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ERROR ESTIMATES IN APPROXIMATING FUNCTIONS OF SELFADJOINT OPERATORS IN HILBERT SPACES VIA A MONTGOMERY'S TYPE EXPANSION

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Abstract. On utilizing the spectral representation of selfadjoint operators in Hilbert spaces, some error estimates in approximating n-time differentiable functions of selfadjoint operators in Hilbert spaces via a Montgomery's type expansion are given.

1. Introduction

Let U be a selfadjoint operator on the complex Hilbert space $(H, \langle .,. \rangle)$ with the spectrum $Sp(U)$ included in the interval $[m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its *spectral family*. Then for any continuous function $f : [m, M] \to \mathbb{C}$, it is well known that we have the following *spectral representation in terms of the Riemann-Stieltjes integral*:

$$f(U) = \int_{m-0}^{M} f(\lambda)\, dE_\lambda, \tag{1.1}$$

which in terms of vectors can be written as

$$\langle f(U)x, y\rangle = \int_{m-0}^{M} f(\lambda)\, d\langle E_\lambda x, y\rangle, \tag{1.2}$$

for any $x, y \in H$. The function $g_{x,y}(\lambda) := \langle E_\lambda x, y\rangle$ is of *bounded variation* on the interval $[m, M]$ and

$$g_{x,y}(m-0) = 0 \text{ and } g_{x,y}(M) = \langle x, y\rangle$$

for any $x, y \in H$. It is also well known that $g_x(\lambda) := \langle E_\lambda x, x\rangle$ is *monotonic nondecreasing* and *right continuous* on $[m, M]$.

For a recent monograph devoted to various inequalities for continuous functions of selfadjoint operators, see [10] and the references therein.

For other recent results see [3], [4], [5], [12], [13], [14] and [15].

Utilising the spectral representation from (1.2) we have established in [7] the following Ostrowski type vector inequality:

Theorem 1. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral*

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family. If $f:[m,M]\to\mathbb{C}$ is a continuous function of bounded variation on $[m,M]$, then we have the inequality

$$
\begin{aligned}
(1.3)\qquad & |f(s)\langle x,y\rangle-\langle f(A)x,y\rangle| \\
& \leq \langle E_s x,x\rangle^{1/2}\langle E_s y,y\rangle^{1/2}\bigvee_m^s(f) \\
& +\langle(1_H-E_s)x,x\rangle^{1/2}\langle(1_H-E_s)y,y\rangle^{1/2}\bigvee_s^M(f) \\
& \leq \|x\|\|y\|\left(\frac{1}{2}\bigvee_m^M(f)+\frac{1}{2}\left|\bigvee_m^s(f)-\bigvee_s^M(f)\right|\right)\left(\leq\|x\|\|y\|\bigvee_m^M(f)\right)
\end{aligned}
$$

for any $x,y\in H$ and for any $s\in[m,M]$.

Another result that compares the function of a selfadjoint operator with the integral mean is embodied in the following theorem [8]:

Theorem 2. *With the assumptions in Theorem 1 we have the inequalities*

$$
\begin{aligned}
(1.4)\qquad & \left|\langle x,y\rangle\cdot\frac{1}{M-m}\int_m^M f(s)\,ds-\langle f(A)x,y\rangle\right| \\
& \leq \frac{1}{M-m}\bigvee_m^M(f)\max_{t\in[m,M]}\Big[(M-t)\langle E_t x,x\rangle^{1/2}\langle E_t y,y\rangle^{1/2} \\
& +(t-m)\langle(1_H-E_t)x,x\rangle^{1/2}\langle(1_H-E_t)y,y\rangle^{1/2}\Big] \\
& \leq \|x\|\|y\|\bigvee_m^M(f)
\end{aligned}
$$

for any $x,y\in H$.

The trapezoid version of the above result has been obtained in [6] and is as follows:

Theorem 3. *With the assumptions in Theorem 1 we have the inequalities*

$$
\begin{aligned}
(1.5)\qquad & \left|\frac{f(M)+f(m)}{2}\cdot\langle x,y\rangle-\langle f(A)x,y\rangle\right| \\
& \leq \frac{1}{2}\max_{\lambda\in[m,M]}\Big[\langle E_\lambda x,x\rangle^{1/2}\langle E_\lambda y,y\rangle^{1/2} \\
& +\langle(1_H-E_\lambda)x,x\rangle^{1/2}\langle(1_H-E_\lambda)y,y\rangle^{1/2}\Big]\bigvee_m^M(f) \\
& \leq \frac{1}{2}\|x\|\|y\|\bigvee_m^M(f)
\end{aligned}
$$

for any $x,y\in H$.

A generalized trapezoid type inequality was obtained in [9] and is as follows:

Theorem 4. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family.*

1. If $f : [m, M] \to \mathbb{C}$ is continuous and of bounded variation on $[m, M]$, then

$$\left| \left\langle \left[\frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M - m} \right] x, y \right\rangle - \langle f(A)x, y\rangle \right| \tag{1.6}$$
$$\le \sup_{t\in[m,M]} \left[\frac{t-m}{M-m} \bigvee_{m-0}^{t} \left(\left\langle E_{(\cdot)}x, y\right\rangle\right) + \frac{M-t}{M-m} \bigvee_{t}^{M} \left(\left\langle E_{(\cdot)}x, y\right\rangle\right) \right] \bigvee_{m}^{M}(f)$$
$$\le \bigvee_{m}^{M} \left(\left\langle E_{(\cdot)}x, y\right\rangle\right) \bigvee_{m}^{M}(f) \le \|x\| \|y\| \bigvee_{m}^{M}(f)$$

for any $x, y \in H$.

2. If $f : [m, M] \to \mathbb{C}$ is Lipschitzian with the constant $L > 0$ on $[m, M]$, then

$$\left| \left\langle \left[\frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M - m} \right] x, y \right\rangle - \langle f(A)x, y\rangle \right| \tag{1.7}$$
$$\le L \int_{m-0}^{M} \left[\frac{t-m}{M-m} \bigvee_{m}^{t} \left(\left\langle E_{(\cdot)}x, y\right\rangle\right) + \frac{M-t}{M-m} \bigvee_{t}^{M} \left(\left\langle E_{(\cdot)}x, y\right\rangle\right) \right] dt$$
$$\le L(M-m) \bigvee_{m}^{M} \left(\left\langle E_{(\cdot)}x, y\right\rangle\right) \le L(M-m)\|x\| \|y\|$$

for any $x, y \in H$.

3. If $f : [m, M] \to \mathbb{R}$ is continuous and monotonic nondecreasing on $[m, M]$, then

$$\left| \left\langle \left[\frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M - m} \right] x, y \right\rangle - \langle f(A)x, y\rangle \right| \tag{1.8}$$
$$\le \int_{m-0}^{M} \left[\frac{t-m}{M-m} \bigvee_{m-0}^{t} \left(\left\langle E_{(\cdot)}x, y\right\rangle\right) + \frac{M-t}{M-m} \bigvee_{t}^{M} \left(\left\langle E_{(\cdot)}x, y\right\rangle\right) \right] df(t)$$
$$\le \bigvee_{m}^{M} \left(\left\langle E_{(\cdot)}x, y\right\rangle\right) [f(M) - f(m)] \le \|x\| \|y\| [f(M) - f(m)]$$

for any $x, y \in H$.

In this paper, by utilizing the spectral representation of selfadjoint operators in Hilbert spaces, some error estimates in approximating n-time differentiable functions of selfadjoint operators in Hilbert spaces via a Montgomery's type expansion are given. The obtained results generalize the inequality from Theorem 2. Applications for some elementary functions of interest including the exponential and logarithmic functions are also provided.

2. Some Identities

In [2], the authors obtained the following integral identity:

Lemma 1. *Let $f : [a, b] \to \mathbb{R}$ be a mapping such that the $(n-1)$-derivative $f^{(n-1)}$ (where $n \ge 1$) is absolutely continuous on $[a, b]$. Then for all $x \in [a, b]$, we have the* *identity:*

$$\int_a^b f(t)\,dt = \sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x) + (-1)^n\int_a^b K_n(x,t)\,f^{(n)}(t)\,dt \tag{2.1}$$

where the kernel $K_n:[a,b]^2\to\mathbb{R}$ *is given by*

$$K_n(x,t):=\begin{cases}\frac{(t-a)^n}{n!}, & a\le t\le x\le b\\ \\ \frac{(t-b)^n}{n!}, & a\le x<t\le b.\end{cases} \tag{2.2}$$

The identity (2.2) can be written in the following equivalent form as:

$$\begin{aligned} f(z) &= \frac{1}{b-a}\int_a^b f(t)\,dt \\ &\quad -\frac{1}{b-a}\sum_{k=1}^{n-1}\frac{1}{(k+1)!}\left[(b-z)^{k+1}+(-1)^k(z-a)^{k+1}\right]f^{(k)}(z) \\ &\quad +\frac{(-1)^{n-1}}{(b-a)\,n!}\left[\int_a^z (t-a)^n f^{(n)}(t)\,dt+\int_z^b (t-b)^n f^{(n)}(t)\,dt\right] \end{aligned} \tag{2.3}$$

for all $z\in[a,b]$.

Note that for $n=1$, the sum $\sum_{k=1}^{n-1}$ is empty and we obtain the well known *Montgomery's identity* (see for example [1])

$$\begin{aligned} f(z) &= \frac{1}{b-a}\int_a^b f(t)\,dt \\ &\quad +\frac{1}{b-a}\left[\int_a^z (t-a)\,f^{(1)}(t)\,dt+\int_z^b (t-b)\,f^{(1)}(t)\,dt\right], \end{aligned} \tag{2.4}$$

for any $z\in[a,b]$.

In a slightly more general setting, by the use of the identity (2.3), we can state the following result as well:

Lemma 2. *Let* $f:[a,b]\to\mathbb{R}$ *be a mapping such that the n-derivative* $f^{(n)}$ *(where* $n\ge 1$*) is of bounded variation on* $[a,b]$*. Then for all* $\lambda\in[a,b]$*, we have the identity:*

$$\begin{aligned} f(\lambda) &= \frac{1}{b-a}\int_a^b f(t)\,dt \\ &\quad -\frac{1}{b-a}\sum_{k=1}^{n}\frac{1}{(k+1)!}\left[(b-\lambda)^{k+1}+(-1)^k(\lambda-a)^{k+1}\right]f^{(k)}(\lambda) \\ &\quad +\frac{(-1)^n}{(b-a)(n+1)!} \\ &\quad \times\left[\int_a^\lambda (t-a)^{n+1}\,d\left(f^{(n)}(t)\right)+\int_\lambda^b (t-b)^{n+1}\,d\left(f^{(n)}(t)\right)\right]. \end{aligned} \tag{2.5}$$

Now we can state the following representation result for functions of selfadjoint operators:

Theorem 5. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$, $\{E_\lambda\}_\lambda$ be its spectral family, I be a closed subinterval on $\mathbb{R}$ with $[m, M] \subset \mathring{I}$ (the interior of I) and let n be an integer with $n \geq 1$. If $f : I \to \mathbb{C}$ is such that the n-th derivative $f^{(n)}$ is of bounded variation on the interval $[m, M]$, then we have the representation*

$$f(A) = \left(\frac{1}{M-m}\int_m^M f(t)\,dt\right)1_H - \frac{1}{M-m} \times \sum_{k=1}^{n} \frac{1}{(k+1)!}\left[(M1_H - A)^{k+1} + (-1)^k (A - m1_H)^{k+1}\right] f^{(k)}(A) + T_n(A, m, M) \tag{2.6}$$

where the remainder is given by

$$T_n(A, m, M) := \frac{(-1)^n}{(M-m)(n+1)!} \times \left[\int_{m-0}^{M}\left(\int_m^\lambda (t-m)^{n+1}\, d\left(f^{(n)}(t)\right)\right) dE_\lambda + \int_{m-0}^{M}\left(\int_\lambda^M (t-M)^{n+1}\, d\left(f^{(n)}(t)\right)\right) dE_\lambda\right]. \tag{2.7}$$

In particular, if the n-th derivative $f^{(n)}$ is absolutely continuous on $[m, M]$, then the remainder can be represented as

$$T_n(A, m, M) = \frac{(-1)^n}{(M-m)(n+1)!} \times \int_{m-0}^{M}\left[(\lambda - m)^{n+1}(1_H - E_\lambda) + (\lambda - M)^{n+1} E_\lambda\right] f^{(n+1)}(\lambda)\, d\lambda. \tag{2.8}$$

Proof. By Lemma 2 we have

$$f(\lambda) = \frac{1}{M-m}\int_m^M f(t)\,dt - \frac{1}{M-m} \times \sum_{k=1}^{n}\frac{1}{(k+1)!}\left[(M-\lambda)^{k+1} + (-1)^k(\lambda - m)^{k+1}\right] f^{(k)}(\lambda) + \frac{(-1)^n}{(M-m)(n+1)!} \times \left[\int_m^\lambda (t-m)^{n+1}\, d\left(f^{(n)}(t)\right) + \int_\lambda^M (t-M)^{n+1}\, d\left(f^{(n)}(t)\right)\right] \tag{2.9}$$

for any $\lambda \in [m, M]$.

Integrating the identity (2.9) in the Riemann-Stieltjes sense with the integrator E_λ we get

$$
\begin{aligned}
(2.10)\qquad & \int_{m-0}^{M} f(\lambda)\, dE_\lambda \\
&= \frac{1}{M-m}\int_m^M f(t)\,dt \int_{m-0}^{M} dE_\lambda - \frac{1}{M-m} \\
&\times \sum_{k=1}^{n} \frac{1}{(k+1)!}\int_{m-0}^{M}\left[(M-\lambda)^{k+1} + (-1)^k (\lambda-m)^{k+1}\right] f^{(k)}(\lambda)\, dE_\lambda \\
&+ T_n(A,m,M).
\end{aligned}
$$

Since, by the spectral representation (1.1) we have

$$
\int_{m-0}^{M} f(\lambda)\, dE_\lambda = f(A),\ \int_{m-0}^{M} dE_\lambda = 1_H
$$

and

$$
\begin{aligned}
&\int_{m-0}^{M}\left[(M-\lambda)^{k+1} + (-1)^k (\lambda-m)^{k+1}\right] f^{(k)}(\lambda)\, dE_\lambda \\
&= \left[(M1_H - A)^{k+1} + (-1)^k (A - m1_H)^{k+1}\right] f^{(k)}(A),
\end{aligned}
$$

then by (2.10) we deduce the representation (2.6).

Now, if the n-th derivative $f^{(n)}$ is absolutely continuous on $[m,M]$, then

$$
\int_m^\lambda (t-m)^{n+1}\, d\left(f^{(n)}(t)\right) = \int_m^\lambda (t-m)^{n+1} f^{(n+1)}(t)\,dt
$$

and

$$
\int_\lambda^M (t-M)^{n+1}\, d\left(f^{(n)}(t)\right) = \int_\lambda^M (t-M)^{n+1} f^{(n+1)}(t)\,dt
$$

where the integrals in the right hand side are taken in the Lebesgue sense.

Utilising the integration by parts formula for the Riemann-Stieltjes integral and the differentiation rule for the Stieltjes integral we have successively

$$
\begin{aligned}
&\int_{m-0}^{M}\left(\int_m^\lambda (t-m)^{n+1} f^{(n+1)}(t)\,dt\right) dE_\lambda \\
&= \left(\int_m^\lambda (t-m)^{n+1} f^{(n+1)}(t)\,dt\right) E_\lambda \Big|_{m-0}^{M} - \int_{m-0}^{M} (\lambda-m)^{n+1} f^{(n+1)}(\lambda)\, E_\lambda d\lambda \\
&= \left(\int_m^M (t-m)^{n+1} f^{(n+1)}(t)\,dt\right) 1_H - \int_{m-0}^{M} (\lambda-m)^{n+1} f^{(n+1)}(\lambda)\, E_\lambda d\lambda \\
&= \int_{m-0}^{M} (\lambda-m)^{n+1} f^{(n+1)}(\lambda)\,(1_H - E_\lambda)\, d\lambda
\end{aligned}
$$

and

$$\begin{aligned}&\int_{m-0}^{M}\left(\int_{\lambda}^{M}(t-M)^{n+1}f^{(n+1)}(t)\,dt\right)dE_{\lambda}\\ =&\left.\left(\int_{\lambda}^{M}(t-M)^{n+1}f^{(n+1)}(t)\,dt\right)E_{\lambda}\right|_{m-0}^{M}+\int_{m-0}^{M}(\lambda-M)^{n+1}f^{(n+1)}(\lambda)\,E_{\lambda}d\lambda\\ =&\int_{m-0}^{M}(\lambda-M)^{n+1}f^{(n+1)}(\lambda)\,E_{\lambda}d\lambda\end{aligned}$$

and the representation (2.8) is thus obtained. □

Remark 1. *Let A be a positive selfadjoint operator in the Hilbert space H with the spectrum $Sp(A)\subseteq[m,M]$ for some positive real numbers $0<m<M$ and $\{E_{\lambda}\}_{\lambda}$ be its spectral family. Then, for $n\geq 1$, we have the equality*

$$\begin{aligned}\ln A=&\left[\ln I(m,M)\right]1_H+\frac{1}{M-m}\\ &\times\sum_{k=1}^{n}\frac{1}{k(k+1)}\left[(A-m1_H)^{k+1}+(-1)^k(M1_H-A)^{k+1}\right]A^{-k}\\ &+\frac{1}{(M-m)(n+1)}\\ &\times\left[\int_{m-0}^{M}\left[(\lambda-m)^{n+1}(1_H-E_{\lambda})+(\lambda-M)^{n+1}E_{\lambda}\right]\lambda^{-n-1}d\lambda\right],\end{aligned}\tag{2.11}$$

where $I(m,M)$ is the identric mean and is defined by

$$I(m,M)=\begin{cases}\frac{1}{e}\left(\frac{M^M}{m^m}\right)^{1/(M-m)} & \text{if } M\neq m;\\ M \text{ if } M=m.\end{cases}$$

Remark 2. *If we introduce the exponential mean by*

$$E(m,M)=\begin{cases}\frac{\exp M-\exp m}{M-m} & \text{if } M\neq m;\\ M \text{ if } M=m\end{cases}$$

and applying the identity (2.6) for the exponential function, we have

$$\begin{aligned}&\left[1_H+\frac{1}{M-m}\sum_{k=1}^{n}\frac{1}{(k+1)!}\left[(M1_H-A)^{k+1}+(-1)^k(A-m1_H)^{k+1}\right]\right]\\ &\times\exp A-E(m,M)\,1_H\\ &=\frac{(-1)^n}{(M-m)(n+1)!}\int_{m-0}^{M}\left[(\lambda-m)^{n+1}(1_H-E_{\lambda})+(\lambda-M)^{n+1}E_{\lambda}\right]e^{\lambda}d\lambda\end{aligned}\tag{2.12}$$

where A is a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A)\subseteq[m,M]$ for some real numbers $m<M$ and $\{E_{\lambda}\}_{\lambda}$ is its spectral family.

3. Error Bounds for $f^{(n)}$ of Bounded Variation

From the identity (2.6), we define for any $x,y\in H$

$$\begin{aligned}(3.1)\quad & T_n(A,m,M;x,y) \\ & := \langle f(A)x,y\rangle + \frac{1}{M-m}\sum_{k=1}^{n}\frac{1}{(k+1)!} \\ & \times\left[\left\langle (M1_H-A)^{k+1}f^{(k)}(A)x,y\right\rangle + (-1)^k\left\langle (A-m1_H)^{k+1}f^{(k)}(A)x,y\right\rangle\right] \\ & -\left(\frac{1}{M-m}\int_m^M f(t)\,dt\right)\langle x,y\rangle .\end{aligned}$$

We have the following result concerning bounds for the absolute value of $T_n(A,m,M;x,y)$ when the n-th derivative $f^{(n)}$ is of bounded variation:

Theorem 6. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A)\subseteq[m,M]$ for some real numbers $m<M$, $\{E_\lambda\}_\lambda$ be its spectral family, I be a closed subinterval on $\mathbb{R}$ with $[m,M]\subset\mathring{I}$ and let n be an integer with $n\geq 1$.*

1. If $f:I\to\mathbb{C}$ is such that the n-th derivative $f^{(n)}$ is of bounded variation on the interval $[m,M]$, then we have the inequalities

$$\begin{aligned}(3.2)\quad & |T_n(A,m,M;x,y)| \\ & \leq \frac{1}{(M-m)(n+1)!}\bigvee_m^M\left(\langle E_{(\cdot)}x,y\rangle\right) \\ & \times\max_{\lambda\in[m,M]}\left[(\lambda-m)^{n+1}\bigvee_m^\lambda\left(f^{(n)}\right)+(M-\lambda)^{n+1}\bigvee_\lambda^M\left(f^{(n)}\right)\right] \\ & \leq\frac{(M-m)^n}{(n+1)!}\bigvee_m^M\left(\langle E_{(\cdot)}x,y\rangle\right)\bigvee_m^M\left(f^{(n)}\right)\leq\frac{(M-m)^n}{(n+1)!}\bigvee_m^M\left(f^{(n)}\right)\|x\|\,\|y\|\end{aligned}$$

for any $x,y\in H$.

2. If $f:I\to\mathbb{C}$ is such that the n-th derivative $f^{(n)}$ is Lipschitzian with the constant $L_n>0$ on the interval $[m,M]$, then we have the inequalities

$$\begin{aligned}(3.3)\quad |T_n(A,m,M;x,y)| & \leq\frac{L_n(M-m)^{n+1}}{(n+2)!}\bigvee_m^M\left(\langle E_{(\cdot)}x,y\rangle\right) \\ & \leq\frac{L_n(M-m)^{n+1}}{(n+2)!}\|x\|\,\|y\|\end{aligned}$$

for any $x,y\in H$.

3. If $f : I \to \mathbb{R}$ is such that the n-th derivative $f^{(n)}$ is monotonic nondecreasing on the interval $[m, M]$, then we have the inequalities

$$
\begin{aligned}
(3.4) \qquad & \left|T_n(A, m, M; x, y)\right| \\
& \leq \frac{1}{(M-m)(n+1)!} \bigvee_m^M \left(\left\langle E_{(\cdot)} x, y\right\rangle\right) \\
& \times \max_{\lambda \in [m, M]} \left[f^{(n)}(\lambda) \left((\lambda - m)^{n+1} - (M-\lambda)^{n+1} \right) \right. \\
& \left. + (n+1) \left[\int_\lambda^M (M-t)^n f^{(n)}(t)\, dt - \int_m^\lambda (t-m)^n f^{(n)}(t)\, dt \right] \right] \\
& \leq \frac{1}{(M-m)(n+1)!} \max_{\lambda \in [m, M]} \left[(\lambda - m)^{n+1} \left[f^{(n)}(\lambda) - f^{(n)}(m) \right] \right. \\
& \left. + (M-\lambda)^{n+1} \left[f^{(n)}(M) - f^{(n)}(\lambda) \right] \right] \bigvee_m^M \left(\left\langle E_{(\cdot)} x, y\right\rangle\right) \\
& \leq \frac{(M-m)^n}{(n+1)!} \bigvee_m^M \left(\left\langle E_{(\cdot)} x, y\right\rangle\right) \left[f^{(n)}(M) - f^{(n)}(m) \right] \\
& \leq \frac{(M-m)^n}{(n+1)!} \left[f^{(n)}(M) - f^{(n)}(m) \right] \|x\| \|y\|
\end{aligned}
$$

for any $x, y \in H$.

Proof. 1. By the identity (2.7) we have for any $x, y \in H$ that

$$
\begin{aligned}
(3.5) \qquad T_n(A, m, M; x, y) := & \frac{(-1)^n}{(M-m)(n+1)!} \\
& \times \left[\int_{m-0}^M \left(\int_m^\lambda (t-m)^{n+1} d\left(f^{(n)}(t) \right) \right) d\left\langle E_\lambda x, y\right\rangle \right. \\
& \left. + \int_{m-0}^M \left(\int_\lambda^M (t-M)^{n+1} d\left(f^{(n)}(t) \right) \right) d\left\langle E_\lambda x, y\right\rangle \right].
\end{aligned}
$$

It is well known that if $p : [a, b] \to \mathbb{C}$ is a continuous function, $v : [a, b] \to \mathbb{C}$ is of bounded variation then the Riemann-Stieltjes integral $\int_a^b p(t)\, dv(t)$ exists and the following inequality holds

$$
(3.6) \qquad \left| \int_a^b p(t)\, dv(t) \right| \leq \max_{t \in [a,b]} |p(t)| \bigvee_a^b (v),
$$

where $\bigvee_a^b (v)$ denotes the total variation of v on $[a, b]$.

Taking the modulus in (3.5) and utilizing the property (3.6), we have successively that

$$|T_n(A,m,M;x,y)| = \frac{1}{(M-m)(n+1)!} \tag{3.7}$$
$$\times\left|\int_{m-0}^{M}\left[\left(\int_m^\lambda (t-m)^{n+1} d\left(f^{(n)}(t)\right) + \left(\int_\lambda^M (t-M)^{n+1} d\left(f^{(n)}(t)\right)\right)\right)\right] d\langle E_\lambda x, y\rangle\right|$$
$$\leq \frac{1}{(M-m)(n+1)!}\bigvee_m^M\left(\langle E_{(\cdot)}x,y\rangle\right)$$
$$\times \max_{\lambda\in[m,M]}\left|\int_m^\lambda (t-m)^{n+1} d\left(f^{(n)}(t)\right) + \int_\lambda^M (t-M)^{n+1} d\left(f^{(n)}(t)\right)\right|$$

for any $x, y \in H$.

By the same property (3.6) we have for $\lambda \in (m, M)$ that

$$\left|\int_m^\lambda (t-m)^{n+1} d\left(f^{(n)}(t)\right)\right| \leq \max_{t\in[m,\lambda]}(t-m)^{n+1}\bigvee_m^\lambda\left(f^{(n)}\right) = (\lambda-m)^{n+1}\bigvee_m^\lambda\left(f^{(n)}\right)$$

and

$$\left|\int_\lambda^M (t-M)^{n+1} d\left(f^{(n)}(t)\right)\right| \leq \max_{t\in[\lambda,M]}(M-t)^{n+1}\bigvee_\lambda^M\left(f^{(n)}\right) = (M-\lambda)^{n+1}\bigvee_\lambda^M\left(f^{(n)}\right)$$

which produce the inequality

$$\left|\int_m^\lambda (t-m)^{n+1} d\left(f^{(n)}(t)\right) + \int_\lambda^M (t-M)^{n+1} d\left(f^{(n)}(t)\right)\right| \leq (\lambda-m)^{n+1}\bigvee_m^\lambda\left(f^{(n)}\right) + (M-\lambda)^{n+1}\bigvee_\lambda^M\left(f^{(n)}\right). \tag{3.8}$$

Taking the maximum over $\lambda \in [m, M]$ in (3.8) and utilizing (3.7) we deduce the first inequality in (3.2).

Now observe that

$$\begin{aligned}
&(\lambda-m)^{n+1}\bigvee_{m}^{\lambda}\left(f^{(n)}\right)+(M-\lambda)^{n+1}\bigvee_{\lambda}^{M}\left(f^{(n)}\right) \\
&\leq \max\left\{(\lambda-m)^{n+1},(M-\lambda)^{n+1}\right\}\left[\bigvee_{m}^{\lambda}\left(f^{(n)}\right)+\bigvee_{\lambda}^{M}\left(f^{(n)}\right)\right] \\
&=\max\left\{(\lambda-m)^{n+1},(M-\lambda)^{n+1}\right\}\bigvee_{m}^{M}\left(f^{(n)}\right) \\
&=\left[\frac{1}{2}(M-m)+\left|\lambda-\frac{m+M}{2}\right|\right]^{n+1}\bigvee_{m}^{M}\left(f^{(n)}\right)
\end{aligned}$$

giving that

$$\begin{aligned}
&\max_{\lambda\in[m,M]}\left[(\lambda-m)^{n+1}\bigvee_{m}^{\lambda}\left(f^{(n)}\right)+(M-\lambda)^{n+1}\bigvee_{\lambda}^{M}\left(f^{(n)}\right)\right] \\
&\leq (M-m)^{n+1}\bigvee_{m}^{M}\left(f^{(n)}\right)
\end{aligned}$$

and the second inequality in (3.2) is proved.

If P is a nonnegative operator on H, i.e., $\langle Px,x\rangle\geq 0$ for any $x\in H$, then the following inequality is a generalization of the Schwarz inequality in H

$$|\langle Px,y\rangle|^{2}\leq\langle Px,x\rangle\langle Py,y\rangle \tag{3.9}$$

for any $x,y\in H$.

Now, if $d: m=t_0<t_1<...<t_{n-1}<t_n=M$ is an arbitrary partition of the interval $[m,M]$, then we have by Schwarz's inequality for nonnegative operators (3.9) that

$$\begin{aligned}
&\bigvee_{m}^{M}\left(\langle E_{(\cdot)}x,y\rangle\right) \\
&=\sup_{d}\left\{\sum_{i=0}^{n-1}\left|\left\langle\left(E_{t_{i+1}}-E_{t_i}\right)x,y\right\rangle\right|\right\} \\
&\leq\sup_{d}\left\{\sum_{i=0}^{n-1}\left[\left\langle\left(E_{t_{i+1}}-E_{t_i}\right)x,x\right\rangle^{1/2}\left\langle\left(E_{t_{i+1}}-E_{t_i}\right)y,y\right\rangle^{1/2}\right]\right\}:=B.
\end{aligned}$$

By the Cauchy-Buniakovski-Schwarz inequality for sequences of real numbers we also have that

$$
\begin{aligned}
B &\leq \sup_{d}\left\{\left[\sum_{i=0}^{n-1}\left\langle\left(E_{t_{i+1}}-E_{t_i}\right)x,x\right\rangle\right]^{1/2}\left[\sum_{i=0}^{n-1}\left\langle\left(E_{t_{i+1}}-E_{t_i}\right)y,y\right\rangle\right]^{1/2}\right\} \\
&\leq \sup_{d}\left\{\left[\sum_{i=0}^{n-1}\left\langle\left(E_{t_{i+1}}-E_{t_i}\right)x,x\right\rangle\right]^{1/2}\sup_{d}\left[\sum_{i=0}^{n-1}\left\langle\left(E_{t_{i+1}}-E_{t_i}\right)y,y\right\rangle\right]^{1/2}\right\} \\
&= \left[\bigvee_{m}^{M}\left(\left\langle E_{(\cdot)}x,x\right\rangle\right)\right]^{1/2}\left[\bigvee_{m}^{M}\left(\left\langle E_{(\cdot)}y,y\right\rangle\right)\right]^{1/2} = \|x\|\,\|y\|
\end{aligned}
$$

for any $x, y \in H$.

These prove the last part of (3.2).

2. Now, recall that if $p : [a, b] \to \mathbb{C}$ is a Riemann integrable function and $v : [a, b] \to \mathbb{C}$ is Lipschitzian with the constant $L > 0$, i.e.,

$$|f(s) - f(t)| \leq L|s - t| \text{ for any } t, s \in [a, b],$$

then the Riemann-Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and the following inequality holds

$$\left|\int_a^b p(t)\,dv(t)\right| \leq L\int_a^b |p(t)|\,dt. \tag{3.10}$$

By the property (3.10) we have for $\lambda \in (m, M)$ that

$$\left|\int_m^\lambda (t-m)^{n+1}\,d\left(f^{(n)}(t)\right)\right| \leq L_n\int_m^\lambda (t-m)^{n+1}\,d(t) = \frac{L_n}{n+2}(\lambda - m)^{n+2}$$

and

$$\left|\int_\lambda^M (t-M)^{n+1}\,d\left(f^{(n)}(t)\right)\right| \leq L_n\int_\lambda^M (M-t)^{n+1}\,dt = \frac{L_n}{n+2}(M-\lambda)^{n+2}.$$

By the inequality (3.7) we then have

$$
\begin{aligned}
&|T_n(A, m, M; x, y)| \\
&\leq \frac{1}{(M-m)(n+1)!}\bigvee_{m}^{M}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \\
&\times \max_{\lambda\in[m,M]}\left[\frac{L_n}{n+2}(\lambda-m)^{n+2}+\frac{L_n}{n+2}(M-\lambda)^{n+2}\right] \\
&= \frac{L_n(M-m)^{n+1}}{(n+2)!}\bigvee_{m}^{M}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \leq \frac{L_n(M-m)^{n+1}}{(n+2)!}\|x\|\,\|y\|
\end{aligned}
\tag{3.11}
$$

for any $x, y \in H$ and the inequality (3.3) is proved.

3. Further, from the theory of Riemann-Stieltjes integral it is also well known that if $p : [a, b] \to \mathbb{C}$ is continuous and $v : [a, b] \to \mathbb{R}$ is monotonic nondecreasing,

then the Riemann-Stieltjes integrals $\int_a^b p(t)\,dv(t)$ and $\int_a^b |p(t)|\,dv(t)$ exist and

$$\left|\int_a^b p(t)\,dv(t)\right| \le \int_a^b |p(t)|\,dv(t) \le \max_{t\in[a,b]} |p(t)|\,[v(b)-v(a)]. \tag{3.12}$$

On making use of (3.12) we have

$$\begin{aligned}\left|\int_m^\lambda (t-m)^{n+1}\,d\left(f^{(n)}(t)\right)\right| &\le \int_m^\lambda (t-m)^{n+1}\,d\left(f^{(n)}(t)\right) \\ &\le (\lambda-m)^{n+1}\left[f^{(n)}(\lambda)-f^{(n)}(m)\right]\end{aligned} \tag{3.13}$$

and

$$\begin{aligned}\left|\int_\lambda^M (t-M)^{n+1}\,d\left(f^{(n)}(t)\right)\right| &\le \int_\lambda^M (M-t)^{n+1}\,d\left(f^{(n)}(t)\right) \\ &\le (M-\lambda)^{n+1}\left[f^{(n)}(M)-f^{(n)}(\lambda)\right]\end{aligned} \tag{3.14}$$

for any $\lambda \in (m, M)$.

Integrating by parts in the Riemann-Stieltjes integral, we also have

$$\begin{aligned}&\int_m^\lambda (t-m)^{n+1}\,d\left(f^{(n)}(t)\right) \\ &= (\lambda-m)^{n+1} f^{(n)}(\lambda) - (n+1)\int_m^\lambda (t-m)^n f^{(n)}(t)\,dt\end{aligned}$$

and

$$\begin{aligned}&\int_\lambda^M (M-t)^{n+1}\,d\left(f^{(n)}(t)\right) \\ &= (n+1)\int_\lambda^M (M-t)^n f^{(n)}(t)\,dt - (M-\lambda)^{n+1} f^{(n)}(\lambda)\end{aligned}$$

for any $\lambda \in (m, M)$.

Therefore, by adding (3.13) with (3.14) we get

$$\begin{aligned}&\left|\int_m^\lambda (t-m)^{n+1}\,d\left(f^{(n)}(t)\right)\right| + \left|\int_\lambda^M (t-M)^{n+1}\,d\left(f^{(n)}(t)\right)\right| \\ &\le \left[f^{(n)}(\lambda)\left((\lambda-m)^{n+1}-(M-\lambda)^{n+1}\right)\right] \\ &+ (n+1)\left[\int_\lambda^M (M-t)^n f^{(n)}(t)\,dt - \int_m^\lambda (t-m)^n f^{(n)}(t)\,dt\right] \\ &\le (\lambda-m)^{n+1}\left[f^{(n)}(\lambda)-f^{(n)}(m)\right] + (M-\lambda)^{n+1}\left[f^{(n)}(M)-f^{(n)}(\lambda)\right]\end{aligned}$$

for any $\lambda \in (m, M)$.

Now, on making use of the inequality (3.7) we deduce (3.4). □

Remark 3. *If we use the inequality (3.2) for the function* ln, *then we get the inequality*

$$\begin{aligned} (3.15)\qquad & |L_n(A,m,M;x,y)| \\ & \leq \frac{1}{(M-m)\,n\,(n+1)} \bigvee_m^M (\langle E_{(\cdot)}x,y\rangle) \\ & \times \max_{\lambda\in[m,M]} \left[(\lambda-m)^{n+1}\frac{\lambda^n-m^n}{\lambda^n m^n} + (M-\lambda)^{n+1}\frac{M^n-\lambda^n}{M^n\lambda^n}\right] \\ & \leq \frac{(M-m)^n(M^n-m^n)}{n\,(n+1)\,M^n m^n} \bigvee_m^M (\langle E_{(\cdot)}x,y\rangle) \\ & \leq \frac{(M-m)^n(M^n-m^n)}{n\,(n+1)\,M^n m^n}\|x\|\,\|y\| \end{aligned}$$

for any $x,y\in H$, *where*

$$\begin{aligned} (3.16)\qquad & L_n(A,m,M;x,y) \\ & := \langle \ln Ax,y\rangle - [\ln I(m,M)]\langle x,y\rangle \\ & - \frac{1}{M-m}\sum_{k=1}^n \frac{1}{k\,(k+1)} \\ & \times \left[\left\langle (A-m1_H)^{k+1}A^{-k}x,y\right\rangle + (-1)^k\left\langle (M1_H-A)^{k+1}A^{-k}x,y\right\rangle\right]. \end{aligned}$$

If we use the inequality (3.3) for the function ln *we get the following bound as well*

$$\begin{aligned} (3.17)\qquad & |L_n(A,m,M;x,y)| \\ & \leq \frac{1}{(n+1)(n+2)}\left(\frac{M}{m}-1\right)^{n+1}\bigvee_m^M (\langle E_{(\cdot)}x,y\rangle) \\ & \leq \frac{1}{(n+1)(n+2)}\left(\frac{M}{m}-1\right)^{n+1}\|x\|\,\|y\| \end{aligned}$$

for any $x,y\in H$.

Remark 4. *If we define*

$$\begin{aligned} (3.18)\quad & E_n(A,m,M;x,y) \\ := & \left\langle \left[1_H + \frac{1}{M-m}\sum_{k=1}^n \frac{1}{(k+1)!}\left[(M1_H-A)^{k+1} + (-1)^k (A-m1_H)^{k+1}\right]\right]\exp Ax, y\right\rangle \\ & - E(m,M)\langle x,y\rangle, \end{aligned}$$

then by the inequality (3.2) we have

$$\begin{aligned}(3.19)\quad & |E_n(A,m,M;x,y)| \\ & \leq \frac{1}{(M-m)(n+1)!}\bigvee_m^M(\langle E_{(\cdot)}x,y\rangle) \\ & \times \max_{\lambda\in[m,M]}\left[(\lambda-m)^{n+1}\left(e^{\lambda}-e^m\right)+(M-\lambda)^{n+1}\left(e^M-e^{\lambda}\right)\right] \\ & \leq \frac{(M-m)^n}{(n+1)!}\bigvee_m^M(\langle E_{(\cdot)}x,y\rangle)\left(e^M-e^m\right) \leq \frac{(M-m)^n}{(n+1)!}\left(e^M-e^m\right)\|x\|\|y\|\end{aligned}$$

for any $x,y\in H$.

If we use the inequality (3.3) for the function exp *we get the following bound as well*

$$\begin{aligned}(3.20)\quad |E_n(A,m,M;x,y)| & \leq \frac{e^M(M-m)^{n+1}}{(n+2)!}\bigvee_m^M(\langle E_{(\cdot)}x,y\rangle) \\ & \leq \frac{e^M(M-m)^{n+1}}{(n+2)!}\|x\|\|y\|\end{aligned}$$

for any $x,y\in H$.

4. Error Bounds for $f^{(n)}$ Absolutely Continuous

We consider the Lebesgue norms defined by

$$\|g\|_{[a,b],\infty} := ess\sup_{t\in[a,b]}|g(t)| \text{ if } g\in L_\infty[a,b]$$

and

$$\|g\|_{[a,b],p} := \left(\int_a^b|g(t)|^p\,dt\right)^{1/p} \text{ if } g\in L_p[a,b],\, p\geq 1.$$

Theorem 7. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A)\subseteq[m,M]$ for some real numbers $m<M$, $\{E_\lambda\}_\lambda$ be its spectral family, I be a closed subinterval on $\mathbb{R}$ with $[m,M]\subset\mathring{I}$ and let n be an integer with $n\geq 1$. If the n-th derivative $f^{(n)}$ is absolutely continuous on $[m,M]$, then*

$$\begin{aligned}(4.1)\quad |T_n(A,m,M;x,y)| & \leq \frac{1}{(M-m)(n+1)!} \\ \times\int_{m-0}^M & \left|(\lambda-m)^{n+1}\langle(1_H-E_\lambda)x,y\rangle+(\lambda-M)^{n+1}\langle E_\lambda x,y\rangle\right|\left|f^{(n+1)}(\lambda)\right|d\lambda. \\ & \leq \frac{1}{(M-m)(n+1)!} \\ \times & \begin{cases} B_{n,1}(A,m,M;x,y)\left\|f^{(n)}\right\|_{[m,M],\infty} & \text{if } f^{(n)}\in L_\infty[m,M], \\ B_{n,p}(A,m,M;x,y)\left\|f^{(n)}\right\|_{[m,M],q} & \text{if } f^{(n)}\in L_q[m,M],\, p>1,\, \frac{1}{p}+\frac{1}{q}=1, \\ B_{n,\infty}(A,m,M;x,y)\left\|f^{(n)}\right\|_{[m,M],1}, & \end{cases}\end{aligned}$$

for any $x, y \in H$, *where*

$$B_{n,p}(A, m, M; x, y)$$
$$:= \left(\int_{m-0}^{M} \left| (\lambda - m)^{n+1} \langle (1_H - E_\lambda) x, y \rangle + (\lambda - M)^{n+1} \langle E_\lambda x, y \rangle \right|^p d\lambda \right)^{1/p}, p \geq 1$$

and

$$B_{n,\infty}(A, m, M; x, y)$$
$$:= \sup_{t \in [m,M]} \left| (\lambda - m)^{n+1} \langle (1_H - E_\lambda) x, y \rangle + (\lambda - M)^{n+1} \langle E_\lambda x, y \rangle \right|.$$

Proof. Follows from the representation

$$T_n(A, m, M; x, y)$$
$$= \frac{(-1)^n}{(M - m)(n + 1)!}$$
$$\times \int_{m-0}^{M} \left[(\lambda - m)^{n+1} \langle (1_H - E_\lambda) x, y \rangle + (\lambda - M)^{n+1} \langle E_\lambda x, y \rangle \right] f^{(n+1)}(\lambda) d\lambda$$

for any $x, y \in H$, by taking the modulus and utilizing the Hölder integral inequality. The details are omitted. □

The bounds provided by $B_{n,p}(A, m, M; x, y)$ are not useful for applications, therefore we will establish in the following some simpler, however coarser bounds.

Proposition 1. *With the above notations, we have*

$$B_{n,\infty}(A, m, M; x, y) \leq (M - m)^{n+1} \|x\| \|y\|, \tag{4.2}$$

$$B_{n,1}(A, m, M; x, y) \leq \frac{(2^{n+2} - 1)}{(n + 2) 2^{n+1}} (M - m)^{n+2} \|x\| \|y\| \tag{4.3}$$

and for $p > 1$

$$B_{n,p}(A, m, M; x, y) \leq \frac{(2^{(n+1)p+1} - 1)^{1/p}}{2^{n+1} [(n + 1) p + 1]^{1/p}} (M - m)^{n+1+1/p} \|x\| \|y\| \tag{4.4}$$

for any $x, y \in H$.

Proof. Utilising the triangle inequality for the modulus we have

$$\begin{aligned} & \left| (\lambda - m)^{n+1} \langle (1_H - E_\lambda) x, y \rangle + (\lambda - M)^{n+1} \langle E_\lambda x, y \rangle \right| \\ & \leq (\lambda - m)^{n+1} |\langle (1_H - E_\lambda) x, y \rangle| + (M - \lambda)^{n+1} |\langle E_\lambda x, y \rangle| \\ & \leq \max \left\{ (\lambda - m)^{n+1}, (M - \lambda)^{n+1} \right\} [|\langle (1_H - E_\lambda) x, y \rangle| + |\langle E_\lambda x, y \rangle|] \end{aligned} \tag{4.5}$$

for any $x, y \in H$.

Utilising the generalization of Schwarz's inequality for nonnegative selfadjoint operators (3.9) we have

$$|\langle (1_H - E_\lambda) x, y \rangle| \leq \langle (1_H - E_\lambda) x, x \rangle^{1/2} \langle (1_H - E_\lambda) y, y \rangle^{1/2}$$

and

$$|\langle E_\lambda x, y \rangle| \leq \langle E_\lambda x, x \rangle^{1/2} \langle E_\lambda y, y \rangle^{1/2}$$

for any $x, y \in H$ and $\lambda \in [m, M]$.

Further, by making use of the elementary inequality

$$ac + bd \leq \left(a^2 + b^2\right)^{1/2} \left(c^2 + d^2\right)^{1/2}, a, b, c, d \geq 0$$

we have

$$\begin{aligned} (4.6) \qquad & |\langle (1_H - E_\lambda) x, y\rangle| + |\langle E_\lambda x, y\rangle| \\ & \leq \langle (1_H - E_\lambda) x, x\rangle^{1/2} \langle (1_H - E_\lambda) y, y\rangle^{1/2} + \langle E_\lambda x, x\rangle^{1/2} \langle E_\lambda y, y\rangle^{1/2} \\ & \leq (\langle (1_H - E_\lambda) x, x\rangle + \langle E_\lambda x, x\rangle)^{1/2} (\langle (1_H - E_\lambda) y, y\rangle + \langle E_\lambda y, y\rangle)^{1/2} \\ & = \|x\| \|y\| \end{aligned}$$

for any $x, y \in H$ and $\lambda \in [m, M]$.

Combining (4.5) with (4.6) we deduce that

$$\begin{aligned} (4.7) \qquad & \left| (\lambda - m)^{n+1} \langle (1_H - E_\lambda) x, y\rangle + (\lambda - M)^{n+1} \langle E_\lambda x, y\rangle \right| \\ & \leq \max\left\{ (\lambda - m)^{n+1}, (M - \lambda)^{n+1} \right\} \|x\| \|y\| \end{aligned}$$

for any $x, y \in H$ and $\lambda \in [m, M]$.

Taking the supremum over $\lambda \in [m, M]$ in (4.7) we deduce the inequality (4.2).

Now, if we take the power $r \geq 1$ in (4.7) and integrate, then we get

$$\begin{aligned} (4.8) \qquad & \int_{m-0}^{M} \left| (\lambda - m)^{n+1} \langle (1_H - E_\lambda) x, y\rangle + (\lambda - M)^{n+1} \langle E_\lambda x, y\rangle \right|^r d\lambda \\ & \leq \|x\|^r \|y\|^r \int_m^M \max\left\{ (\lambda - m)^{(n+1)r}, (M - \lambda)^{(n+1)r} \right\} d\lambda \\ & = \|x\|^r \|y\|^r \left[\int_m^{\frac{M+m}{2}} (M - \lambda)^{(n+1)r} d\lambda + \int_{\frac{M+m}{2}}^M (\lambda - m)^{(n+1)r} d\lambda \right] \\ & = \frac{\left(2^{(n+1)r+1} - 1\right)}{[(n+1)r + 1] 2^{(n+1)r}} (M - m)^{(n+1)r+1} \|x\|^r \|y\|^r \end{aligned}$$

for any $x, y \in H$.

Utilizing (4.8) for $r = 1$ we deduce the bound (4.3). Also, by making $r = p$ and then taking the power $1/p$, we deduce the last inequality (4.4). □

The following result provides refinements of the inequalities in Proposition 1:

Proposition 2. *With the above notations, we have*

$$\begin{aligned} (4.9) \quad & B_{n,\infty}(A, m, M; x, y) \\ & \leq \|y\| \max_{\lambda \in [m,M]} \left[(\lambda - m)^{2(n+1)} \langle (1_H - E_\lambda) x, x\rangle + (M - \lambda)^{2(n+1)} \langle E_\lambda x, x\rangle \right]^{1/2} \\ & \qquad \leq (M - m)^{n+1} \|x\| \|y\|, \end{aligned}$$

$$B_{n,1}(A,m,M;x,y) \le \|y\| \int_{m-0}^{M} \left[(\lambda-m)^{2(n+1)} \langle (1_H - E_\lambda) x, x\rangle + (M-\lambda)^{2(n+1)} \langle E_\lambda x, x\rangle \right]^{1/2} d\lambda \le \frac{\left(2^{n+2}-1\right)}{(n+2)\,2^{n+1}} (M-m)^{n+2} \|x\| \|y\| \tag{4.10}$$

and for $p>1$

$$B_{n,p}(A,m,M;x,y) \le \|y\| \left(\int_{m-0}^{M} \left[(\lambda-m)^{2(n+1)} \langle (1_H - E_\lambda) x, x\rangle + (M-\lambda)^{2(n+1)} \langle E_\lambda x, x\rangle \right]^{p/2} d\lambda \right)^{1/p} \le \frac{\left(2^{(n+1)p+1}-1\right)^{1/p}}{2^{n+1}\left[(n+1)p+1\right]^{1/p}} (M-m)^{n+1+1/p} \|x\| \|y\| \tag{4.11}$$

for any $x,y\in H$.

Proof. Utilising the Schwarz inequality in H, we have

$$\left| \left\langle (\lambda-m)^{n+1} (1_H - E_\lambda) x + (\lambda-M)^{n+1} E_\lambda x, y \right\rangle \right| \le \|y\| \left\| (\lambda-m)^{n+1} (1_H - E_\lambda) x + (\lambda-M)^{n+1} E_\lambda x \right\| \tag{4.12}$$

for any $x,y\in H$.

Since E_λ are projectors for each $\lambda\in[m,M]$, then we have

$$\begin{aligned} &\left\| (\lambda-m)^{n+1} (1_H - E_\lambda) x + (\lambda-M)^{n+1} E_\lambda x \right\|^2 \\ &= (\lambda-m)^{2(n+1)} \|(1_H - E_\lambda) x\|^2 \\ &+ 2(\lambda-m)^{n+1} (\lambda-M)^{n+1} \operatorname{Re}\langle (1_H - E_\lambda) x, E_\lambda x\rangle \\ &+ (M-\lambda)^{2(n+1)} \|E_\lambda x\|^2 \\ &= (\lambda-m)^{2(n+1)} \|(1_H - E_\lambda) x\|^2 + (M-\lambda)^{2(n+1)} \|E_\lambda x\|^2 \\ &= (\lambda-m)^{2(n+1)} \langle (1_H - E_\lambda) x, x\rangle + (M-\lambda)^{2(n+1)} \langle E_\lambda x, x\rangle \\ &\le \|x\|^2 \max\left\{ (\lambda-m)^{2(n+1)}, (M-\lambda)^{2(n+1)} \right\} \end{aligned} \tag{4.13}$$

for any $x,y\in H$ and $\lambda\in[m,M]$.

On making use of (4.12) and (4.13) we obtain the following refinement of (4.7)

$$\begin{aligned} &\left| \left\langle (\lambda-m)^{n+1} (1_H - E_\lambda) x + (\lambda-M)^{n+1} E_\lambda x, y \right\rangle \right| \\ &\le \|y\| \left[(\lambda-m)^{2(n+1)} \langle (1_H - E_\lambda) x, x\rangle + (M-\lambda)^{2(n+1)} \langle E_\lambda x, x\rangle \right]^{1/2} \\ &\le \max\left\{ (\lambda-m)^{n+1}, (M-\lambda)^{n+1} \right\} \|x\| \|y\| \end{aligned} \tag{4.14}$$

for any $x,y\in H$ and $\lambda\in[m,M]$.

The proof now follows the lines of the proof from Proposition 1 and we omit the details. □

Remark 5. *One can apply Theorem 7 and Proposition 1 for particular functions including the exponential and logarithmic function. However the details are left to the interested reader.*

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