Dragomir [**13**]*):*

$$|E_G(f,u;a,b)| \leq \frac{1}{4}\left[u'_-(b)-u'_+(a)\right](b-a)\bigvee_a^b(f). \tag{5.10}$$

The constant $\frac{1}{4}$ is best possible in (5.10).

(ii) If $f:[a,b]\to\mathbb{R}$ a nondecreasing function on $[a,b]$. Then (see Dragomir [**13**]*):*

$$\begin{aligned} 0 &\leq E_G(f,u;a,b) \\ &\leq 2\cdot\frac{u'_-(b)-u'_+(a)}{b-a}\int_a^b\left(t-\frac{a+b}{2}\right)f(t)\,dt \\ &\leq \left[u'_-(b)-u'_+(a)\right]\times\begin{cases}\frac{1}{2}\max\{|f(a)|,|f(b)|\}(b-a); \\ \frac{1}{(q+1)^{\frac{1}{q}}}\|f\|_p(b-a)^{\frac{1}{q}} \\ \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \|f\|_1. \end{cases}\end{aligned} \tag{5.11}$$

The constants 2 and $\frac{1}{2}$ are best possible.

(iii) If f an $L-$Lipschitzian function on $[a,b]$, then (see Dragomir [**13**]*):*

$$|E_G(f,u;a,b)| \leq \frac{1}{6}L\left[u'_-(b)-u'_+(a)\right](b-a)^2. \tag{5.12}$$

Remark 3. *For other similar results, see the survey paper* [**15**].

Remark 4. *Applications of the above results for functions of sefadjoint operators in Hilbert spaces may be found in the recent book due to the author "**Inequalities for Functions of Selfadjoint Operators in Hilbert Spaces**", RGMIA Monographs, 2011:*

[ONLINE : http : //rgmia.org/monographs.php].

References

[1] N.S. Barnett, P. Cerone and S.S. Dragomir, Majorisation inequalities for Stieltjes integrals, *Applied Mathematics Letters* **22** (2009) 416 421.

[2] N.S. Barnett, W.-S. Cheung, S.S. Dragomir and A. Sofo, Ostrowski and trapezoid type inequalities for the Stieltjes integral with Lipschitzian integrands or integrators, *Computers and Mathematics with Applications* **57** (2009) 195 201.

[3] N.S. Barnett and S.S. Dragomir, The Beesack–Darst–Pollard inequalities and approximations of the Riemann–Stieltjes integral, *Applied Mathematics Letters* **22** (2009) 58–63.

[4] W.-S. Cheung and S. S. Dragomir, Two Ostrowski type inequalities for the Stieltjes integral of monotonic functions. *Bull. Austral. Math. Soc.* **75** (2007), no. 2, 299–311.

[5] P. Cerone and S.S. Dragomir, Bounding the Čebyšev functional for the Riemann Stieltjes integral via a Beesack inequality and applications, *Computers and Mathematics with Applications* **58** (2009) 1247 1252.

[6] P. Cerone and S.S. Dragomir, Approximating the Riemann Stieltjes integral via some moments of the integrand, *Mathematical and Computer Modelling* **49** (2009) 242 248.

[7] P. Cerone, W.S. Cheung and S.S. Dragomir, On Ostrowski type inequalities for Stieltjes integrals with absolutely continuous integrands and integrators of bounded variation, *Computers and Mathematics with Applications* **54** (2007) 183–191.
[8] K. Diethelm, A note on the midpoint rectangle formula for Riemann-Stieltjes integrals. *J. Stat. Comput. Simul.* **74** (2004), no. 12, 920–922.
[9] S.S. Dragomir, Some inequalities for Riemann-Stieltjes integral and applications, in: A. Rubinov and B. Glover (eds.), *Optimization and Related Topics*, 197-235, Kluwer Academic Publishers, 2001.
[10] S.S. Dragomir, On the Ostrowski's inequality for Riemann-Stieltjes integral, *Korean J. Appl. Math.,* **7** (2000), 477–485.
[11] S.S. Dragomir, Approximating the Riemann–Stieltjes integral by a trapezoidal quadrature rule with applications, *Mathematical and Computer Modelling,* In Press, Corrected Proof, Available online 18 February 2011.
[12] S.S. Dragomir, Approximating the Riemann Stieltjes integral in terms of generalised trapezoidal rules, *Nonlinear Analysis* **71** (2009) e62 -e72.
[13] S.S. Dragomir, Inequalities for Stieltjes integrals with convex integrators and applications, *Applied Mathematics Letters* **20** (2007) 123–130.
[14] S.S. Dragomir, Accurate approximations of the Riemann-Stieltjes integral with (l, L)-Lipschitzian integrators, *AIP Conf. Proc. 939, Numerical Anal. & Appl. Math.* , Ed. T.H. Simos et al., pp. 686-690. Preprint *RGMIA Res. Rep. Coll.* **10**(2007), No. 3, Article 5. `[Online http://rgmia.vu.edu.au/v10n3.html].`
[15] S. S. Dragomir, Accurate approximations for the Riemann-Stieltjes integral via theory of inequalities. *J. Math. Inequal.* **3** (2009), no. 4, 663–681.
[16] S.S. Dragomir and I. Fedotov, An inequality of Grüss type for the Riemann-Stieltjes integral and applications for special means, *Tamkang J. Math.,* **29**(4) (1998), 287-292.
[17] S.S. Dragomir and I. Fedotov, A Grüss type inequality for mappings of bounded variation and applications to numerical analysis, *Nonlinear Funct. Anal. Appl.*, **6**(3) (2001), 425-433.
[18] S. Dubuc and F. Todor, La règle du trapèze pour l'intégrale de Riemann-Stieltjes. I, II. (French) [The trapezoid formula for the Riemann-Stieltjes integral. I, II] *Ann. Sci. Math. Québec* **8** (1984), no. 2, 135–140, 141–153.
[19] S. Dubuc and F. Todor, La règle optimale du trapèze pour l'intégrale de Riemann-Stieltjes d'une fonction donnée. (French) [The optimal trapezoidal rule for the Riemann-Stieltjes integral of a given function] *C. R. Math. Rep. Acad. Sci. Canada* **9** (1987), no. 5, 213–218.
[20] Z. Liu, Refinement of an inequality of Grüss type for Riemann-Stieltjes integral. *Soochow J. Math.* 30 (2004), no. 4, 483–489.
[21] P.R. Mercer, Hadamard's inequality and trapezoid rules for the Riemann-Stieltjes integral. *J. Math. Anal. Appl.* **344** (2008), no. 2, 921–926.
[22] M. Munteanu, Quadrature formulas for the generalized Riemann-Stieltjes integral. *Bull. Braz. Math. Soc.* (N.S.) **38** (2007), no. 1, 39–50.
[23] M. Tortorella, Closed Newton-Cotes quadrature rules for Stieltjes integrals and numerical convolution of life distributions. *SIAM J. Sci. Statist. Comput.* **11** (1990), no. 4, 732–748.
[24] D. Mozyrska, E. Pawłuszewicz and Delfim F. M. Torres, The Riemann-Stieltjes integral on time scales. *Aust. J. Math. Anal. Appl.* **7** (2010), no. 1, Art. 10, 14 pp.

School of Computational & Applied Mathematics, University of the Witwatersrand, Johannesburg, SA and School of Engineering and Science, Victoria University, Melbourne, AU