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EXTENSION OF INEQUALITY SIMILAR TO HARDY-HILBERT'S INTEGRAL INEQUALITY

V. ADIYASUREN AND TS. BATBOLD

ABSTRACT. In this note, by introducing some parameters we establish new extension of inequality similar to Hardy-Hilbert's integral inequality.

1. INTRODUCTION

If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, f, g \geq 0$, satisfy $0 < \int_0^\infty f^p(x)dx < \infty$ and $0 < \int_0^\infty g^q(x)dx < \infty$, then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \frac{\pi}{\sin(\frac{\pi}{p})} \left\{ \int_0^\infty f^p(x)dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty g^q(x)dx \right\}^{\frac{1}{q}} \tag{1.1}$$

where the constant factor $\pi/(\sin \pi/p)$ is the best possible. Inequality (1.1) is called Hardy-Hilbert's inequality (see [1]) and is important in analysis and application (cf. Mitrinović et al. [2]). Yang [3, 4] gave the following two distinct generalizations of (1.1):

$$\int_\alpha^\infty \int_\alpha^\infty \frac{f(x)g(y)}{(x+y-2\alpha)^\lambda} dxdy < k_\lambda(p) \left\{ \int_\alpha^\infty (t-\alpha)^{1-\lambda} f^p(t)dt \right\}^{\frac{1}{p}} \left\{ \int_\alpha^\infty (t-\alpha)^{1-\lambda} g^q(t)dt \right\}^{\frac{1}{q}} \tag{1.2}$$

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x^\lambda + y^\lambda} dxdy < \frac{\pi}{\lambda \sin(\frac{\pi}{p})} \left\{ \int_0^\infty t^{(p-1)(1-\lambda)} f^p(t)dt \right\}^{\frac{1}{p}} \left\{ \int_0^\infty t^{(q-1)(1-\lambda)} g^q(t)dt \right\}^{\frac{1}{q}} \tag{1.3}$$

where the constant factors $k_\lambda(p) = B\left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\right)$ $(\lambda > 2 - \min\{p,q\})$ in (1.2) and $\frac{\pi}{\lambda \sin(\frac{\pi}{p})}$ $(\lambda > 0)$ in (1.3) are the best possible ($B(u;v)$ is the Beta function). In the recent years a lot of results with extensions of (1.1) of multiple integral inequalities were obtained. B. Yang and L. Debnath [5] proved the following integral inequality:

If $\alpha \in \mathbb{R}, n \geq 2, a > 0, p_i > 1, \lambda > n - \min_{1 \leq i \leq n}\{p_i\}$, and $\sum_{i=1}^n \frac{1}{p_i} = 1$, and $f_i(x) \geq 0$ such that

$$0 < \int_\alpha^\infty (t-\alpha)^{a(n-\lambda)-1-(a-1)p_i} f_i^{p_i}(t)dt < \infty \ (i = 1, 2, \ldots, n)$$

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then

$$\int_\alpha^\infty \cdots \int_\alpha^\infty \frac{1}{[\sum_{i=1}^n (x_i-\alpha)^a]^\lambda} \prod_{i=1}^n f_i(x_i) dx_1 \cdots dx_n \tag{1.4}$$
$$< \frac{1}{a^{n-1}\Gamma(\lambda)} \prod_{i=1}^n \Gamma\left(1-\frac{n-\lambda}{p_i}\right) \left\{\int_\alpha^\infty (t-\alpha)^{a(n-\lambda)-1-(a-1)p_i} f_i^{p_i}(t) dt\right\}^{\frac{1}{p_i}},$$

where the constant factor $\frac{1}{a^{n-1}\Gamma(\lambda)}\prod_{i=1}^n \Gamma\left(1-\frac{n-\lambda}{p_i}\right)$ is the best possible.

Sulaiman [6] derived a new integral inequality similar to Hardy-Hilbert's integral inequality as follows:

Let $n \in \mathbb{N}\setminus\{1\}, p > 1, \frac{1}{p}+\frac{1}{q} = 1, a_i > 0, 1 \le i \le n, \lambda > \sum_{i=r+1}^n a_i, 1 \le r < n$, $\lambda_j = (a_j-1)(1-q), r+1 \le j \le n, K_{r+1} = \left(\prod_{j=r+1}^n \Gamma(a_i)\right)\Gamma(\lambda - \sum_{i=r+1}^n a_i)/\Gamma(\lambda)$, then

$$\left(\frac{\int_0^\infty \cdots \int_0^\infty \frac{f_1(x_1)\cdots f_n(x_n)}{(x_1+\cdots+x_n)^\lambda} dx_1\cdots dx_n}{K_{r+1}\int_0^\infty \cdots \int_0^\infty f_1^p(x_1)\cdots f_r^p(x_r) dx_1 \cdots dx_r}\right)^q \tag{1.5}$$
$$\le \frac{\int_0^\infty \cdots \int_0^\infty \frac{(x_1+\cdots+x_r)^{\sum_{i=r+1}^n a_i - \lambda} x_{r+1}^{\lambda_{r+1}} f_{r+1}^q(x_{r+1})\cdots x_n^{\lambda_n} f_n^q(x_n)}{(x_1+\cdots+x_n)^\lambda} dx_1 \cdots dx_n}{K_{r+1}\int_0^\infty \cdots \int_0^\infty f_1^p(x_1)\cdots f_r^p(x_r) dx_1\cdots dx_r}.$$

The main objective of this note is to build a new extension of inequality (1.5) similar to inequality (1.4). Some particular results are obtained.

2. Main Results

Theorem 2.1. *If* $\alpha \in \mathbb{R}, b > 0, n \in \mathbb{N}\setminus\{1\}, p > 1, \frac{1}{p}+\frac{1}{q} = 1, a_i > 0, 1 \le i \le n$, $\lambda > \sum_{i=r+1}^n a_i, 1 \le r < n$, $\lambda_j = (ba_j - 1)(1-q), r+1 \le j \le n$,

$$C_{r+1} = \frac{\Gamma(\lambda - \sum_{i=r+1}^n a_i)}{b^{n-r}\Gamma(\lambda)} \left(\prod_{j=r+1}^n \Gamma(a_i)\right),$$

then

$$\left(\frac{\int_\alpha^\infty \cdots \int_\alpha^\infty \frac{\prod_{i=1}^n f_i(x_i)}{\left(\sum_{i=1}^n (x_i-\alpha)^b\right)^\lambda} dx_1 \cdots dx_n}{C_{r+1}\int_\alpha^\infty \cdots \int_\alpha^\infty \prod_{i=1}^r f_i^p(x_i) dx_1\cdots dx_r}\right)^q \tag{2.1}$$
$$\le \frac{\int_\alpha^\infty \cdots \int_\alpha^\infty \frac{(\sum_{i=1}^r (x_i-\alpha)^b)^{\sum_{i=r+1}^n a_i - \lambda} \prod_{i=r+1}^n (x_i-\alpha)^{\lambda_i} f_i^q(x_i)}{(\sum_{i=1}^n (x_i-\alpha)^b)^\lambda} dx_1\cdots dx_n}{C_{r+1}\int_\alpha^\infty \cdots \int_\alpha^\infty \prod_{i=1}^r f_i^p(x_i) dx_1 \cdots dx_r}.$$

Proof. Using Hölder's inequality twice, we have

$$\begin{aligned}
\int_{\alpha}^{\infty}\cdots\int_{\alpha}^{\infty}\frac{\prod_{i=1}^{n}f_i(x_i)}{\left(\sum_{i=1}^{n}(x_i-\alpha)^b\right)^{\lambda}}dx_1\cdots dx_n &= \int_{\alpha}^{\infty}\cdots\int_{\alpha}^{\infty}\prod_{i=1}^{r}f_i(x_i) \\
&\times\left(\int_{\alpha}^{\infty}\cdots\int_{\alpha}^{\infty}\frac{\prod_{i=r+1}^{n}f_i(x_i)}{\left(\sum_{i=1}^{n}(x_i-\alpha)^b\right)^{\lambda}}dx_{r+1}\cdots dx_n\right)dx_1\cdots dx_r \\
&\le\left\{\int_{\alpha}^{\infty}\cdots\int_{\alpha}^{\infty}\prod_{i=1}^{r}f_i^p(x_i)dx_1\cdots dx_r\right\}^{1/p} \\
&\times\left\{\int_{\alpha}^{\infty}\cdots\int_{\alpha}^{\infty}\left(\int_{\alpha}^{\infty}\cdots\int_{\alpha}^{\infty}\frac{\prod_{i=r+1}^{n}f_i(x_i)}{\left(\sum_{i=1}^{n}(x_i-\alpha)^b\right)^{\lambda}}dx_{r+1}\cdots dx_n\right)^{q}dx_1\cdots dx_r\right\}^{1/q} \\
&\le\left\{\int_{\alpha}^{\infty}\cdots\int_{\alpha}^{\infty}\prod_{i=1}^{r}f_i^p(x_i)dx_1\cdots dx_r\right\}^{1/p} \\
&\times\left\{\int_{\alpha}^{\infty}\cdots\int_{\alpha}^{\infty}\left(\int_{\alpha}^{\infty}\cdots\int_{\alpha}^{\infty}\frac{\prod_{i=r+1}^{n}(x_i-\alpha)^{\lambda_i}f_i^q(x_i)}{\left(\sum_{i=1}^{n}(x_i-\alpha)^b\right)^{\lambda}}dx_{r+1}\cdots dx_n\right)\right. \\
&\times\left.\left(\int_{\alpha}^{\infty}\cdots\int_{\alpha}^{\infty}\frac{\prod_{i=r+1}^{n}(x_i-\alpha)^{ba_i-1}}{\left(\sum_{i=1}^{n}(x_i-\alpha)^b\right)^{\lambda}}dx_{r+1}\cdots dx_n\right)^{q-1}dx_1\cdots dx_r\right\}^{1/q}
\end{aligned}$$

Now, we consider

$$\begin{aligned}
I &= \int_{\alpha}^{\infty}\cdots\int_{\alpha}^{\infty}\frac{\prod_{i=r+1}^{n}(x_i-\alpha)^{ba_i-1}}{\left(\sum_{i=1}^{n}(x_i-\alpha)^b\right)^{\lambda}}dx_{r+1}\cdots dx_n \\
&= \int_{\alpha}^{\infty}\cdots\int_{\alpha}^{\infty}\frac{\prod_{i=r+1}^{n-1}(x_i-\alpha)^{ba_i-1}}{\left(\sum_{i=1}^{n-1}(x_i-\alpha)^b\right)^{\lambda-a_n}}dx_{r+1}\cdots dx_{n-1} \\
&\times\frac{1}{b}\int_{\alpha}^{\infty}\frac{\left(\frac{(x_n-\alpha)^b}{\sum_{i=1}^{n-1}(x_i-\alpha)^b}\right)^{a_n-1}}{\left(1+\frac{(x_n-\alpha)^b}{\sum_{i=1}^{n-1}(x_i-\alpha)^b}\right)^{\lambda}}\,d\frac{(x_n-\alpha)^b}{\sum_{i=1}^{n-1}(x_i-\alpha)^b} \\
&= \frac{B(a_n,\lambda-a_n)}{b}\int_{\alpha}^{\infty}\cdots\int_{\alpha}^{\infty}\frac{\prod_{i=r+1}^{n-1}(x_i-\alpha)^{ba_i-1}}{\left(\sum_{i=1}^{n-1}(x_i-\alpha)^b\right)^{\lambda-a_n}}dx_{r+1}\cdots dx_{n-1}.
\end{aligned}$$

Proceeding in this manner, we obtain

$$\begin{aligned}
I &= \frac{1}{b^{n-r}}\prod_{j=r+1}^{n}B\left(a_i,\lambda-\sum_{i=j}^{n}a_i\right)\left(\sum_{i=1}^{r}(x_i-\alpha)^b\right)^{\sum_{i=r+1}^{n}a_i-\lambda} \\
&= \frac{\Gamma(\lambda-\sum_{i=r+1}^{n}a_i)}{b^{n-r}\Gamma(\lambda)}\left(\prod_{j=r+1}^{n}\Gamma(a_i)\right)\left(\sum_{i=1}^{r}(x_i-\alpha)^b\right)^{\sum_{i=r+1}^{n}a_i-\lambda}.
\end{aligned}$$

Therefore, we have

$$\int_\alpha^\infty \cdots \int_\alpha^\infty \frac{\prod_{i=1}^n f_i(x_i)}{\left(\sum_{i=1}^n (x_i-\alpha)^b\right)^\lambda} dx_1\cdots dx_n$$
$$\le C_{r+1}^{1/p} \left\{ \int_\alpha^\infty \cdots \int_\alpha^\infty \prod_{i=1}^r f_i^p(x_i) dx_1 \cdots dx_r \right\}^{1/p}$$
$$\times \left(\int_\alpha^\infty \cdots \int_\alpha^\infty \frac{\left(\sum_{i=1}^r (x_i-\alpha)^b\right)^{\sum_{i=r+1}^n a_i - \lambda} \prod_{i=r+1}^n (x_i-\alpha)^{\lambda_i} f_i^q(x_i)}{\left(\sum_{i=1}^n (x_i-\alpha)^b\right)^\lambda} dx_1 \cdots dx_n \right)^{1/q}.$$

Hence

$$\left(\frac{\int_\alpha^\infty \cdots \int_\alpha^\infty \frac{\prod_{i=1}^n f_i(x_i)}{\left(\sum_{i=1}^n (x_i-\alpha)^b\right)^\lambda} dx_1\cdots dx_n}{C_{r+1}\int_\alpha^\infty \cdots \int_\alpha^\infty \prod_{i=1}^r f_i^p(x_i) dx_1 \cdots dx_r} \right)^q$$
$$\le \frac{\int_\alpha^\infty \cdots \int_\alpha^\infty \frac{\left(\sum_{i=1}^r (x_i-\alpha)^b\right)^{\sum_{i=r+1}^n a_i - \lambda} \prod_{i=r+1}^n (x_i-\alpha)^{\lambda_i} f_i^q(x_i)}{\left(\sum_{i=1}^n (x_i-\alpha)^b\right)^\lambda} dx_1 \cdots dx_n}{C_{r+1}\int_\alpha^\infty \cdots \int_\alpha^\infty \prod_{i=1}^r f_i^p(x_i) dx_1 \cdots dx_r}.$$

The theorem is proved. □

Corollary 2.2. *(i) In Theorem 2.1, if we take $\alpha = 0$, then inequality (2.1) reduces to*

$$\left(\frac{\int_0^\infty \cdots \int_0^\infty \frac{\prod_{i=1}^n f_i(x_i)}{\left(\sum_{i=1}^n x_i^b\right)^\lambda} dx_1\cdots dx_n}{C_{r+1}\int_0^\infty \cdots \int_0^\infty \prod_{i=1}^r f_i^p(x_i) dx_1 \cdots dx_r} \right)^q \tag{2.2}$$
$$\le \frac{\int_0^\infty \cdots \int_0^\infty \frac{\left(\sum_{i=1}^r x_i^b\right)^{\sum_{i=r+1}^n a_i - \lambda} \prod_{i=r+1}^n x_i^{\lambda_i} f_i^q(x_i)}{\left(\sum_{i=1}^n x_i^b\right)^\lambda} dx_1 \cdots dx_n}{C_{r+1}\int_0^\infty \cdots \int_0^\infty \prod_{i=1}^r f_i^p(x_i) dx_1 \cdots dx_r}.$$

(ii) In (i), if we take $b = 1$, then inequality (2.2) reduces to inequality (1.5).

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Department of Mathematical Analysis, National University of Mongolia, Ulaanbaatar, Mongolia

E-mail address: V_Adiyasuren@yahoo.com

Institute of Mathematics, National University of Mongolia, Ulaanbaatar, Mongolia

E-mail address: tsbatbold@hotmail.com