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INEQUALITIES FOR $h-$CONVEX FUNCTIONS VIA FURTHER PROPERTIES

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ABSTRACT. In this paper, we establish some new inequalities for $h-$convex functions which are supermultiplicative or superadditive. We also give some applications to special means.

1. INTRODUCTION

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping and $a, b \in I$ with $a < b$. The following double inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

is well-known in the literature as Hadamard's inequality for convex mapping. Note that some of the classical inequalities for means can be derived from (1.1) for appropriate particular selections of the mapping f. Both inequalities hold in the reversed direction if f is concave.

Definition 1. *[See [2]] We say that $f : I \to \mathbb{R}$ is Godunova-Levin function or that f belongs to the class $Q(I)$ if f is non-negative and for all $x, y \in I$ and $t \in (0,1)$ we have*

$$f(tx + (1-t)y) \leq \frac{f(x)}{t} + \frac{f(y)}{1-t}. \tag{1.2}$$

Definition 2. *[See [3]] Let $s \in (0,1]$. A function $f : (0,\infty] \to [0,\infty]$ is said to be $s-$convex in the second sense if*

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y) \tag{1.3}$$

for all $x, y \in (0,b)$ and $t \in [0,1]$.

In 1978, Breckner introduced $s-$convex functions as a generalization of convex functions in [6]. Also, in that one work Breckner proved the important fact that the set valued map is $s-$convex only if the associated support function is $s-$convex function in [7]. A number of properties and connections with $s-$convex in the first sense are discussed in paper [3]. Of course, $s-$convexity means just convexity when $s = 1$.

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Definition 3. *[See [1]] We say that $f: I \subseteq \mathbb{R} \to \mathbb{R}$ is a $P-$ function or that f belongs to the class $P(I)$ if f is nonnegative and for all $x, y \in I$ and $t \in [0,1]$, we have*

$$f(tx + (1-t)y) \leq f(x) + f(y). \tag{1.4}$$

Definition 4. *[See [4]] Let $h: J \subseteq \mathbb{R} \to \mathbb{R}$ be a non-negative function. We say that $f: I \subseteq \mathbb{R} \to \mathbb{R}$ is an $h-$convex function or that f belongs to the class $SX(h, I)$, if f is nonnegative and for all $x, y \in I$ and $\alpha \in [0,1]$ we have*

$$f(\alpha x + (1-\alpha)y) \leq h(\alpha) f(x) + h(1-\alpha) f(y). \tag{1.5}$$

If inequality (1.5) is reversed, then f is said to be $h-$concave, i.e., $f \in SV(h, I)$. Obviously, if $h(\alpha) = \alpha$, then all nonnegative convex functions belong to $SX(h, I)$ and all nonnegative concave functions belong to $SV(h, I)$; if $h(\alpha) = \frac{1}{\alpha}$, then $SX(h, I) = Q(I)$; if $h(\alpha) = 1$, $SX(h, I) \supseteq P(I)$; and if $h(\alpha) = \alpha^s$, where $s \in (0,1)$, then $SX(h, I) \supseteq K_s^2$.

Furthermore, in [5] Bombardelli and Varošanec wrote some generalizations of the Hermite-Hadamard inequalitiesand some properties of functions H and F.

More about those inequalities can be found in a number of papers (for example: see [4, 8, 9, 10]).

Definition 5. *[See [4]]A function $h: J \to \mathbb{R}$ is said to be a supermultiplicative function if*

$$h(xy) \geq h(x)h(y) \tag{1.6}$$

for all $x, y \in J$.

If inequality (1.6) is reversed, then h is said to be a submultiplicative function. If the equality holds in (1.6), then h is said to be a multiplicative function.

Definition 6. *[See [11]]A function $h: J \to \mathbb{R}$ is said to be a superadditive function if*

$$h(x+y) \geq h(x) + h(y) \tag{1.7}$$

for all $x, y \in J$.

Definition 7. *[See [12]] Two functions $f: X \to \mathbb{R}$ and $g: X \to \mathbb{R}$ are said to be similarly ordered, shortly f s.o. g, if*

$$(f(x) - f(y))(g(x) - g(y)) \geq 0$$

for every $x, y \in X$.

The main purpose of this paper is to give some inequalities under the special assumptions of $h-$convex functions by using the elementary analysis. We also give some applications to special means. Throughout the paper we will imply $M(a,b) = f(a)g(a) + f(b)g(b)$ and $N(a,b) = f(a)g(b) + f(b)g(a)$.

2. MAIN RESULTS

Theorem 1. *Let $f, g \in SX(h, I)$, h is supermultiplicative and f, g be similarly ordered functions on I for all $x, y \in I \subseteq \mathbb{R}$. Then for all $\alpha \in (0,1)$ and $\alpha + \beta = 1$ we have the following inequalities;*

$$(fg)(\alpha x + \beta y) \leq \left(h(\alpha\beta) + h^2(\alpha)\right) f(x)g(x) + \left(h(\alpha\beta) + h^2(\beta)\right) f(y)g(y) \tag{2.1}$$

and

$$\frac{1}{b-a}\int_a^b (fg)(x)dx \leq f(a)g(a)\int_0^1 (h(t(1-t))+h^2(t))dt$$
$$+f(b)g(b)\int_0^1 (h(t(1-t))+h^2(1-t))dt. \tag{2.2}$$

Proof. First of all, it is easy to observe that

$$\int_0^1 (fg)(ta+(1-t)b)dt = \frac{1}{b-a}\int_a^b (fg)(x)dx.$$

Since f,g are $h-$convex functions on I, we have

$$\begin{aligned} f(\alpha x+\beta y) &\leq h(\alpha)f(x)+h(\beta)f(y) \\ g(\alpha x+\beta y) &\leq h(\alpha)g(x)+h(\beta)g(y) \end{aligned}$$

for all $\alpha,\beta\in(0,1)$, $\alpha+\beta=1$. On the other hand, f and g are similarly ordered functions, thus we have

$$f(\alpha x+\beta y)g(\alpha x+\beta y)\leq (h(\alpha)f(x)+h(\beta)f(y))\,(h(\alpha)g(x)+h(\beta)g(y))\,.$$

Using the other properties of f,g and h in Theorem 1, we get

$$\begin{aligned} & f(\alpha x+\beta y)g(\alpha x+\beta y) \\ \leq\;& h^2(\alpha)f(x)g(x)+h(\alpha)h(\beta)f(x)g(y)+h(\alpha)h(\beta)f(y)g(x)+h^2(\beta)f(y)g(y) \\ \leq\;& h^2(\alpha)f(x)g(x)+h(\alpha\beta)f(x)g(y)+h(\alpha\beta)f(y)g(x)+h^2(\beta)f(y)g(y) \\ =\;& h^2(\alpha)f(x)g(x)+h^2(\beta)f(y)g(y)+h(\alpha\beta)\left[f(x)g(y)+f(y)g(x)\right] \\ \leq\;& h^2(\alpha)f(x)g(x)+h^2(\beta)f(y)g(y)+h(\alpha\beta)\left[f(x)g(x)+f(y)g(y)\right] \\ =\;& \left(h^2(\alpha)+h(\alpha\beta)\right)(f(x)g(x))+\left(h^2(\beta)+h(\alpha\beta)\right)(f(y)g(y)) \end{aligned}$$

which completes the proof of (2.1).

To get the second inequality, if we choose $x=a$, $y=b$ and $\alpha=t$, $\beta=1-t$ in (2.1), we have

$$(fg)(ta+(1-t)b)\leq\left[h(t(1-t))+h^2(t)\right]f(a)g(a)+\left[h(t(1-t))+h^2(1-t)\right]f(b)g(b).$$

By integrating the result with respect to t over $[0,1]$, we get (2.2). □

Corollary 1. *If we choose $h(t)=1$ in (2.2), then we obtain an integral inequality for $P-$functions*

$$\frac{1}{b-a}\int_a^b (fg)(x)dx\leq 2M(a,b).$$

Corollary 2. *If we choose $h(t) = t$ in (2.2), then we obtain an integral inequality for ordinary convex functions*

$$\begin{aligned}\frac{1}{b-a}\int_a^b (fg)(x)dx &\leq f(a)g(a)\int_0^1 (t(1-t)+t^2)dt \\ &\quad +f(b)g(b)\int_0^1 (t(1-t)+(1-t)^2)dt \\ &= \frac{f(a)g(a)}{2}+\frac{f(b)g(b)}{2} \\ &= \frac{M(a,b)}{2}.\end{aligned}$$

Corollary 3. *If we choose $h(t) = t^s$ in (2.2), then we obtain an integral inequality for $s-$convex functions in the second sense with use of the Beta function of Euler type*

$$\begin{aligned}\frac{1}{b-a}\int_a^b (fg)(x)dx &\leq f(a)g(a)\left\{\int_0^1 t^s(1-t)^s dt+\int_0^1 t^{2s}dt\right\} \\ &\quad +f(b)g(b)\left\{\int_0^1 t^s(1-t)^s dt+\int_0^1 (1-t)^{2s}dt\right\}\end{aligned}$$

$$= f(a)g(a)\left\{\beta(s+1,s+1)+\beta(2s+1,1)\right\}$$

$$+f(b)g(b)\left\{\beta(s+1,s+1)+\beta(1,2s+1)\right\} \tag{2.3}$$

$$\leq M(a,b)\left[\frac{\beta\left(s+1,\frac{1}{2}\right)}{2^{2s+1}}+\beta(2s+1,1)\right]. \tag{2.4}$$

Corollary 4. *By using some properties of Gamma mapping on (2.3) we have*

$$\frac{1}{b-a}\int_a^b (fg)(x)dx \leq M(a,b)\left[\frac{H\left(\Gamma(s+1),\Gamma(\frac{1}{2})\right)}{2^{2s+2}}+\frac{H\left(\Gamma(2s+1),\Gamma(1)\right)}{2}\right].$$

Proof. We know that Gamma mapping is superadditive and there is a relation between Gamma and Beta mappings with $x, y > 0$; i.e;

$$\begin{aligned}\beta(x,y) &= \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \\ \Gamma(x)+\Gamma(y) &\leq \Gamma(x+y)\end{aligned}$$

and

$$H(x,y)=\frac{2xy}{x+y}.$$

Thus we can write

$$\begin{aligned}\frac{1}{b-a}\int_a^b (fg)(x)dx &\leq M(a,b)\left[\frac{\beta\left(s+1,\frac{1}{2}\right)}{2^{2s+1}}+\beta(2s+1,1)\right]\\ &= M(a,b)\left[\frac{\Gamma(s+1)\Gamma\left(\frac{1}{2}\right)}{2^{2s+1}\Gamma\left(s+1+\frac{1}{2}\right)}+\frac{\Gamma(2s+1)\Gamma(1)}{\Gamma(2s+2)}\right]\\ &\leq M(a,b)\left[\frac{\Gamma(s+1)\Gamma\left(\frac{1}{2}\right)}{2^{2s+1}\left(\Gamma(s+1)+\Gamma\left(\frac{1}{2}\right)\right)}+\frac{\Gamma(2s+1)\Gamma(1)}{\Gamma(2s+1)+\Gamma(1)}\right]\\ &= M(a,b)\left[\frac{H\left(\Gamma(s+1),\Gamma\left(\frac{1}{2}\right)\right)}{2^{2s+2}}+\frac{H\left(\Gamma(2s+1),\Gamma(1)\right)}{2}\right]\end{aligned}$$

which completes the proof. □

Theorem 2. *Let $f,g\in SX(h,I)$, h is superadditive and f,g be similarly ordered functions on I. For all $x,y\in I\subseteq\mathbb{R}$ and $\alpha\in(0,1)$, $\alpha+\beta=1$ we have,*

$$(fg)(\alpha x+\beta y)\leq h(1)\left[h(\alpha)f(x)g(x)+h(\beta)f(y)g(y)\right] \tag{2.5}$$

and

$$\frac{1}{b-a}\int_a^b (fg)(x)dx\leq h(1)\left[f(a)g(a)\int_0^1 h(t)dt+f(b)g(b)\int_0^1 h(1-t)dt\right]. \tag{2.6}$$

Proof. Since f,g are $h-$convex functions on I, we have

$$f(\alpha x+\beta y)g(\alpha x+\beta y)\leq\left(h(\alpha)f(x)+h(\beta)f(y)\right)\left(h(\alpha)g(x)+h(\beta)g(y)\right)$$

or

$$\begin{aligned}&f(\alpha x+\beta y)g(\alpha x+\beta y)\\ \leq\;& h^2(\alpha)f(x)g(x)+h(\alpha)h(\beta)f(x)g(y)+h(\alpha)h(\beta)f(y)g(x)+h^2(\beta)f(y)g(y).\end{aligned}$$

Since h is superadditive function and f and g are similarly ordered functions, we get

$$\begin{aligned}&f(\alpha x+\beta y)g(\alpha x+\beta y)\\ \leq\;& h^2(\alpha)f(x)g(x)+h(\alpha)h(\beta)f(x)g(y)+h(\alpha)h(\beta)f(y)g(x)+h^2(\beta)f(y)g(y)\\ =\;& h^2(\alpha)f(x)g(x)+h^2(\beta)f(y)g(y)+h(\alpha)h(\beta)\left[f(x)g(y)+f(y)g(x)\right]\\ \leq\;& h^2(\alpha)f(x)g(x)+h^2(\beta)f(y)g(y)+h(\alpha)h(\beta)\left[f(x)g(x)+f(y)g(y)\right]\\ =\;& \left[h^2(\alpha)+h(\alpha)h(\beta)\right](f(x)g(x))+\left[h^2(\beta)+h(\alpha)h(\beta)\right](f(y)g(y))\\ \leq\;& \left[h(\alpha)h(\alpha+\beta)\right](f(x)g(x))+\left[h(\beta)h(\alpha+\beta)\right](f(y)g(y))\\ =\;& h(\alpha+\beta)\left[h(\alpha)f(x)g(x)+h(\beta)f(y)g(y)\right]\\ =\;& h(1)\left[h(\alpha)f(x)g(x)+h(\beta)f(y)g(y)\right]\end{aligned}$$

which completes the proof of (2.5).

To get the second inequality in Theorem 2, by taking $x=a$, $y=b$ and $\alpha=t$, $\beta=1-t$ in (2.5) as in the proof of the inequality (2.1), we have

$$(fg)(ta+(1-t)b)\leq h(1)\left[h(t)f(a)g(a)+h(1-t)f(b)g(b)\right].$$

By integrating the result with respect to t over $[0,1]$ we get

$$\begin{aligned}\int_0^1 (fg)(ta+(1-t)b)dt &= \frac{1}{b-a}\int_a^b (fg)(x)dx \\ &\leq h(1)\left[f(a)g(a)\int_0^1 h(t)dt + f(b)g(b)\int_0^1 h(1-t)dt\right].\end{aligned}$$

□

Here, we can't choose the some special cases of $h(t)$ as in Corollary 1-4. For instance, if we choose $h(t)=1$, then we don't obtain the superadditive.

Theorem 3. *Let $I=[a,b]\subseteq\mathbb{R}$, $f\in SX(h,I)$, f is symmetric about $\frac{a+b}{2}$ and h is superadditive, then for all $x\in I$ and $\alpha\in[0,1]$ we have*

$$\frac{1}{b-a}\int_a^b f(x)dx \leq h(1)\frac{f(a)+f(b)}{2}. \tag{2.7}$$

Proof. With the above assumptions for f, h and take into account the change of variable $x=\alpha a+(1-\alpha)b$, we have

$$\begin{aligned} f(a+b-x) &= f(a+b-\alpha a-(1-\alpha)b) = f((1-\alpha)a+\alpha b) \\ &\leq h(1-\alpha)f(a)+h(\alpha)f(b) \\ &\leq (h(\alpha)+h(1-\alpha))(f(a)+f(b)) - f(\alpha a+(1-\alpha)b) \\ &\leq h(1)(f(a)+f(b)) - f(\alpha a+(1-\alpha)b).\end{aligned}$$

Since

$$f(x) = f(a+b-x) = f(\alpha a+(1-\alpha)b)$$

we have

$$2f(x) \leq h(1)(f(a)+f(b)). \tag{2.8}$$

By integrating (2.8) with respect to x over $[a,b]$ and simple calculation we get the desired result. □

Theorem 4. *Let $I=[a,b]\subseteq\mathbb{R}$, $f,g\in SX(h,I)$, f,g are symmetric about $\frac{a+b}{2}$ and h is supermultiplicative, then for all $x\in I$ and $\alpha\in[0,1]$ we have*

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)g(x)dx &\leq \frac{1}{2}M(a,b)\left[\left(h(\alpha^2)+h\left((1-\alpha)^2\right)\right)\right] \\ &\quad + N(a,b)\left[h(\alpha-\alpha^2)\right].\end{aligned}$$

Proof. Since f and g are symmetric about $\frac{a+b}{2}$ we can write

$$\begin{aligned} f(a+b-x) &= f(a+b-\alpha a-(1-\alpha)b) = f((1-\alpha)a+\alpha b) \\ &\leq h(1-\alpha)f(a)+h(\alpha)f(b)\end{aligned}$$

and

$$\begin{aligned} g(a+b-x) &= g(a+b-\alpha a-(1-\alpha)b) = g((1-\alpha)a+\alpha b) \\ &\leq h(1-\alpha)g(a)+h(\alpha)g(b).\end{aligned}$$

If we multiply the above two inequalities, we have that

$$\begin{aligned} f(a+b-x)\,g(a+b-x) &\leq h^2(1-\alpha)f(a)g(a)+h^2(\alpha)f(b)g(b) \\ &\quad +h(\alpha)h(1-\alpha)\left[f(a)g(b)+f(b)g(a)\right] \\ &\leq \left(h^2(\alpha)+h^2(1-\alpha)\right)\left[f(a)g(a)+f(b)g(b)\right] \\ &\quad +2h(\alpha)h(1-\alpha)\left[f(a)g(b)+f(b)g(a)\right] \\ &\quad -f(\alpha a+(1-\alpha)b)\,g(\alpha a+(1-\alpha)b)\,. \end{aligned}$$

Since

$$f(x)=f(a+b-x)=f(\alpha a+(1-\alpha)b)$$

and

$$g(x)=g(a+b-x)=g(\alpha a+(1-\alpha)b)$$

we have

$$\begin{aligned} 2f(x)g(x) &\leq \left(h^2(\alpha)+h^2(1-\alpha)\right)\left[f(a)g(a)+f(b)g(b)\right] \\ &\quad +2h(\alpha)h(1-\alpha)\left[f(a)g(b)+f(b)g(a)\right]. \end{aligned} \tag{2.9}$$

Because of h is supermultiplicative we can write

$$\begin{aligned} 2f(x)g(x) &\leq \left(h(\alpha^2)+h\left((1-\alpha)^2\right)\right)\left[f(a)g(a)+f(b)g(b)\right] \\ &\quad +2h(\alpha-\alpha^2)\left[f(a)g(b)+f(b)g(a)\right]. \end{aligned}$$

By integrating both sides respect to x over the interval $[a,b]$, we get

$$\begin{aligned} \frac{1}{b-a}\int_a^b f(x)g(x)dx &\leq \frac{1}{2}M(a,b)\left[\left(h(\alpha^2)+h\left((1-\alpha)^2\right)\right)\right] \\ &\quad +N(a,b)\left[h(\alpha-\alpha^2)\right] \end{aligned}$$

which is the desired result. □

Theorem 5. *Under the assumptions of Theorem 4, if f and g are similarly ordered functions, we have*

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \leq \frac{M(a,b)}{2}\left[h(\alpha^2)+h\left((1-\alpha)^2\right)+2h(\alpha-\alpha^2)\right].$$

Proof. Since h is supermultiplicative we can write (2.9) as

$$\begin{aligned} 2f(x)g(x) &\leq \left(h(\alpha^2)+h\left((1-\alpha)^2\right)\right)\left[f(a)g(a)+f(b)g(b)\right] \\ &\quad +2h(\alpha-\alpha^2)\left[f(a)g(b)+f(b)g(a)\right]. \end{aligned}$$

By using the similarly ordered properties of f and g, we get

$$\begin{aligned} 2f(x)g(x) &\leq \left(h(\alpha^2)+h\left((1-\alpha)^2\right)\right)\left[f(a)g(a)+f(b)g(b)\right] \\ &\quad +2h(\alpha-\alpha^2)\left[f(a)g(a)+f(b)g(b)\right] \\ &= \left[h(\alpha^2)+h\left((1-\alpha)^2\right)+2h(\alpha-\alpha^2)\right]\left[f(a)g(a)+f(b)g(b)\right]. \end{aligned}$$

By integrating both sides respect to x over the interval $[a,b]$, we get

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \leq \frac{M(a,b)}{2}\left[h(\alpha^2)+h\left((1-\alpha)^2\right)+2h(\alpha-\alpha^2)\right]$$

which completes the proof. □

3. APPLICATIONS TO SOME SPECIAL MEANS

We now consider the applications of our Theorems to the following special means

a) The arithmetic mean:

$$A = A(a,b) := \frac{a+b}{2}, \quad a,b \geq 0,$$

b) The geometric mean:

$$G = G(a,b) := \sqrt{ab}, \quad a,b \geq 0,$$

c) The harmonic mean:

$$H = H(a,b) := \frac{2ab}{a+b}, \quad a,b \geq 0,$$

d) The logarithmic mean:

$$L = L(a,b) := \begin{cases} a & \text{if} \quad a=b \\ \frac{b-a}{\ln b - \ln a} & \text{if} \quad a \neq b \end{cases}, \quad a,b \geq 0,$$

e) The p-logarithmic mean:

$$L_p = L_p(a,b) := \begin{cases} \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{1/p} & \text{if} \quad a \neq b \\ a & \text{if} \quad a=b \end{cases}, \quad p \in \mathbb{R} \backslash \{-1,0\}; \quad a,b>0.$$

These means are often used in numerical approximation. However, only the following simple relationships are known in the literature.

$$H \leq G \leq L \leq A$$

We now derive some sophisticated bounds of the above means.

Proposition 1. *Let $a,b \in \mathbb{R}$, $0<a<b$ and $n \in \mathbb{Z}$, $|n| \geq 1$. Then, we have:*

$$L_{2n}^{2n}(a,b) \leq A\left(a^{2n}, b^{2n}\right).$$

Proof. The proof is immediate from Theorem 1 applied for $f(x) = g(x) = x^n$, $h(\alpha) = \alpha$ where $x \in \mathbb{R}$, $n \in \mathbb{Z}$, $|n| \geq 1$. □

Proposition 2. *Let $a,b \in \mathbb{R}$, $0<a<b$. Then, we have:*

$$\begin{aligned} G^2(a,b) &\leq A\left(a^2,b^2\right)\{\beta(s+1,s+1)+\beta(2s+1,1)\} \\ &\leq A\left(a^2,b^2\right)\left[\frac{\beta(s+1,\frac{1}{2})}{2^{2s+1}}+\beta(2s+1,1)\right]. \end{aligned}$$

Proof. The assertion follows from Corollary 3 applied to $f(x) = g(x) = \frac{1}{x}$, $x \in [a,b]$. □

Proposition 3. *Let $a,b \in \mathbb{R}$, $0<a<b$. Then, we have:*

$$G^2(a,b) \leq A\left(a^2,b^2\right)\left[\frac{H\left(\Gamma(s+1),\Gamma(\frac{1}{2})\right)}{2^{2s+2}}+\frac{H\left(\Gamma(2s+1),\Gamma(1)\right)}{2}\right].$$

Proof. The assertion follows from Corollary 4 applied to $f(x) = g(x) = \frac{1}{x}$, $x \in [a,b]$. □

Proposition 4. *Let $a,b \in \mathbb{R}$, $0 < a < b$. Then, we have:*

$$G^2(a,b) \leq A\left(a^2,b^2\right).$$

Proof. The assertion follows from Theorem 2 applied to $f(x) = g(x) = \frac{1}{x}$, $x \in [a,b]$ and $h(\alpha) = \alpha$. □

We may develop analogous results based on $h-$convexity in our next paper.

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