Received 25/03/11

Modified Noor iterative procedure for uniformly continuous mappings in Banach spaces[*][†]

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Abstract. In this paper, a strong convergence theorem is obtained for three uniformly continuous mappings in real Banach spaces. Our results extend, improve and generalize the recent results of Chang et al.(2009) among others.

1 Introduction

Let E be an arbitrary real Banach Space and let $J : E \to 2^{E^*}$ be the normalized duality mapping defined by

$$J(x) = \{f \in E^* :< x, f >= \|x\|^2 = \|f\|^2\}, \forall x \in E$$

where E^* denotes the dual space of E and $< .,. >$ denotes the generalized duality pairing between E and E^*. The single-valued normalized duality mapping is denoted by j. Let K be a nonempty closed convex subset of E and $T : K \to K$ be a map.
The mapping T is said to be uniformly L- Lipschitzian if there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \leq L\|x - y\|$$

for any $x, y \in K$ and $\forall n \geq 1$.
The mapping T is said to be asymptotically pseudocontractive if there exists a sequence $(k_n) \subset [1, \infty)$ with $lim_{n\to\infty} k_n = 1$ and for any $x, y \in K$ there exists $j(x - y) \in J(x - y)$ such that

$$< T^n x - T^n y, j(x - y) >\leq k_n\|x - y\|^2, \forall n \geq 1.$$

The concept of asymptotically pseudocontractive mappings was introduced by Schu[14].

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[*]2000 AMS Mathematics Classification: 47H10, 46A03

[†]KEYWORD AND PHRASES: Noor iteration; Strongly accretive mappings; Strongly pseudocontractive mappings; Strongly Φ- pseudocontractive operators; Banach spaces

Recently, Chang et al.[4] pointed out some gaps in the proofs of result in [12] and then proved a strong convergence theorem for a pair of L-Lipschitzian mappings instead of a single map used in [12]. In fact,they proved the following theorem :

Theorem 1.1 ([4]). Let E be a real Banach space, K be a nonempty closed convex subset of E, $T_i : K \to K$,$(i = 1, 2)$ be two uniformly L_i-Lipschitzian mappings with $F(T_1) \cap F(T_2) \neq \phi$, where $F(T_i)$ is the set of fixed points of T_i in K and ρ be a point in $F(T_1) \cap F(T_2)$. Let $k_n \subset [1, \infty)$ be a sequence with $k_n \to 1$. Let $\{x_n\}$ and $\{\beta_n\}$ be two sequences in $[0, 1]$ satisfying the following conditions:
(i).$\sum_{n=1}^{\infty} \alpha_n = \infty$ (ii).$\sum_{n=1}^{\infty} \alpha_n^2 < \infty$ (iii).$\sum_{n=1}^{\infty} \beta_n < \infty$
(iv).$\sum_{n=1}^{\infty} \alpha_n(k_n - 1) < \infty$.
For any $x_0 \in K$, let $\{x_n\}$ be the iterative sequence defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T_1^n y_n$$
$$y_n = (1 - \beta_n)x_n + \beta_n T_2^n x_n.$$

If there exists a strictly increasing function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$ such that

$$< T_1^n x_n - \rho, j(x_n - \rho) > \leq k_n \|x_n - \rho\|^2 - \varphi(\|x_n - \rho\|)$$

for all $j(x - \rho) \in J(x - \rho)$ and $x \in K$, (i=1,2), then $\{x_n\}$ converges strongly to ρ.
The result above extends and improves the corresponding results of [12] from one uniformly Lipschitzian asymptotically pseudocontractive mapping to two uniformly Lipschitzian asymptotically pseudocontractive mappings. In fact, if the iteration parameter $\{\beta_n\}$ in Theorem 1 above is equal to zero for all n and $T_1 = T_2 = T$ then we have the main result of Ofoedu [12].
Rafiq [13], introduced a new type of iteration- the modified three-step iteration process, to approximate the common fixed point of three strongly pseudocontractive mappings in a real Banach space. It is defined as follows:
Let $T_1, T_2, T_3 : K \to K$ be three mappings. For any given $x_0 \in K$, the modified Noor iteration $\{x_n\}_{n=0}^{\infty} \subset K$ is defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T_1 y_n$$
$$y_n = (1 - \beta_n)x_n + \beta_n T_2 z_n$$
$$z_n = (1 - \gamma_n)x_n + \gamma_n T_3 x_n, \quad \geq 0 \tag{1.1}$$

where $\{\alpha_n\}_{n=0}^{\infty}$,$\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are three real sequences satisfying some conditions. It is clear that the iteration scheme (1.1) includes iterations defined in the theorems of Ofoedu[12] as special cases.
The purpose of this paper is to extend and improve the recent results of Chang et al.[4] which in turn is a correction, improvement and generalization of results in Ofoedu[12]. We remove the conditions $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$ and $\sum_{n=1}^{\infty} \alpha_n(k_n - 1) < \infty$ from Theorem 1.1 and, replace them with a weaker

condition $\alpha_n, \beta_n \to 0$ as $n \to \infty$. We equally extend their pair of maps to three maps. Furthermore, we use a more general iteration procedure. Also, the L- Lipschitzian assumption imposed on T_i in Theorem 1.1 is replaced by more general uniformly continuous mappings. Our method is different from [4].
In order to obtain the main results, the following lemmas are needed.
Lemma 1.2[2].Let E be real Banach Space and $J : E \to 2^{E^*}$ be the normalized duality mapping.Then,for any $x, y \in E$

$$\|x + y\|^2 \leq \|x\|^2 + 2 < y, j(x + y) >, \forall j(x + y) \in J(x + y)$$

Lemma 1.3[13]. Let $\Phi : [0, \infty) \to [0, \infty)$ be an increasing function with $\Phi(x) = 0 \Leftrightarrow x = 0$ and let $\{b_n\}_{n=0}^{\infty}$ be a positive real sequence satisfying

$$\sum_{n=0}^{\infty} b_n = +\infty \quad \text{and} \quad \lim_{n\to\infty} b_n = 0.$$

Suppose that $\{a_n\}_{n=0}^{\infty}$ is a nonnegative real sequence. If there exists an integer $N_0 > 0$ satisfying

$$a_{n+1}^2 < a_n^2 + o(b_n) - b_n\Phi(a_{n+1}), \quad \forall n \geq N_0$$

where $\lim_{n\to\infty} \frac{o(b_n)}{b_n} = 0$, then $\lim_{n\to\infty} a_n = 0$.

2.Main results

Theorem 2.1. Let X be a real Banach space , K a nonempty closed and convex subset of X and $T_1, T_2, T_3 : K \to K$ be uniformly continuous mappings such that $T_1(K)$ is bounded with $\rho \in F(T_1) \cap F(T_2) \cap F(T_3)$. Let $k_n \subset [1, \infty)$ be a sequence with $k_n \to 1$ and $\{x_n\}$ be a sequence defined by (1.1) where $\{\alpha_n\}_{n=0}^{\infty}$,$\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are three sequences in [0,1] satisfying

(i) $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = \lim_{n\to\infty} \gamma_n = 0$
and

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$.

If there exists a strictly increasing function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$ such that

$$< T_i^n x_n - \rho, j(x_n - \rho) >\leq k_n\|x_n - \rho\|^2 - \varphi(\|x_n - \rho\|)$$

for all $j(x - \rho) \in J(x - \rho)$ and $x \in K$, (i=1,2,3), then $\{x_n\}$ converges strongly to ρ the unique common fixed point of T_1, T_2, T_3.
Proof. By assumption, we have $F(T_1) \cap F(T_2) \cap F(T_3) = \rho$, say. Let $D_1 = \|x_0 - \rho\| + \sup_{n\geq 0} \|T_1^n y_n - \rho\|$. We prove by induction that $\|x_n - \rho\| \leq D_1$ for

all n. It is clear that, $\|x_0-\rho\|\le D_1$. Assume that $\|x_n-\rho\|\le D_1$ holds. We will prove that $\|x_{n+1}-\rho\|\le D_1$. Indeed, from (1.1), we obtain

$$\begin{aligned}\|x_{n+1}-\rho\| &= \|(1-\alpha_n)(x_n-\rho)+\alpha_n(T_1^n y_n-\rho)\| \\ &\le (1-\alpha_n)\|x_n-\rho\|+\alpha_n\|T_1^n y_n-\rho\| \\ &\le (1-\alpha_n)D_1+\alpha_n D_1 = D_1.\end{aligned}$$

Hence the sequence $\{x_n\}$ is bounded.
Using the uniformly continuity of T_3, we have $\{T_3^n x_n\}$ is bounded. Denote $D_2=\max\{D_1,\sup\{\|T_3^n x_n-\rho\|\}\}$, then

$$\begin{aligned}\|z_n-\rho\| &\le (1-\gamma_n)\|x_n-\rho\|+\gamma_n\|T_3^n x_n-\rho\| \\ &\le (1-\gamma_n)D_1+\gamma_n D_2 \\ &\le (1-\gamma_n)D_2+\gamma_n D_2 = D_2\end{aligned}$$

By the virtue of the uniform continuity of T_2, we get that $\{T_2^n z_n\}$ is bounded. Set $D=\sup_{n\ge 0}\|T_2^n z_n-\rho\|+D_2$. From equation (1.1) we have, in view of Lemma 1.1, that

$$\begin{aligned}\|x_{n+1}-\rho\|^2 &= \langle x_{n+1}-\rho, j(x_{n+1}-\rho)\rangle \\ &\le (1-\alpha_n)\|x_n-\rho\|\|x_{n+1}-\rho\|+\alpha_n\langle T_1^n y_n-\rho, j(x_{n+1}-\rho)\rangle \\ &= (1-\alpha_n)\|x_n-\rho\|\|x_{n+1}-\rho\| \\ &\quad +\alpha_n\langle T_1^n x_{n+1}-\rho, j(x_{n+1}-\rho)\rangle \\ &\quad +\alpha_n\langle T_1^n y_n-T_1^n x_{n+1}, j(x_{n+1}-p)\rangle \\ &\le (1-\alpha_n)\|x_n-\rho\|\|x_{n+1}-\rho\|+\alpha_n\sigma_n\|x_{n+1}-\rho\| \\ &\quad +\alpha_n(k_n\|x_{n+1}-\rho\|^2-\Phi(\|x_{n+1}-\rho\|))\end{aligned} \tag{2.1}$$

where $\sigma_n=\|T_1^n y_n-T_1^n x_{n+1}\|$. Observe that

$$\begin{aligned}\|x_{n+1}-y_n\| &= \beta_n\|x_n-T_2^n z_n\|+\alpha_n\|x_n-T_1^n y_n\| \\ &\le \beta_n(\|x_n-\rho\|+\|T_2^n z_n-\rho\|)+\alpha_n(\|x_n-\rho\|+\|T_1^n y_n-\rho\|) \\ &\le \beta_n(D_1+D)+\alpha_n(D_1+D)\end{aligned}$$

This implies that $\lim_{n\to\infty}\|x_{n+1}-y_n\|=0$, since $\lim_{n\to\infty}\alpha_n=0$, $\lim_{n\to\infty}\beta_n=0$. Since T_1 is uniformly continuous, we have

$$\sigma_n=\|T_1^n x_{n+1}-T_1^n y_n\|\to 0,\quad (n\to\infty) \tag{2.2}$$

In view of the fact that $(a-1)^2\ge 0$, if $a=\|x_{n+1}-\rho\|$ then

$$\|x_{n+1}-\rho\|\le\frac{1}{2}(1+\|x_{n+1}-\rho\|^2). \tag{2.3}$$

Substituting (2.3) into (2.1), we obtain

$$\begin{aligned}\|x_{n+1}-\rho\|^2 &\le \tfrac{1}{2}((1-\alpha_n)^2\|x_n-\rho\|^2+\|x_{n+1}-\rho\|^2)+k_n\alpha_n\|x_{n+1}-\rho\|^2 \\ &\quad -\alpha_n\Phi(\|x_{n+1}-\rho\|)+\alpha_n\sigma_n.\tfrac{1}{2}(1+\|x_{n+1}-\rho\|^2)\end{aligned}$$

$$(1-2k_n\alpha_n-\alpha_n\sigma_n)\|x_{n+1}-\rho\|^2 \leq (1-\alpha_n)^2\|x_n-\rho\|^2-2\alpha_n\Phi(\|x_{n+1}-\rho\|)+\alpha_n\sigma_n. \tag{2.4}$$

Since $\lim_{n\to\infty} k_n\alpha_n = \lim_{n\to\infty}\alpha_n\sigma_n = 0$, there exists a natural number N_0 such that

$$\frac{1}{2} < 1-2k_n\alpha_n-\alpha_n\sigma_n < 1$$

for all $n > N_0$. Then, (2.4) implies that

$$\begin{aligned} \|x_{n+1}-\rho\|^2 &\leq \tfrac{(1-\alpha_n)^2}{1-2k_n\alpha_n-\alpha_n\sigma_n}\|x_n-\rho\|^2 - \tfrac{2\alpha_n}{1-2k_n\alpha_n-\alpha_n\sigma_n}\Phi(\|x_{n+1}-\rho\|) \\ &\quad + \tfrac{\alpha_n\sigma_n}{1-2k_n\alpha_n-\alpha_n\sigma_n} \\ &\leq \|x_n-\rho\|^2 + \alpha_n\tfrac{(\alpha_n+\sigma_n-2(1-k_n))}{1-2k_n\alpha_n-\alpha_n\sigma_n}\|x_n-\rho\|^2 \\ &\quad - \tfrac{2\alpha_n}{1-2k_n\alpha_n-\alpha_n\sigma_n}\Phi(\|x_{n+1}-\rho\|) + \tfrac{\alpha_n\sigma_n}{1-2k_n\alpha_n-\alpha_n\sigma_n} \end{aligned} \tag{2.5}$$

Since $\|x_n-\rho\| \leq D_1$, it follows from (2.5) that $\forall\ n \geq N_0$,

$$\begin{aligned} \|x_{n+1}-\rho\|^2 &\leq \|x_n-\rho\|^2 + 2\alpha_n(\alpha_n+\sigma_n-2(1-k_n))D_1^2 \\ &\quad -2\alpha_n\Phi(\|x_{n+1}-\rho\|) + 2\alpha_n\sigma_n \\ &= \|x_n-\rho\|^2 - 2\alpha_n\Phi(\|x_{n+1}-\rho\|) \\ &\quad +2\alpha_n((\alpha_n+\sigma_n-2(1-k_n))D_1^2+\sigma_n) \\ &\leq \|x_n-\rho\|^2 - 2\alpha_n\Phi(\|x_{n+1}-\rho\|) \\ &\quad +2\alpha_n((\alpha_n+\sigma_n-2(1-k_n))D_1^2+\sigma_n), \quad \forall n \geq N_0 \end{aligned} \tag{2.6}$$

Taking $b_n = 2\alpha_n$ and observing that

$$\lim_{n\to\infty}\frac{2\alpha_n((\alpha_n+\sigma_n-2(1-k_n))D_1^2+\sigma_n)}{2\alpha_n}$$

$$= \lim_{n\to\infty}((\alpha_n+\sigma_n-2(1-k_n))D_1^2+\sigma_n) = 0$$

then (2.6) becomes

$$a_{n+1}^2 \leq a_n^2 - b_n\Phi(a_{n+1}) + o(b_n), \quad \forall n \geq N_0$$

This, with Lemma 1.1, showed that $a_n \to 0$ as $n \to \infty$, that is ,

$$\lim_{n\to\infty}\|x_n-p\| = 0.$$

This completes the proof.

Theorem 2.2. Let X be a real Banach space , K a nonempty closed and convex subset of X and $T : K \to K$ be uniformly continuous mappings such that $T(K)$ is bounded and $F(T) \neq \phi$ where $F(T)$ is the set of fixed points of T.

Let $k_n \subset [1,\infty)$ be a sequence with $k_n \to 1$ and let $\{x_n\}$ be a sequence defined by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n y_n$$

$$y_n = (1-\beta_n)x_n + \beta_n T^n z_n$$

$$z_n = (1-\gamma_n)x_n + \gamma_n T^n x_n, \quad n \geq 0$$

where $\{\alpha_n\}_{n=0}^{\infty}$,$\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are three sequences in [0,1] satisfying

(i) $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$.

If there exists a strictly increasing function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(0) = 0$ such that

$$< T^n x_n - \rho, j(x_n - \rho) > \leq k_n \|x_n - \rho\|^2 - \varphi(\|x_n - \rho\|)$$

for all $j(x-\rho) \in J(x-\rho)$ and $x \in K$, then $\{x_n\}$ cconverges strongly to the fixed point of T.

Corollary 2.3. Let X be a real Banach space , K a nonempty closed and convex subset of X and $T_1, T_2 : K \to K$ be uniformly continuous mappings such that $T_1(K)$ is bounded with $F(T_1) \cap F(T_2) \neq \phi$. Let $k_n \subset [1,\infty)$ be a sequence with $k_n \to 1$ and let $\{x_n\}$ be a sequence defined by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_1^n y_n$$

$$y_n = (1-\beta_n)x_n + \beta_n T_2^n x_n$$

where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are two sequences in [0,1] satisfying

(i) $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$.

If there exists a strictly increasing function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(0) = 0$ such that

$$< T_i^n x_n - \rho, j(x_n - \rho) > \leq k_n \|x_n - \rho\|^2 - \varphi(\|x_n - \rho\|)$$

for all $j(x-\rho) \in J(x-\rho)$ and $x \in K$, (i=1,2), then $\{x_n\}$ converges strongly to the unique common fixed point of T_1, T_2.

Corollary 2.4. Let X be a real Banach space , K a nonempty closed and convex subset of X and $T : K \to K$ be uniformly continuous mappings such that $T(K)$ is bounded and $F(T) \neq \phi$ where $F(T)$ is the set of fixed points of T. Let $k_n \subset [1,\infty)$ be a sequence with $k_n \to 1$ and let $\{x_n\}$ be a sequence defined by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n y_n$$

$$y_n = (1-\beta_n)x_n + \beta_n T^n x_n$$

where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are two sequences in [0,1] satisfying

(i) $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$.

If there exists a strictly increasing function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(0) = 0$ such that

$$< T^n x_n - \rho, j(x_n - \rho) > \leq k_n \|x_n - \rho\|^2 - \varphi(\|x_n - \rho\|)$$

for all $j(x-\rho) \in J(x-\rho)$ and $x \in K$, then $\{x_n\}$ converges strongly to the fixed point of T.

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