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QUASILINEARITY OF SOME FUNCTIONALS ASSOCIATED WITH MONOTONIC CONVEX FUNCTIONS

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Abstract. Some quasilinearity properties of composite functionals generated by monotonic and convex/concave functions and their applications in improving some classical inequalities such as the Jensen, Hölder and Minkowski inequalities are given.

1. Introduction

The problem of studying the quasilinearity properties of functionals associated to some celebrated inequalities such as the Jensen, Cauchy-Bunyakowsky-Schwarz, Hölder, Minkowski and other famous inequalities has been investigated by many authors during the last 50 years.

In the following, in order to provide a natural background that will enable us to construct composite functionals out of simple ones and to investigate their quasilinearity properties we recall a number of concepts and simple results that are of importance for the task.

Let X be a linear space. A subset $C\subseteq X$ is called a *convex cone* in X provided the following conditions hold:

(i) $x,y\in C$ imply $x+y\in C$;
(ii) $x\in C$, $\alpha\geq 0$ imply $\alpha x\in C$.

A functional $h:C\rightarrow\mathbb{R}$ is called *superadditive (subadditive)* on C if

(iii) $h(x+y)\geq(\leq)\,h(x)+h(y)$ for any $x,y\in C$

and *nonnegative (strictly positive)* on C if, obviously, it satisfies

(iv) $h(x)\geq(>)\,0$ for each $x\in C$.

The functional h is $s-$*positive homogeneous* on C, for a given $s>0$, if

(v) $h(\alpha x)=\alpha^{s}h(x)$ for any $\alpha\geq 0$ and $x\in C$.
If $s=1$, we simply call it positive homogeneous.

In [4], the following result has been obtained:

Theorem 1. *Let $x,y\in C$ and $h:C\rightarrow\mathbb{R}$ be a nonnegative, superadditive and $s-$positive homogeneous functional on C. If $M\geq m\geq 0$ are such that $x-my$ and $My-x\in C$, then*

$$M^{s}h(y)\geq h(x)\geq m^{s}h(y). \tag{1.1}$$

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Now, consider $v : C \to \mathbb{R}$ an *additive* and *strictly positive* functional on C which is also *positive homogeneous* on C, i.e.,

(vi) $v(\alpha x) = \alpha v(x)$ for any $\alpha > 0$ and $x \in C$.

In [5] we obtained further results concerning the quasilinearity of some composite functionals:

Theorem 2. *Let C be a convex cone in the linear space X and $v : C \to (0, \infty)$ an additive functional on C. If $h : C \to [0, \infty)$ is a superadditive (subadditive) functional on C and $p, q \geq 1$ $(0 < p, q < 1)$ then the functional*

$$\Psi_{p,q} : C \to [0, \infty), \Psi_{p,q}(x) = h^q(x) v^{q\left(1-\frac{1}{p}\right)}(x) \tag{1.2}$$

is superadditive (subadditive) on C.

Theorem 3. *Let C be a convex cone in the linear space X and $v : C \to (0, \infty)$ an additive functional on C. If $h : C \to [0, \infty)$ is a superadditive functional on C and $0 < p, q < 1$ then the functional*

$$\Phi_{p,q} : C \to [0, \infty), \Phi_{p,q}(x) = \frac{v^{q\left(1-\frac{1}{p}\right)}(x)}{h^q(x)} \tag{1.3}$$

is subadditive on C.

Another result similar to Theorem 1 has been obtained in [5] as well, namely

Theorem 4. *Let $x, y \in C$, $h : C \to \mathbb{R}$ be a nonnegative, superadditive and $s-$positive homogeneous functional on C and v an additive, strictly positive and positive homogeneous functional on C. If $p, q \geq 1$ and $M \geq m \geq 0$ are such that $x - my$, $My - x \in C$, then*

$$M^{sq+q\left(1-\frac{1}{p}\right)} \Psi_{p,q}(y) \geq \Psi_{p,q}(x) \geq m^{sq+q\left(1-\frac{1}{p}\right)} \Psi_{p,q}(y) \tag{1.4}$$

where $\Psi_{p,q}$ is defined by (1.2).

As shown in [4] and [5], the above results can be applied to obtain refinements of the Jensen, Hölder, Minkowski and Schwarz inequalities for weights satisfying certain conditions.

The main aim of the present paper is to study quasilinearity properties of other composite functionals generated by monotonic and convex/concave functions and to apply the obtained results in improving some classical inequalities as those mentioned above.

2. Some General Results

We start with the following general result:

Theorem 5 (Quasilinearity Theorem)**.** *Let C be a convex cone in the linear space X and $v : C \to (0, \infty)$ an additive functional on C.*

(i) *If $h : C \to [0, \infty)$ is a superadditive (**subadditive**) functional on C and $\Phi : [0, \infty) \to \mathbb{R}$ is concave (**convex**) and monotonic nondecreasing on $[0, \infty)$, then the composite functional $\eta_\Phi : C \to \mathbb{R}$ defined by*

$$\eta_\Phi(x) := v(x) \Phi\left(\frac{h(x)}{v(x)}\right) \tag{2.1}$$

*is superadditive (**subadditive**) on C.*

(ii) *If $h: C \to [0,\infty)$ is a superadditive (**subadditive**) functional on C and $\Phi: [0,\infty) \to \mathbb{R}$ is convex (**concave**) and monotonic nonincreasing on $[0,\infty)$, then the composite functional η_Φ is subadditive (**superadditive**) on C.*

Proof. (i) Assume that h is superadditive and $\Phi: [0,\infty) \to \mathbb{R}$ is concave and monotonic nondecreasing on $[0,\infty)$. Then

$$h(x+y) \geq h(x) + h(y) \text{ for any } x, y \in C$$

and since $v(x+y) = v(x) + v(y)$ for any $x, y \in C$, then by the monotonicity of Φ we have

$$\begin{aligned}\Phi\left(\frac{h(x+y)}{v(x+y)}\right) &= \Phi\left(\frac{h(x+y)}{v(x)+v(y)}\right) \\ &\geq \Phi\left(\frac{h(x)+h(y)}{v(x)+v(y)}\right) \\ &= \Phi\left(\frac{v(x)\cdot\frac{h(x)}{v(x)} + v(y)\cdot\frac{h(y)}{v(y)}}{v(x)+v(y)}\right)\end{aligned} \tag{2.2}$$

for any $x, y \in C$.

Now, since $\Phi: [0,\infty) \to \mathbb{R}$ is concave then

$$\begin{aligned}&\Phi\left(\frac{v(x)\cdot\frac{h(x)}{v(x)} + v(y)\cdot\frac{h(y)}{v(y)}}{v(x)+v(y)}\right) \\ &\geq \frac{v(x)\Phi\left(\frac{h(x)}{v(x)}\right) + v(y)\Phi\left(\frac{h(y)}{v(y)}\right)}{v(x)+v(y)} \\ &= \frac{v(x)\Phi\left(\frac{h(x)}{v(x)}\right) + v(y)\Phi\left(\frac{h(y)}{v(y)}\right)}{v(x+y)}\end{aligned} \tag{2.3}$$

for any $x, y \in C$.

Utilising (2.2) and (2.3) we get

$$v(x+y)\Phi\left(\frac{h(x+y)}{v(x+y)}\right) \geq v(x)\Phi\left(\frac{h(x)}{v(x)}\right) + v(y)\Phi\left(\frac{h(y)}{v(y)}\right) \tag{2.4}$$

for any $x, y \in C$, which shows that the functional η_Φ is superadditive on C.

Now, if $h: C \to \mathbb{R}$ is a subadditive functional on C and $\Phi: [0,\infty) \to [0,\infty)$ is convex and monotonic nondecreasing on $[0,\infty)$, then the inequalities (2.2), (2.3) and (2.4) hold with the reverse sign for any $x, y \in C$, which shows that the functional η_Φ is subadditive on C.

(ii) Follows in a similar manner and the details are omitted. □

Corollary 1 (Boundedness Property)**.** *Let C be a convex cone in the linear space X and $v: C \to (0,\infty)$ an additive and positive homogeneous functional on C. Let $x, y \in C$ and assume that there exist $M \geq m > 0$ such that $x - my$ and $My - x \in C$.*

(a) *If $h: C \to [0,\infty)$ is a superadditive and positive homogeneous functional on C and $\Phi: [0,\infty) \to [0,\infty)$ is concave and monotonic nondecreasing on $[0,\infty)$, then*

$$M\eta_\Phi(y) \geq \eta_\Phi(x) \geq m\eta_\Phi(y). \tag{2.5}$$

(aa) *If* $h : C \to [0,\infty)$ *is a subadditive and positive homogeneous functional on* C *and* $\Phi : [0,\infty) \to [0,\infty)$ *is concave and monotonic nonincreasing on* $[0,\infty)$, *then (2.5) is valid as well.*

Proof. We observe that if v and h are positive homogeneous functionals, then η_Φ is also a positive homogeneous functional and by the Quasilinearity theorem above it follows that in both cases η_Φ is a superadditive functional on C. By applying Theorem 1 for $s = 1$ we deduce the desired result. □

Remark 1 (Monotonicity Property)**.** *Let* C *be a convex cone in the linear space* X. *We say, for* $x, y \in X$, *that* $x \geq_C y$ *(*x *is greater than* y *relative to the cone* C*) if* $x - y \in C$. *Now, observe that if* $x, y \in C$ *and* $x \geq_C y$, *then, under the assumptions of Corollary 1, by (2.5), we have that* $\eta_\Phi (x) \geq \eta_\Phi (y)$, *which is a monotonicity property for the functional* η_Φ.

There are various possibilities to build such functionals. For instance, for the finite families of functionals $v_i : C \to (0,\infty)$ and $h_i : C \to [0,\infty)$ with $i \in I$, (I is a *finite family of indices*) and $\Phi : [0,\infty) \to \mathbb{R}$ is concave/convex and monotonic, then the composite functional $\sigma_\Phi : C \to \mathbb{R}$ defined by

$$\sigma_\Phi (x) := \sum_{i \in I} v_i (x) \Phi \left(\frac{h_i (x)}{v_i (x)} \right) \tag{2.6}$$

has the same properties as the functional η_Φ.

If, for a given cone C we consider the *Cartesian product* $C^n := C \times ... \times C \subset X^n$ and define for the vector $\overline{x} := (x_1, ..., x_n) \in C^n$ the functional $\varpi_\Phi : C^n \to \mathbb{R}$ given by

$$\varpi_\Phi (\overline{x}) := \sum_{i=1}^{n} v (x_i) \Phi \left(\frac{h (x_i)}{v (x_i)} \right) \tag{2.7}$$

where v and h defined on C are as above, then we observe that ϖ_Φ has the same properties as η_Φ.

There are some natural examples of composite functionals that are embodied in the propositions below.

Proposition 1. *Let* C *be a convex cone in the linear space* X *and* $v : C \to (0,\infty)$ *an additive functional on* C.

(i) *If* $h : C \to (0,\infty)$ *is a superadditive functional on* C *and* $r > 0$ *then the composite functional* $\eta_r : C \to (0,\infty)$ *defined by*

$$\eta_r (x) := \frac{[v (x)]^{1+r}}{[h (x)]^r} \tag{2.8}$$

is subadditive on C. *In particular,* $\eta_1 (x) = \frac{v^2(x)}{h(x)}$ *is subadditive.*

(ii) *If* $h : C \to [0,\infty)$ *is a superadditive functional on* C *and* $q \in (0,1)$, *then the composite functional* $\eta_q : C \to [0,\infty)$ *defined by*

$$\eta_q (x) := [v (x)]^{1-q} [h (x)]^q \tag{2.9}$$

is superadditive on C. *In particular,* $\eta_{1/2} (x) = \sqrt{v (x) h (x)}$ *is superadditive.*

(iii) *If* $h : C \to [0,\infty)$ *is a subadditive functional on* C *and* $p \geq 1$, *then the composite functional* $\eta_p : C \to [0,\infty)$ *defined by*

$$\eta_p(x) := \frac{[h(x)]^p}{[v(x)]^{p-1}} \tag{2.10}$$

is subadditive on C. *In particular,* $\eta_2(x) = \frac{h^2(x)}{v(x)}$ *is subadditive.*

Proof. Follows from Theorem 5 for the function $\Phi : (0,\infty) \to (0,\infty), \Phi(t) = t^s$ which is convex and decreasing for $s \in (-\infty, 0)$, concave and increasing for $s \in (0,1)$ and convex and increasing for $s \in [1,\infty)$. The details are omitted. □

The following boundedness property also holds:

Corollary 2. *Let* C *be a convex cone in the linear space* X *and* $v : C \to (0,\infty)$ *an additive and positive homogeneous functional on* C. *Let* $x, y \in C$ *and assume that there exist* $M \geq m > 0$ *such that* $x - my$ *and* $My - x \in C$. *If* $h : C \to [0,\infty)$ *is a superadditive and positive homogeneous functional on* C *and* $q \in (0,1)$, *then*

$$M[v(y)]^{1-q}[h(y)]^q \geq [v(x)]^{1-q}[h(x)]^q \geq m[v(y)]^{1-q}[h(y)]^q. \tag{2.11}$$

In particular

$$M^2 v(y) h(y) \geq v(x) h(x) \geq m^2 v(y) h(y). \tag{2.12}$$

Proposition 2. *Let* C *be a convex cone in the linear space* X *and* $v : C \to (0,\infty)$ *an additive functional on* C.

(i) *If* $h : C \to [0,\infty)$ *is a subadditive functional on* C, *then the composite functional* $\varepsilon_\alpha : C \to (0,\infty)$ *defined by*

$$\varepsilon_\alpha(x) := v(x)\exp\left(\frac{\alpha h(x)}{v(x)}\right) \tag{2.13}$$

is subadditive on C *provided* $\alpha > 0$.

(ii) *If* $h : C \to [0,\infty)$ *is a superadditive functional on* C, *then the composite functional* ε_α *is also subadditive on* C *when* $\alpha < 0$.

The proof follows by Theorem 5. The details are omitted.

Remark 2. *Similar composite functionals can be considered for the functions* $\Phi : [0,\infty) \to \mathbb{R}$ *defined as follows*

$$\Phi(t) = \arctan(t), \text{ which is increasing and concave on } [0,\infty);$$

$$\Phi(t) = \sinh(t) := \frac{1}{2}\left(e^t - e^{-t}\right), \text{ which is increasing and convex on } [0,\infty);$$

$$\Phi(t) = \cosh(t) := \frac{1}{2}\left(e^t + e^{-t}\right), \text{ which is increasing and convex on } [0,\infty);$$

$$\Phi(t) = \tanh(t) := \frac{e^t - e^{-t}}{e^t + e^{-t}}, \text{ which is increasing and concave on } [0,\infty);$$

$$\Phi(t) = \coth(t) := \frac{e^t + e^{-t}}{e^t - e^{-t}}, \text{ which is decreasing and convex on } (0,\infty).$$

For instance, if we consider the composite functional

$$\eta_{\arctan}(x) := v(x)\arctan\left(\frac{h(x)}{v(x)}\right),$$

where $h : C \to [0, \infty)$ is a superadditive functional on C, $v : C \to (0, \infty)$ is an additive functional on C and C is a convex cone in the linear space X, then by the Quasilinearity Theorem we conclude that $\eta_{\arctan} : C \to [0, \infty)$ is superadditive on C. Moreover, if $v : C \to (0, \infty)$ is an additive and positive homogeneous functional on C, $h : C \to [0, \infty)$ is a superadditive and positive homogeneous functional on C and $x, y \in C$ such that there exist $M \geq m > 0$ with the property that $x - my$ and $My - x \in C$, then

$$Mv(y)\arctan\left(\frac{h(y)}{v(y)}\right) \geq v(x)\arctan\left(\frac{h(x)}{v(x)}\right) \geq mv(y)\arctan\left(\frac{h(y)}{v(y)}\right).$$

The same properties hold for the composite functional generated by the hyperbolic tangent function, namely

$$\eta_{\tanh}(x) := v(x)\left[\frac{\exp\left(\frac{h(x)}{v(x)}\right) - \exp\left(-\frac{h(x)}{v(x)}\right)}{\exp\left(\frac{h(x)}{v(x)}\right) + \exp\left(-\frac{h(x)}{v(x)}\right)}\right],$$

however the details are omitted.

Taking into account the above result and its applications for various concrete examples of convex functions, it is therefore natural to investigate the corresponding results for the case of *log-convex functions*, namely functions $\Psi : I \to (0, \infty)$, I is an interval of real numbers, for which $\ln \Psi$ is convex.

We observe that such functions satisfy the elementary inequality

$$\Psi((1-t)a + tb) \leq [\Psi(a)]^{1-t}[\Psi(b)]^{t}$$

for any $a, b \in I$ and $t \in [0, 1]$. Also, due to the fact that the weighted geometric mean is less than the weighted arithmetic mean, it follows that any log-convex function is a convex functions. However, obviously, there are functions that are convex but not log-convex.

Theorem 6 (Quasimultiplicity Theorem)**.** *Let C be a convex cone in the linear space X and $v : C \to (0, \infty)$ an additive functional on C.*

(i) *If $h : C \to [0, \infty)$ is a superadditive (**subadditive**) functional on C and $F : [0, \infty) \to (0, \infty)$ is log-concave (**log-convex**) and monotonic nondecreasing on $[0, \infty)$, then the composite functional $\xi_F : C \to (0, \infty)$ defined by*

$$\xi_F(x) := \left[F\left(\frac{h(x)}{v(x)}\right)\right]^{v(x)} \tag{2.14}$$

*is supermultiplicative (**submultiplicative**) on C, i.e., we recall that*

$$\xi_F(x+y) \geq (\leq)\, \xi_F(x)\,\xi_F(y) \tag{2.15}$$

for any $x, y \in C$.

(ii) *If $h : C \to [0, \infty)$ is a superadditive (**subadditive**) functional on C and $F : [0, \infty) \to (0, \infty)$ is log-convex (**log-concave**) and monotonic nonincreasing on $[0, \infty)$, then the composite functional η_F is submultiplicative (**supemultiplicative**) on C.*

Proof. We observe that

$$\log \xi_F(x) = v(x)\log\left[F\left(\frac{h(x)}{v(x)}\right)\right] = \eta_{\log(F)}(x)$$

for any $x \in C$.

Applying now the Quasilinearity Theorem for the functions $\log(F)$ we deduce the desired result.

The details are omitted. □

Corollary 3 (Exponential boundedness)**.** *Let C be a convex cone in the linear space X and $v : C \to (0,\infty)$ an additive and positive homogeneous functional on C. Let $x, y \in C$ and assume that there exist $M \geq m > 0$ such that $x - my$ and $My - x \in C$.*

(a) *If $h : C \to [0,\infty)$ is a superadditive and positive homogeneous functional on C and $F : [0,\infty) \to [1,\infty)$ is log-concave and monotonic nondecreasing on $[0,\infty)$, then*

$$\left[F\left(\frac{h(y)}{v(y)}\right)\right]^{Mv(y)} \geq \left[F\left(\frac{h(x)}{v(x)}\right)\right]^{v(x)} \geq \left[F\left(\frac{h(y)}{v(y)}\right)\right]^{mv(y)}. \tag{2.16}$$

(aa) *If $h : C \to [0,\infty)$ is a subadditive and positive homogeneous functional on C and $F : [0,\infty) \to [1,\infty)$ is log-concave and monotonic nonincreasing on $[0,\infty)$, then (2.16) is valid as well.*

There are numerous examples of log-convex (log-concave) functions of interest that can provide some nice examples.

Following [2], we consider the following *Dirichlet series*:

$$\psi(s) := \sum_{n=1}^{\infty} \frac{a_n}{n^s} \tag{2.17}$$

for which we assume that the coefficients $a_n \geq 0$ for $n \geq 1$ and the series is uniformly convergent for $s > 1$.

It is obvious that in this class we can find the *Zeta function*

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$$

and the *Lambda function*

$$\lambda(s) := \sum_{n=0}^{\infty} \frac{1}{(2n+1)^s} = \left(1 - 2^{-s}\right)\zeta(s),$$

where $s > 1$.

If $\Lambda(n)$ is the *von Mangoldt function*, where

$$\Lambda(n) := \begin{cases} \log p, & n = p^k \quad (p \text{ prime}, \, k \geq 1) \\ 0, & \text{otherwise}, \end{cases}$$

then [8, p. 3]:

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}, \quad s > 1.$$

If $d(n)$ is the number of divisors of n, we have [8, p. 35] the following relationships with the Zeta function:

$$\zeta^2(s) = \sum_{n=1}^{\infty} \frac{d(n)}{n^s}, \qquad \frac{\zeta^3(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{d(n^2)}{n^s}, \qquad \frac{\zeta^4(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{d^2(n)}{n^s},$$

and [8, p. 36]

$$\frac{\zeta^2(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{2^{\omega(n)}}{n^s}, \quad s > 1,$$

where $\omega(n)$ is the number of distinct prime factors of n.

We use the following result, see [2]

Lemma 1. *The function ψ defined by (2.17) is nonincreasing and log-convex on $(1, \infty)$.*

Utilising the Quasimultiplicity Theorem and this lemma, we can state the following result as well:

Proposition 3. *Let C be a convex cone in the linear space X and $v : C \to (0, \infty)$ an additive functional on C. If $h : C \to [0, \infty)$ is a subadditive functional on C and $\psi : (1, \infty) \to (0, \infty)$ is defined by (2.17) then the composite functional $\xi_\psi : C \to (0, \infty)$ defined by*

$$\xi_\psi(x) := \left[\psi\left(\frac{h(x) + v(x)}{v(x)}\right)\right]^{v(x)} \tag{2.18}$$

is submultiplicative on C.

Proof. We observe that the function $F(t) := \psi(t+1)$ is well defined on $(0, \infty)$ and is nonincreasing and log-convex on this interval. Applying Theorem 6 we deduce the desired result. □

3. Applications

3.1. Applications for Jensen's Inequality. Let C be a convex subset of the real linear space X and let $f : C \to \mathbb{R}$ be a convex mapping. Here we consider the following well-known form of *Jensen's discrete inequality*:

$$f\left(\frac{1}{P_I}\sum_{i\in I} p_i x_i\right) \le \frac{1}{P_I}\sum_{i\in I} p_i f(x_i), \tag{3.1}$$

where I denotes a finite subset of the set $\mathbb{N}$ of natural numbers, $x_i \in C$, $p_i \ge 0$ for $i \in I$ and $P_I := \sum_{i\in I} p_i > 0$.

Let us fix $I \in \mathcal{P}_f(\mathbb{N})$ (the class of finite parts of $\mathbb{N}$) and $x_i \in C$ $(i \in I)$. Now consider the functional $J : S_+(I) \to \mathbb{R}$ given by

$$J_I(\mathbf{p}) := \sum_{i\in I} p_i f(x_i) - P_I f\left(\frac{1}{P_I}\sum_{i\in I} p_i x_i\right) \ge 0 \tag{3.2}$$

where $S_+(I) := \left\{\mathbf{p} = (p_i)_{i\in I} \middle| p_i \ge 0,\ i \in I \text{ and } P_I > 0\right\}$ and f is convex on C.

We observe that $S_+(I)$ is a convex cone and the functional J_I is nonnegative and positive homogeneous on $S_+(I)$.

Lemma 2 ([7])**.** *The functional $J_I(\cdot)$ is a superadditive functional on $S_+(I)$.*

For a function $\Phi : [0, \infty) \to \mathbb{R}$, define the following functional $\xi_{\Phi,I} : S_+(I) \to \mathbb{R}$

$$\xi_{\Phi,I}(\mathbf{p}) := P_I \Phi\left(\frac{1}{P_I}\sum_{i\in I} p_i f(x_i) - f\left(\frac{1}{P_I}\sum_{i\in I} p_i x_i\right)\right). \tag{3.3}$$

By the use of Theorem 5 we can state the following proposition:

Proposition 4. *If $\Phi : [0,\infty) \to \mathbb{R}$ is concave and monotonic nondecreasing on $[0,\infty)$, then the composite functional $\xi_{\Phi,I} : S_+(I) \to \mathbb{R}$ defined by (3.3) is superadditive on $S_+(I)$.*

If $\Phi : [0,\infty) \to \mathbb{R}$ is convex and monotonic nonincreasing on $[0,\infty)$, then the composite functional $\xi_{\Phi,I}$ is subadditive on $S_+(I)$.

Proof. Consider the functionals $v(\mathbf{p}) := P_I$ and $h(\mathbf{p}) := J_I(\mathbf{p})$. We observe that v is additive, h is superadditive and

$$\eta_\Phi(\mathbf{p}) = v(\mathbf{p})\,\Phi\left(\frac{h(\mathbf{p})}{v(\mathbf{p})}\right) = \xi_{\Phi,I}(\mathbf{p}).$$

Applying Theorem 5 we deduce the desired result. □

Corollary 4. *If $\mathbf{p},\mathbf{q} \in S_+(I)$ and $M \geq m \geq 0$ are such that $M\mathbf{p} \geq \mathbf{q} \geq m\mathbf{p}$, i.e., $Mp_i \geq q_i \geq mp_i$ for each $i \in I$, then:*

$$\begin{aligned}(3.4)\qquad & M\frac{P_I}{Q_I}\Phi\left(\frac{1}{P_I}\sum_{i\in I}p_i f(x_i) - f\left(\frac{1}{P_I}\sum_{i\in I}p_i x_i\right)\right)\\ &\geq \Phi\left(\frac{1}{Q_I}\sum_{i\in I}q_i f(x_i) - f\left(\frac{1}{Q_I}\sum_{i\in I}q_i x_i\right)\right)\\ &\geq m\frac{P_I}{Q_I}\Phi\left(\frac{1}{P_I}\sum_{i\in I}p_i f(x_i) - f\left(\frac{1}{P_I}\sum_{i\in I}p_i x_i\right)\right) \geq 0\end{aligned}$$

for any $\Phi : [0,\infty) \to [0,\infty)$ a concave and monotonic nondecreasing function on $[0,\infty)$.

The proof follows from Corollary 1 and the details are omitted.

On utilizing Proposition 1, statement (ii), we observe that the functional $\xi_{q,I} : S_+(I) \to [0,\infty)$, where $q \in (0,1)$ and

$$\begin{aligned}(3.5)\qquad \xi_{q,I}(\mathbf{p}) &:= P_I\left(\frac{1}{P_I}\sum_{i\in I}p_i f(x_i) - f\left(\frac{1}{P_I}\sum_{i\in I}p_i x_i\right)\right)^q\\ &= \left(P_I^{q-1}\sum_{i\in I}p_i f(x_i) - P_I^q f\left(\frac{1}{P_I}\sum_{i\in I}p_i x_i\right)\right)^q\end{aligned}$$

is superadditive and monotonic nondecreasing on $S_+(I)$.

If $\mathbf{p},\mathbf{q} \in S_+(I)$ and $M \geq m \geq 0$ are such that $M\mathbf{p} \geq \mathbf{q} \geq m\mathbf{p}$, then

$$\begin{aligned}(3.6)\qquad & M^{1/q}\left(\frac{P_I}{Q_I}\right)^{1/q}\left(\frac{1}{P_I}\sum_{i\in I}p_i f(x_i) - f\left(\frac{1}{P_I}\sum_{i\in I}p_i x_i\right)\right)\\ &\geq \frac{1}{Q_I}\sum_{i\in I}q_i f(x_i) - f\left(\frac{1}{Q_I}\sum_{i\in I}q_i x_i\right)\\ &\geq m^{1/q}\left(\frac{P_I}{Q_I}\right)^{1/q}\left(\frac{1}{P_I}\sum_{i\in I}p_i f(x_i) - f\left(\frac{1}{P_I}\sum_{i\in I}p_i x_i\right)\right) \geq 0\end{aligned}$$

Now, if we consider the following composite functional $\xi_{\arctan,I}:S_{+}(I)\to[0,\infty)$ given by

$$\xi_{\arctan,I}(\mathbf{p})=P_{I}\arctan\left(\frac{1}{P_{I}}\sum_{i\in I}p_{i}f(x_{i})-f\left(\frac{1}{P_{I}}\sum_{i\in I}p_{i}x_{i}\right)\right), \tag{3.7}$$

then by utilizing Remark 2 we conclude that $\xi_{\arctan,I}$ is *superadditive and monotonic nondecreasing* on $S_{+}(I)$.

Moreover, if $\mathbf{p},\mathbf{q}\in S_{+}(I)$ and $M\geq m\geq 0$ are such that $M\mathbf{p}\geq\mathbf{q}\geq m\mathbf{p}$, then

$$\begin{aligned}(3.8)\qquad & M\left(\frac{P_{I}}{Q_{I}}\right)\arctan\left(\frac{1}{P_{I}}\sum_{i\in I}p_{i}f(x_{i})-f\left(\frac{1}{P_{I}}\sum_{i\in I}p_{i}x_{i}\right)\right)\\ &\geq\arctan\left(\frac{1}{Q_{I}}\sum_{i\in I}q_{i}f(x_{i})-f\left(\frac{1}{Q_{I}}\sum_{i\in I}q_{i}x_{i}\right)\right)\\ &\geq m\left(\frac{P_{I}}{Q_{I}}\right)\arctan\left(\frac{1}{P_{I}}\sum_{i\in I}p_{i}f(x_{i})-f\left(\frac{1}{P_{I}}\sum_{i\in I}p_{i}x_{i}\right)\right)\geq 0\end{aligned}$$

It is also well known that if $f:C\to\mathbb{R}$ is a *strictly convex mapping* on C and for a given sequence of vectors $x_{i}\in C$ $(i\in I)$ there exists at least two distinct indices k and j in I so that $x_{k}\neq x_{j}$ then

$$J_{I}(\mathbf{p}):=\sum_{i\in I}p_{i}f(x_{i})-P_{I}f\left(\frac{1}{P_{I}}\sum_{i\in I}p_{i}x_{i}\right)>0$$

for any $\mathbf{p}\in S_{+}(I)=\left\{\mathbf{p}=(p_{i})_{i\in I}\middle|\,p_{i}\geq 0,\ i\in I\ \text{ and }\ P_{I}>0\right\}$.

In this situation for the function f and the sequence $x_{i}\in C$ $(i\in I)$ we can define the functional

$$\eta_{r,I}(\mathbf{p}):=\frac{P_{I}^{1+r}}{\left[\sum_{i\in I}p_{i}f(x_{i})-P_{I}f\left(\frac{1}{P_{I}}\sum_{i\in I}p_{i}x_{i}\right)\right]^{r}} \tag{3.9}$$

that is well-defined on $S_{+}(I)$. Utilising the statement (i) from Proposition 1 we conclude that $\eta_{r,I}(\cdot)$ is a *subadditive functional* on $S_{+}(I)$.

We know that the hyperbolic cotangent function $\coth(t):=\frac{e^{t}+e^{-t}}{e^{t}-e^{-t}}$ is decreasing and convex on $(0,\infty)$. If we consider the composite functional

$$\xi_{\coth,I}(\mathbf{p}):=P_{I}\coth\left(\frac{1}{P_{I}}\sum_{i\in I}p_{i}f(x_{i})-f\left(\frac{1}{P_{I}}\sum_{i\in I}p_{i}x_{i}\right)\right)$$

for a function $f:C\to\mathbb{R}$ that is *strictly convex* on C and for a given sequence of vectors $x_{i}\in C$ $(i\in I)$ for which there exists at least two distinct indices k and j in I so that $x_{k}\neq x_{j}$, then we observe that this functional is well-defined on $S_{+}(I)$ and by the statement (ii) of Theorem 5 we conclude that $\xi_{\coth,I}(\cdot)$ is also a *subadditive functional* on $S_{+}(I)$.

3.2. **Applications for Hölder's Inequality.** Let $(X,\|\cdot\|)$ be a normed space and $I\in\mathcal{P}_{f}(\mathbb{N})$. We define

$$E(I):=\left\{x=(x_{j})_{j\in I}\middle|\,x_{j}\in X,\ j\in I\right\}$$

and

$$\mathbb{K}(I) := \left\{ \lambda = (\lambda_j)_{j\in I} \middle| \lambda_j \in \mathbb{K},\ j \in I \right\}.$$

We consider for $\gamma, \beta > 1$, $\frac{1}{\gamma} + \frac{1}{\beta} = 1$ the functional

$$H_I(\mathbf{p}, \lambda, x; \gamma, \beta) := \left(\sum_{j\in I} p_j |\lambda_j|^\gamma \right)^{\frac{1}{\gamma}} \left(\sum_{j\in I} p_j \|x_j\|^\beta \right)^{\frac{1}{\beta}} - \left\| \sum_{j\in I} p_j \lambda_j x_j \right\|.$$

The following result has been proved in [4]:

Lemma 3. *For any $\mathbf{p}, \mathbf{q} \in S_+(I)$ we have*

$$H_I(\mathbf{p}+\mathbf{q}, \lambda, x; \gamma, \beta) \geq H_I(\mathbf{p}, \lambda, x; \gamma, \beta) + H_I(\mathbf{q}, \lambda, x; \gamma, \beta), \tag{3.10}$$

where $x \in E(I)$, $\lambda \in \mathbb{K}(I)$ and $\gamma, \beta > 1$ with $\frac{1}{\gamma} + \frac{1}{\beta} = 1$.

Remark 3. *The same result can be stated if $(B, \|\cdot\|)$ is a normed algebra and the functional H is defined by*

$$H_I(\mathbf{p}, \lambda, x; \gamma, \beta) := \left(\sum_{i\in I} p_i \|x_i\|^\gamma \right)^{\frac{1}{\gamma}} \left(\sum_{i\in I} p_i \|y_i\|^\beta \right)^{\frac{1}{\beta}} - \left\| \sum_{i\in I} p_i x_i y_i \right\|,$$

where $x = (x_i)_{i\in I}$, $y = (y_i)_{i\in I} \subset B$, $\mathbf{p} \in S_+(I)$ and $\gamma, \beta > 1$ with $\frac{1}{\gamma} + \frac{1}{\beta} = 1$.

For a function $\Phi : [0, \infty) \to \mathbb{R}$, define the following functional $\omega_{\Phi,I} : S_+(I) \to \mathbb{R}$

$$\begin{aligned} &\omega_{\Phi,I}(\mathbf{p}) \\ &:= P_I \Phi \left(\left(\frac{1}{P_I} \sum_{j\in I} p_j |\lambda_j|^\gamma \right)^{\frac{1}{\gamma}} \left(\frac{1}{P_I} \sum_{j\in I} p_j \|x_j\|^\beta \right)^{\frac{1}{\beta}} - \left\| \frac{1}{P_I} \sum_{j\in I} p_j \lambda_j x_j \right\| \right). \end{aligned} \tag{3.11}$$

By the use of Theorem 5 we can state the following proposition:

Proposition 5. *If $\Phi : [0, \infty) \to \mathbb{R}$ is concave and monotonic nondecreasing on $[0, \infty)$, then the composite functional $\omega_{\Phi,I} : S_+(I) \to \mathbb{R}$ defined by (3.11) is superadditive on $S_+(I)$.*

If $\Phi : [0, \infty) \to \mathbb{R}$ is convex and monotonic nonincreasing on $[0, \infty)$, then the composite functional $\omega_{\Phi,I}$ is subadditive on $S_+(I)$.

By choosing various examples of concave and monotonic nondecreasing or convex and monotonic nonincreasing functions Φ on $[0, \infty)$ the reader can provide various examples of superadditive or subadditive functionals on $S_+(I)$. The details are omitted.

3.3. **Applications for Minkowski's Inequality.** Let $(X, \|\cdot\|)$ be a normed space and $I \in \mathcal{P}_f(\mathbb{N})$. We define the functional:

$$\begin{aligned} M_I(\mathbf{p}, x, y; \delta) = {} & \left[\left(\sum_{i\in I} p_i \|x_i\|^\delta \right)^{\frac{1}{\delta}} + \left(\sum_{i\in I} p_i \|y_i\|^\delta \right)^{\frac{1}{\delta}} \right]^\delta \\ & - \sum_{i\in I} p_i \|x_i + y_i\|^\delta, \end{aligned} \tag{3.12}$$

where $\mathbf{p} \in S_+(I)$, $\delta \geq 1$ and $x, y \in E(I)$.

The following result concerning the superadditivity of the functional $M_I(\cdot, x, y; \delta)$ holds [4]:

Lemma 4. *For any* $\mathbf{p}, \mathbf{q} \in S_+(I)$, *we have*

$$M_I(\mathbf{p}+\mathbf{q}, x, y; \delta) \geq M_I(\mathbf{p}, x, y; \delta) + M_I(\mathbf{q}, x, y; \delta),$$

where $x, y \in E(I)$ *and* $\delta \geq 1$.

For a function $\Phi : [0,\infty) \to \mathbb{R}$, define the following functional $\varkappa_{\Phi,I} : S_+(I) \to \mathbb{R}$

$$\varkappa_{\Phi,I}(\mathbf{p}) := P_I \Phi \left(\left[\left(\frac{1}{P_I} \sum_{i\in I} p_i \|x_i\|^{\delta} \right)^{\frac{1}{\delta}} + \left(\frac{1}{P_I} \sum_{i\in I} p_i \|y_i\|^{\delta} \right)^{\frac{1}{\delta}} \right]^{\delta} - \frac{1}{P_I} \sum_{i\in I} p_i \|x_i + y_i\|^{\delta} \right). \tag{3.13}$$

By the use of Theorem 5 we can state the following proposition:

Proposition 6. *If* $\Phi : [0,\infty) \to \mathbb{R}$ *is concave and monotonic nondecreasing on* $[0,\infty)$, *then the composite functional* $\varkappa_{\Phi,I} : S_+(I) \to \mathbb{R}$ *defined by (3.13) is superadditive on* $S_+(I)$.

If $\Phi : [0,\infty) \to \mathbb{R}$ *is convex and monotonic nonincreasing on* $[0,\infty)$, *then the composite functional* $\varkappa_{\Phi,I}$ *is subadditive on* $S_+(I)$.

We notice that by choosing various examples of concave and monotonic nondecreasing or convex and monotonic nonincreasing functions Φ on $[0,\infty)$ the reader can provide various examples of superadditive or subadditive functionals on $S_+(I)$. The details are omitted.

Remark 4. *For other examples of superadditive (subadditive) functionals that can provide interesting inequalities similar to the ones outlined above, we refer to* [1], [9], [10] [11] *and* [12].

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