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NEW WEIGHTED OSTROWSKI AND OSTROWSKI-GRÜSS TYPE INEQUALITIES ON TIME SCALES

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Abstract. In this paper we derive new weighted Ostrowski and Ostrowski-Grüss type inequalities on time scales. Some other interesting inequalities on time scales are also given as special cases.

1. Introduction

In 1938, Ostrowski derived a formula to estimate the absolute deviation of a differentiable function from its integral mean [24], the so-called Ostrowski inequality, which can also be shown by using Montgomery identity [22]. In 1997, by combining Montgomery identity and Grüss's integral inequality, Dragomir and Wang [9] proved the following Ostrowski-Grüss type integral inequality, which is a connection between Ostrowski's inequality and Grüss's inequality.

Theorem 1. *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) such that there exist constants $\gamma, \Gamma \in \mathbb{R}$, with $\gamma \le f'(x) \le \Gamma$, $x \in [a, b]$. Then we have*

$$\left| f(x) - \frac{f(b)-f(a)}{b-a}\left(x - \frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)dt \right| \le \frac{1}{4}(b-a)(\Gamma-\gamma), \tag{1.1}$$

for all $x \in [a, b]$.

In [7], Dragomir and Barnett pointed out a new estimation of the left membership of (1.1) as follows.

Theorem 2. *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and twice differentiable on (a, b), whose second derivative $f'' : (a, b) \to \mathbb{R}$ is bounded on (a, b). Then we have*

$$\left| f(x) - \frac{f(b)-f(a)}{b-a}\left(x - \frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)dt \right| \le \frac{M}{2}\left\{\left[\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2} + \frac{1}{4}\right]^2 + \frac{1}{12}\right\}(b-a)^2 \tag{1.2}$$

for all $x \in [a, b]$, where $M = \sup_{a<t<b} |f''(t)| < \infty$.

Recently, Liu [15] established the following perturbed weighted generalization of three point inequality with a parameter for bounded differentiable mapping.

Theorem 3. *Let $0 \le k \le 1$, $f : [a, b] \to \mathbf{R}$ be a differentiable mapping satisfies there exists a constant $\gamma \in \mathbf{R}$ such that $\gamma \le f'(x)$ for $x \in [a, b]$, $g : [a, b] \to [0, \infty)$ continuous and positive on (a, b) and let $h : [a, b] \to \mathbb{R}$ be differentiable such that $h'(t) = g(t)$ on $[a, b]$. Then for all $x \in [a, b]$, we have*

$$\begin{aligned} &\Bigg| \left\{ (1-k)f(x) + k\left[\frac{\int_a^x g(t)dt}{\int_a^b g(t)dt} f(a) + \frac{\int_x^b g(t)dt}{\int_a^b g(t)dt} f(b)\right]\right\}\int_a^b g(t)dt \\ &- \gamma\left\{(1-k)\left[(h(b)-h(a))\left(x-\frac{a+b}{2}\right) + (b-a)\left(h(x) - \frac{h(a)+h(b)}{2}\right)\right]\right. \\ &\left. + \int_a^b (h(t)-h(x))dt\right\} - \int_a^b f(t)g(t)dt \Bigg| \\ &\le \begin{cases} (1-k)\left[\frac{1}{2}\int_a^b g(t)dt + \left|h(x) - \frac{h(a)+h(b)}{2}\right|\right](S-\gamma)(b-a), & k \in \left[0, \frac{1}{2}\right], \\ k\left[\frac{1}{2}\int_a^b g(t)dt + \left|h(x) - \frac{h(a)+h(b)}{2}\right|\right](S-\gamma)(b-a), & k \in \left(\frac{1}{2}, 1\right], \end{cases} \end{aligned} \tag{1.3}$$

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and

$$\left|(1-k)f(x)+k\left[\frac{x-a}{b-a}f(a)+\frac{b-x}{b-a}f(b)\right]-\gamma(1-2k)\left(x-\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)dt\right|$$
$$\leq\left[\frac{1}{2}+\left|k-\frac{1}{2}\right|\right]\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right](S-\gamma) \tag{1.4}$$

when $h(t)=t$ on $[a,b]$, where $S=(f(b)-f(a))/(b-a)$. Similar inequalities hold for $f'(x)\leq\Gamma$, $x\in[a,b]$, $\Gamma\in\mathbf{R}$.

The development of the theory of time scales was initiated by Hilger [10] in 1988 as a theory capable to contain both difference and differential calculus in a consistent way. Since then, many authors have studied the theory of certain integral inequalities on time scales (see [4, 5, 6, 11, 13, 14, 16, 17, 18, 19, 20, 23, 26, 27, 28, 29, 31]). For examples, by using the Montgomery identity on time scales, Bohner and Matthews [5] established the following Ostrowski inequality on time scales which unify discrete, continuous and many other cases.

Theorem 4. *Let $a,b,s,t\in\mathbb{T}$, $a<b$ and $f:[a,b]\to\mathbb{R}$ be differentiable. Then*

$$\left|f(t)-\frac{1}{b-a}\int_a^b f(\sigma(s))\triangle s\right|\leq\frac{M}{b-a}[h_2(t,a)+h_2(t,b)], \tag{1.5}$$

where $M=\sup_{a<t<b}\left|f^{\triangle}(t)\right|<\infty$. This inequality is sharp in the sense that the right-hand side of (1.5) cannot be replaced by a smaller one.

Recently, Karpuz and Özkan [11] generalized Ostrowski's inequality and Montgomery's identity on arbitrary time scales by the means of generalized polynomials on time scales. By introducing a parameter, Liu, Ngô and Chen [20] also extended a generalization of the above inequality on time scales. In [16], Liu and Ngô derived a inequality of Ostrowski-Grüss type on time scales by using the Grüss inequality on time scales. Then, Ngô and Liu [23] gave a sharp Grüss type inequality on time scales and applied it to the sharp Ostrowski-Grüss inequality on time scales. Motivated by the ideas of [5, 20, 23, 32], Tuna and Daghan [31] studied generalizations of Ostrowski and Ostrowski-Grüss type inequality on time scales. More recently, Liu and Tuna [21] derived a weighted Montgomery identity on time scales and then established weighted Ostrowski, Trapezoid and Grüss type inequalities on time scales, respectively.

Motivated by the above researches, the purpose of this paper is to obtain weighted Ostrowski and Ostrowski-Grüss inequalities on time scales. We also give some other interesting inequalities on time scales as special cases.

This paper is organized as follows. In Section 2, we briefly present the general definitions and theorems related to the time scales calculus. The weighted Ostrowski and Ostrowski-Grüss type inequalities on time scales are derived in Section 3.

2. Time Scales Essentials

In this section we briefly introduce the time scales theory. For further details and proofs we refer the reader to Hilger's Ph.D. thesis [10], the books [2, 3, 12], and the survey [1].

Definition 1. *A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of $\mathbb{R}$.*

We assume throughout that $\mathbb{T}$ has the topology that is inherited from the standard topology on $\mathbb{R}$. It also assumed throughout that in $\mathbb{T}$ the interval $[a,b]_{\mathbb{T}}$ means the set $\{t\in\mathbb{T}:\ s<t\}$ for the points $a<b$ in $\mathbb{T}$. Since a time scale may not be connected, we need the following concept of jump operators.

Definition 2. *For $t\in\mathbb{T}$, we define the forward jump operator $\sigma:\mathbb{T}\to\mathbb{T}$ by $\sigma(t)=\inf\{s\in\mathbb{T}:s>t\}$, while the backward jump operator $\rho:\mathbb{T}\to\mathbb{T}$ is defined by $\rho(t)=\sup\{s\in\mathbb{T}:\ s<t\}$.*

The jump operators σ and ρ allow the classification of points in $\mathbb{T}$ as follows.

Definition 3. *If $\sigma(t)>t$, then we say that t is right-scattered, while if $\rho(t)<t$ then we say that t is left-scattered. Points that are right-scattered and left-scattered at the same time are called isolated. If $\sigma(t)=t$, the t is called right-dense, and if $\rho(t)=t$ then t is called left-dense, Points that both right-dense and left-dense are called dense.*

Definition 4. *The mapping $\mu:\mathbb{T}\to\mathbb{R}^+$ defined by $\mu(t)=\sigma(t)-t$ is called the graininess function. The set $\mathbb{T}^k$ is defined as follows: if $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k=\mathbb{T}-\{m\}$; otherwise, $\mathbb{T}^k=\mathbb{T}$.*

If $\mathbb{T}=\mathbb{R}$, then $\mu(t)=0$, and when $\mathbb{T}=\mathbb{Z}$, we have $\mu(t)=1$.

Definition 5. *Let $f:\mathbb{T}\to\mathbb{R}$. f is called differentiable at $t\in\mathbb{T}^k$, with (delta) derivative $f^{\triangle}(t)\in\mathbb{R}$, if given $\varepsilon>0$ there exists a neighborhood U of t such that,*

$$\left|f(\sigma(t))-f(s)-f^{\triangle}(t)[\sigma(t)-s]\right|\leq\varepsilon\left|\sigma(t)-s\right|,\quad\forall\, s\in U.$$

If $\mathbb{T}=\mathbb{R}$, then $f^{\triangle}(t)=\frac{df(t)}{dt}$, and if $\mathbb{T}=\mathbb{Z}$, then $f^{\triangle}(t)=f(t+1)-f(t)$.

Theorem 5. *Assume $f,g:\mathbb{T}\to\mathbb{R}$ are differfentiable at $t\in\mathbb{T}^k$. Then the prfoduct $fg:\mathbb{T}\to\mathbb{R}$ is differfentiable at t with*

$$(fg)^{\triangle}(t)=f^{\triangle}(t)g(t)+f(\sigma(t))g^{\triangle}(t)=f(t)g^{\triangle}(t)+f^{\triangle}(t)g(\sigma(t)).$$

Definition 6. *The function $f:\mathbb{T}\to\mathbb{R}$ is said to be rd-continuous (denote $f\in C_{rd}(\mathbb{T},\mathbb{R})$), if it is continuous at all right-dense points $t\in\mathbb{T}$ and its left-sided limits exist at all left-dense points $t\in\mathbb{T}$.*

Definition 7. *Let $f\in C_{rd}(\mathbb{T},\mathbb{R})$. Then $g:\mathbb{T}\to\mathbb{R}$ is called the antiderivative of f on $\mathbb{T}$ if it satisfies $g^{\triangle}(t)=f(t)$ for any $t\in\mathbb{T}^k$. In this case, we defined*

$$\int_a^t f(s)\triangle s=g(t)-g(a),\quad t\in\mathbb{T}.$$

Theorem 6. *Let f,g be rd-continuous, $a,b,c\in\mathbb{T}$ and $\alpha,\beta\in\mathbb{R}$. Then*

(1) $\int_a^b[f(t)+g(t)]\triangle t=\int_a^b f(t)\triangle t+\int_a^b g(t)\triangle t$,
(2) $\int_a^b f(t)\triangle t=-\int_b^a f(t)\triangle t$,
(3) $\int_a^b f(t)\triangle t=\int_a^c f(t)\triangle t+\int_c^b f(t)\triangle t$,
(4) $\int_a^b f(t)g^{\triangle}(t)\triangle t=(fg)(b)-(fg)(a)-\int_a^b f^{\triangle}(t)g(\sigma(t))\triangle t$,

Theorem 7. *If f is $\triangle$-integrable on $[a,b]$, then so is $|f|$,and*

$$\left|\int_a^b f(t)\triangle t\right|\leq\int_a^b|f(t)|\triangle t.$$

Definition 8. *Let $g_k:\mathbb{T}^2\to\mathbb{R}$, $k\in\mathbb{N}_0$ be defined by*

$$g_0(t,s)=1\quad\text{for all}\quad s,t\in\mathbb{T}$$

and then recursively by

$$g_{k+1}(t,s)=\int_s^t g_k(\sigma(\tau),s)\triangle s\quad\text{for all}\quad s,t\in\mathbb{T}$$

3. Main Results

Theorem 8. *Let $0\leq k\leq1$, $g:[a,b]\to[0,\infty)$ be rd continuous and positive and $h:[a,b]\to\mathbb{R}$ be differentiable such that $h^{\triangle}(t)=g(t)$ on $[a,b]$. Let $a,b,t,x\in\mathbb{T}$, $a<b$ and $f:[a,b]\to\mathbb{R}$ be twice differentiable. Then for all $x\in[a,b]$, we have*

$$\begin{aligned}&\Bigg|(1-k)^2f(x)-\frac{1}{\left(\int_a^b g(t)\triangle t\right)^2}\left(\int_a^b S(x,t)\Delta t\right)\left(\int_a^b g(t)f^{\triangle}(\sigma(t))\Delta t\right)\\&+\frac{k}{\left(\int_a^b g(t)\triangle t\right)^2}\int_a^b S(x,t)\left(f^{\triangle}(a)\int_a^t g(s)\triangle s+f^{\triangle}(b)\int_t^b g(s)\triangle s\right)\Delta t\\&+\frac{k\,(1-k)}{\int_a^b g(t)\triangle t}\left(f(a)\int_a^x g(t)\triangle t+f(b)\int_x^b g(t)\triangle t\right)-\frac{1-k}{\int_a^b g(t)\triangle t}\int_a^b g(t)f(\sigma(t))\Delta t\Bigg|\\&\leq\frac{M}{\left(\int_a^b g(t)\triangle t\right)^2}\int_a^b\int_a^b|S(x,t)|\,|S(t,s)|\,\triangle s\triangle t,\end{aligned}\tag{3.1}$$

where $M=\sup\limits_{a<t<b}\left|f^{\triangle\triangle}(t)\right|<\infty$ and

$$S(x,t)=\begin{cases}h(t)-((1-k)h(a)+kh(x)),&a\leq t<x,\\h(t)-(kh(x)+(1-k)h(b)),&x\leq t\leq b.\end{cases}\tag{3.2}$$

Proof. Using item (4) of Theorem 6 and (3.2), we have (see also [21])

$$
\begin{aligned}
(1-k)^2 f(x) = {} & \frac{1-k}{\int_a^b g(t)\triangle t}\int_a^b S(x,t) f^{\triangle}(t)\triangle t + \frac{1-k}{\int_a^b g(t)\triangle t}\int_a^b g(t) f(\sigma(t))\Delta t \\
& - \frac{(1-k)\,k}{\int_a^b g(t)\triangle t}\left(f(a)\int_a^x g(t)\triangle t + f(b)\int_x^b g(t)\triangle t\right),
\end{aligned} \tag{3.3}
$$

and

$$
\begin{aligned}
(1-k) f^{\triangle}(t) = {} & \frac{1}{\int_a^b g(t)\triangle t}\int_a^b S(t,s) f^{\triangle\triangle}(s)\triangle s + \frac{1}{\int_a^b g(t)\triangle t}\int_a^b g(s) f^{\triangle}(\sigma(s))\Delta s \\
& - \frac{k}{\int_a^b g(t)\triangle t}\left(f^{\Delta}(a)\int_a^t g(s)\triangle s + f^{\Delta}(b)\int_t^b g(s)\triangle s\right).
\end{aligned} \tag{3.4}
$$

Substituting $(1-k)f^{\triangle}(t)$ in the right hand side of (3.3), we obtain

$$
\begin{aligned}
(1-k)^2 f(x) = {} & \frac{1}{\int_a^b g(t)\triangle t}\int_a^b S(x,t)\left[\frac{1}{\int_a^b g(t)\triangle t}\int_a^b S(t,s) f^{\triangle\triangle}(s)\triangle s + \frac{1}{\int_a^b g(t)\triangle t}\int_a^b g(s) f^{\triangle}(\sigma(s))\Delta s\right. \\
& \left. - \frac{k}{\int_a^b g(t)\triangle t}\left(f^{\Delta}(a)\int_a^t g(s)\triangle s + f^{\Delta}(b)\int_t^b g(s)\triangle s\right)\right]\triangle t + \frac{1-k}{\int_a^b g(t)\triangle t}\int_a^b g(t) f(\sigma(t))\Delta t \\
& - \frac{(1-k)\,k}{\int_a^b g(t)\triangle t}\left(f(a)\int_a^x g(t)\triangle t + f(b)\int_x^b g(t)\triangle t\right).
\end{aligned} \tag{3.5}
$$

From (3.5) and using the properties of modulus, the inequality (3.1) is proved. ∎

Corollary 1. *In the case of $\mathbb{T}=\mathbb{R}$ in Theorem 8, we have*

$$
\begin{aligned}
& \left|(1-k)^2 f(x) - \frac{1}{\left(\int_a^b g(t)dt\right)^2}\left(\int_a^b S(x,t)dt\right)\left(\int_a^b g(t) f'(t)dt\right)\right. \\
& + \frac{k}{\left(\int_a^b g(t)dt\right)^2}\int_a^b S(x,t)\left(f'(a)\int_a^t g(s)ds + f'(b)\int_t^b g(s)ds\right)dt \\
& \left. + \frac{k\,(1-k)}{\int_a^b g(t)dt}\left(f(a)\int_a^x g(t)dt + f(b)\int_x^b g(t)dt\right) - \frac{1-k}{\int_a^b g(t)dt}\int_a^b g(t) f(t)dt\right| \\
\leq {} & \frac{M}{\left(\int_a^b g(t)dt\right)^2}\int_a^b\int_a^b |S(x,t)|\,|S(t,s)|\,dsdt,
\end{aligned}
$$

where $g(t)=h'(t)$ on $[a,b]$, $M=\sup_{a<t<b}|f''(t)|<\infty$ and

$$
S(x,t) = \begin{cases} h(t) - ((1-k)h(a) + kh(x)), & a \le t < x, \\ h(t) - (kh(x) + (1-k)h(b)), & x \le t \le b. \end{cases}
$$

Corollary 2. *In the case of $\mathbb{T}=\mathbb{Z}$ in Theorem 8, we have*

$$\Bigg|(1-k)^2 f(x) - \frac{1}{\left(\sum_{t=a}^{b-1} g(t)\right)^2}\left(\sum_{t=a}^{b-1} S(x,t)\right)\left(\sum_{t=a}^{b-1} g(t)\triangle f(t+1)\right)$$
$$+\frac{k}{\left(\sum_{t=a}^{b-1} g(t)\right)^2}\sum_{t=a}^{b-1} S(x,t)\left(\Delta f(a)\sum_{s=a}^{t-1} g(s)+\Delta f(b)\sum_{s=t}^{b-1} g(s)\right)$$
$$+\frac{k(1-k)}{\sum_{t=a}^{b-1} g(t)}\left(f(a)\sum_{t=a}^{x-1} g(t)+f(b)\sum_{t=x}^{b-1} g(t)\right)-\frac{1-k}{\sum_{t=a}^{b-1} g(t)}\sum_{t=a}^{b-1} g(t) f(t+1)\Bigg|$$
$$\leq \frac{M}{\left(\sum_{t=a}^{b-1} g(t)\right)^2}\sum_{t=a}^{b-1}\sum_{s=a}^{b-1}\left|S(x,t)\right|\left|S(t,s)\right|,$$

where $g(t)=h(t+1)-h(t)$ *on* $[a,b-1]$, $M=\sup_{a<t<b}\left|\triangle^2 f(t)\right|<\infty$ *and*

$$S(x,t)=\begin{cases} h(t)-((1-k)h(a)+kh(x)), & a\leq t<x-1, \\ h(t)-(kh(x)+(1-k)h(b)), & x\leq t\leq b-1. \end{cases}$$

Corollary 3. *Let* $f:[a,b]\to\mathbb{R}$, $q>1$, $a=q^m$ *and* $b=q^n$ *with* $m<n$ *in Theorem 8, Then we have*

$$\Bigg|(1-k)^2 f(x) - \frac{1}{\left(\int_{q^m}^{q^n} g(t)\triangle t\right)^2}\left(\sum_{k=m}^{n-1} S(x,q^k)\right)\left(\int_{q^m}^{q^n} g(t) f^{\triangle}(\sigma(t))\Delta t\right)$$
$$+\frac{k}{\left(\int_{q^m}^{q^n} g(t)\triangle t\right)^2}\sum_{k=m}^{n-1} S(x,q^k)\left(f^{\Delta}(q^m)\int_{q^m}^{q^k} g(s)\triangle s+f^{\Delta}(q^n)\int_{q^k}^{q^n} g(s)\triangle s\right)$$
$$+\frac{k(1-k)}{\int_{q^m}^{q^n} g(t)\triangle t}\left(f(q^m)\int_{q^m}^{x} g(t)\triangle t+f(q^n)\int_{x}^{q^n} g(t)\triangle t\right)-\frac{1-k}{\int_{q^m}^{q^n} g(t)\triangle t}\int_{q^m}^{q^n} g(t) f(\sigma(t))\Delta t\Bigg|$$
$$\leq \frac{M}{\left(\int_{q^m}^{q^n} g(t)\triangle t\right)^2}\sum_{k=m}^{n-1}\sum_{s=m}^{n-1}\left|S(x,q^k)\right|\left|S(q^k,q^s)\right|,$$

where $g(t)=\frac{h(qt)-h(t)}{(q-1)t}$ *on* $[q^m,q^n]$, $M=\sup_{q^m<t<q^n}\left|\frac{f(q^2t)-(q+1)f(qt)+qf(t)}{q(q-1)^2t^2}\right|<\infty$ *and*

$$S(x,t)=\begin{cases} h(q^k)-((1-k)h(q^m)+kh(x)), & q^m\leq q^k<x, \\ h(q^k)-(kh(x)+(1-k)h(q^n)), & x\leq q^k\leq q^n. \end{cases}$$

Corollary 4. *In the case of* $h(t)=t$ *in Theorem 8, we have*

$$\Bigg|(1-k)^2 f(x) - \frac{1}{b-a}\left[-h_2(a,(1-k)a+kx)+h_2(x,(1-k)a+kx)\right.$$
$$\left.-h_2(x,kx+(1-k)b)+h_2(b,kx+(1-k)b)\right]\frac{f(\sigma(b))-f(\sigma(a))}{b-a}$$
$$+\frac{k}{(b-a)^2}\int_a^b S(x,t)\left[f^{\Delta}(a)(t-a)+f^{\Delta}(b)(b-t)\right]\Delta t$$
$$+\frac{k(1-k)}{b-a}\left[f(a)(x-a)+f(b)(b-x)\right]-\frac{1-k}{b-a}\int_a^b f(\sigma(t))\Delta t\Bigg|$$
$$\leq \frac{M}{(b-a)^2}\int_a^b \left|S(x,t)\right|\left[h_2(a,(1-k)a+kt)+h_2(t,(1-k)a+kt)\right.$$
$$\left.+h_2(t,kt+(1-k)b)+h_2(b,kt+(1-k)b)\right]\triangle t$$

for all $k \in [0,1]$ *such that* $(1-k)a+kx$ *and* $kx+(1-k)b$ *are in* $\mathbb{T}$, *where* $M = \sup_{a<t<b} \left|f^{\triangle\triangle}(t)\right| < \infty$ *and*

$$S(x,t) = \begin{cases} t-((1-k)a+kx), & a \le t < x, \\ t-(kx+(1-k)b), & x \le t \le b. \end{cases}$$

Corollary 5. *In the case of* $k=0$ *in Corollary 4, we have*

$$\begin{aligned}
&\left|f(x) - \frac{f(\sigma(b))-f(\sigma(a))}{b-a}\frac{h_2(x,a)-h_2(x,b)}{b-a} - \frac{1}{b-a}\int_a^b f(\sigma(t))\Delta t\right| \\
\le&\frac{M}{(b-a)^2}\int_a^b |S(x,t)|\left[h_2(t,a)+h_2(t,b)\right]\triangle t,
\end{aligned}$$

where $M = \sup_{a<t<b} \left|f^{\triangle\triangle}(t)\right| < \infty$ *and*

$$S(x,t) = \begin{cases} t-a, & a \le t < x, \\ t-b, & x \le t \le b. \end{cases}$$

Corollary 6. *In the case of* $\mathbb{T}=\mathbb{R}$ *in Corollary 4, we have*

$$\begin{aligned}
&\Bigg|(1-k)^2 f(x) - (1-2k)\left(x-\frac{a+b}{2}\right)\frac{f(b)-f(a)}{b-a} \\
&+\frac{k}{(b-a)^2}\left\{f'(a)\left[\frac{3k-2}{6}(a-x)^3 - \frac{(b-x)^2}{6}\left[(2-3k)(x-a)+(1-3k)(b-a)\right]\right]\right. \\
&\left.+f'(b)\left[\frac{3k-2}{6}(b-x)^3 - \frac{(a-x)^2}{6}\left[(2-3k)(x-b)-(1-3k)(b-a)\right]\right]\right\} \\
&+\frac{k(1-k)}{b-a}\left[f(a)(x-a)+f(b)(b-x)\right] - \frac{1-k}{b-a}\int_a^b f(s)ds\Bigg| \\
\le&\frac{M}{(b-a)^2}A(x;a,b,k),
\end{aligned}$$

where $M = \sup_{a<t<b} |f''(t)| < \infty$ *and*

$$\begin{aligned}
A(x;a,b,k) :=& \left(\frac{1}{3}k^6 - \frac{2}{3}k^5 + \frac{5}{6}k^4 - \frac{5}{6}k^3 + \frac{2}{3}k^2 - \frac{1}{3}k + \frac{1}{12}\right)\left[(x-a)^4+(b-x)^4\right] \\
&+\left(\frac{2}{3}k^4 - k^3 + \frac{5}{6}k^2 - \frac{1}{3}k + \frac{1}{12}\right)(b-a)^2\left[(x-a)^2+(b-x)^2\right] \\
&+\frac{(2k^2-2k+1)k^3}{6}(b-a)\left[(a-x)^3-(b-x)^3\right] \\
&+\frac{(2k^2-2k+1)(k-1)^2}{3}(x-a)^2(b-x)^2 \\
&+\frac{(2k^2-2k+1)k(k-1)^2}{6}\left[(a-x)^3(b-x)+(b-x)^3(a-x)\right].
\end{aligned}$$

Remark 1. *In the case of* $k=0$ *in Corollary 6, we get (1.2) in Theorem 2.*

Corollary 7. *In the case of* $\mathbb{T}=\mathbb{Z}$ *in Corollary 4, we have*

$$\begin{aligned}
&\Bigg|(1-k)^2 f(x)-\left[(1-2k)\left(x-\frac{a+b}{2}\right)-\frac{1}{2}\right]\frac{f(b+1)-f(a+1)}{b-a}\\
&+\frac{k}{(b-a)^2}\left\{\triangle f(a)\left[\frac{2-3k}{6}(x-a)^3-\frac{1-k}{2}(x-a)^2+\frac{x-a}{6}\right.\right.\\
&\left.+\frac{k}{6}(b-x)^2(-6a-3+3b+3x)+\frac{b-x}{6}\left(3a-3x+3ab-3ax-bx-b^2+2x^2+1\right)\right]\\
&+\triangle f(b)\left[-\frac{2-3k}{6}(b-x)^3-\frac{1-k}{2}(b-x)^2-\frac{b-x}{6}\right.\\
&\left.\left.+\frac{k}{6}(x-a)^2(-6b-3+3a+3x)-\frac{x-a}{6}\left(3b-3x+3ab-ax-3bx-a^2+2x^2+1\right)\right]\right\}\\
&+\frac{k(1-k)}{b-a}\left[f(a)(x-a)+f(b)(b-x)\right]-\frac{1-k}{b-a}\sum_{t=a}^{b-1}f(t+1)\Bigg|\\
\leq&\frac{M}{(b-a)^2}B(x;a,b,k),
\end{aligned}$$

where $M=\sup_{a<t<b}\left|\triangle^2 f(t)\right|<\infty$ *and*

$$\begin{aligned}
&B(x;a,b,k):=3\left(k^2-k+1\right)\left[A_k(x,a)b^2+A_k(x,b)a^2+A_{k-1}(x,a)b^2+A_{k-1}(x,b)a^2\right]\\
&+\left[A_k(x,a)+A_k(x,b)+A_{k-1}(x,a)+A_{k-1}(x,b)\right]\left[ab\left(4k^3-2k^2-2\right)+x\left(-2k^3+11k-7\right)\right.\\
&\left.+x^2\left(2k^4-2k^3+k^2\right)-2(2k-1)\right]-2\left[A_{k-1}(x,a)a+A_{k-1}(x,b)b\right]x\left(4k^3-2k^2+1\right)\\
&+\left[A_k(x,a)b+A_k(x,b)a+A_{k-1}(x,a)b+A_{k-1}(x,b)a\right]\left[\left(4k^2-10k+5\right)+x\left(-4k^3+4k^2-2k\right)\right]\\
&+\left[A_k(x,a)a+A_k(x,b)b+A_{k-1}(x,a)a+A_{k-1}(x,b)b\right]\left[\left(2k^3-4k^2-k+2\right)-2kx\left(2k^3-4k^2+3k-1\right)\right]\\
&+\left[A_k(x,a)+A_k(x,b)\right]\left[ab\left(-8k^2+8k-4\right)+x\left(6k^2-14k+7\right)\right]\\
&+\left[A_k(x,a)a^2+A_k(x,b)b^2+A_{k-1}(x,a)a^2+A_{k-1}(x,b)b^2\right]\left[2k^4-2k^3+2k^2-k+2\right]\\
&+\left[A_k(x,a)a+A_k(x,b)b\right]\left(-6k^2+14k-7\right)+\left[A_k(x,a)a^2+A_k(x,b)b^2\right]\left(-4k^3+6k^2-4k+1\right)\\
&-4\left[A_{k-1}(x,a)b+A_{k-1}(x,b)a\right]x\left(2k^2-2k+1\right)+\left[A_{k-1}(x,a)+A_{k-1}(x,b)\right]x^2\left(4k^3+2k^2-4k+3\right)
\end{aligned}$$

such that

$$A_k(x,y)=\frac{1}{12}k^2(x-y)^2+\frac{1}{12}k^2(x-y)$$

for all $x,y\in\mathbb{T}$.

Remark 2. *In the case of* $k=0$ *in Corollary 7, we have*

$$\begin{aligned}
&\left|f(x)-\frac{f(b+1)-f(a+1)}{b-a}\left(x-\frac{a+b}{2}-\frac{1}{2}\right)-\frac{1}{b-a}\sum_{t=a}^{b-1}f(t+1)\right|\\
\leq&\frac{M}{12\,(b-a)^2}\left\{\left[(x-a)^2+a-x\right]\left[(b-a)^2+(x-a)^2\ +2(b-x)^2+5b+2a-7x+2\right]\right.\\
&\left.+\left[(b-x)^2+b-x\right]\left[(b-a)^2+(b-x)^2+2(x-a)^2+5a+2b-7x+2\right]\right\},
\end{aligned}$$

where $M=\sup_{a<t<b}\left|\triangle^2 f(t)\right|<\infty$.

Theorem 9. *Let* $0\leq k\leq 1$, $g:[a,b]\to[0,\infty)$ *be rd continuous and positive and* $h:[a,b]\to\mathbb{R}$ *be differentiable such that* $h^{\triangle}(t)=g(t)$ *on* $[a,b]$. *Let* $a,b,t,x\in\mathbb{T}$, $a<b$ *and* $f:[a,b]\to\mathbb{R}$ *be differentiable.*

Then for all $x \in [a,b]$, we have

$$\begin{aligned}
&\left| \frac{1}{b-a}\left\{(1-k)f(x)+k\left[\frac{\int_a^x g(t)\triangle t}{\int_a^b g(t)\triangle t}f(a)+\frac{\int_x^b g(t)\triangle t}{\int_a^b g(t)\triangle t}f(b)\right]\right\}\int_a^b g(t)\triangle t \right.\\
&\left. -\frac{f(b)-f(a)}{(b-a)^2}\int_a^b S(x,t)\triangle t-\frac{1}{b-a}\int_a^b f(\sigma(t))g(t)\Delta t \right|\\
&\leq \left[\frac{1}{b-a}\int_a^b S^2(x,t)\triangle t-\left(\frac{1}{b-a}\int_a^b S(x,t)\triangle t\right)^2\right]^{\frac{1}{2}}\\
&\times\left[\frac{1}{b-a}\int_a^b \left(f^{\triangle}(t)\right)^2\triangle t-\left(\frac{f(b)-f(a)}{b-a}\right)^2\right]^{\frac{1}{2}}, \qquad (3.6)
\end{aligned}$$

where

$$S(x,t)=\begin{cases} h(t)-((1-k)h(a)+kh(x)), & a\leq t<x,\\ h(t)-(kh(x)+(1-k)h(b)), & x\leq t\leq b.\end{cases}$$

Proof. We have

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b S(x,t)f^{\triangle}(t)\triangle t-\left(\frac{1}{b-a}\int_a^b S(x,t)\triangle t\right)\left(\frac{1}{b-a}\int_a^b f^{\triangle}(t)\triangle t\right)\\
&=\frac{1}{2(b-a)^2}\int_a^b\int_a^b (S(x,t)-S(x,s))\left(f^{\triangle}(t)-f^{\triangle}(s)\right)\triangle t\triangle s. \qquad (3.7)
\end{aligned}$$

We also have

$$\begin{aligned}
\int_a^b S(x,t)f^{\triangle}(t)\triangle t=&\left\{(1-k)f(x)+k\left[\frac{\int_a^x g(t)\triangle t}{\int_a^b g(t)\triangle t}f(a)+\frac{\int_x^b g(t)\triangle t}{\int_a^b g(t)\triangle t}f(b)\right]\right\}\int_a^b g(t)\triangle t\\
&-\int_a^b g(t)f(\sigma(t))\Delta t \qquad (3.8)
\end{aligned}$$

and

$$\frac{1}{b-a}\int_a^b f^{\triangle}(t)\triangle t=\frac{f(b)-f(a)}{b-a}. \qquad (3.9)$$

Using the Cauchy-Schwartz inequality, we may write

$$\begin{aligned}
&\left|\frac{1}{2(b-a)^2}\int_a^b\int_a^b (S(x,t)-S(x,s))\left(f^{\triangle}(t)-f^{\triangle}(s)\right)\triangle t\triangle s\right|\\
&\leq\left(\frac{1}{2(b-a)^2}\int_a^b\int_a^b (S(x,t)-S(x,s))^2\triangle t\triangle s\right)^{\frac{1}{2}}\left(\frac{1}{2(b-a)^2}\int_a^b\int_a^b \left(f^{\triangle}(t)-f^{\triangle}(s)\right)^2\triangle t\triangle s\right)^{\frac{1}{2}}. \qquad (3.10)
\end{aligned}$$

However

$$\int_a^b\int_a^b (S(x,t)-S(x,s))^2\triangle t\triangle s=\frac{1}{b-a}\int_a^b S^2(x,t)\triangle t-\left(\frac{1}{b-a}\int_a^b S(x,t)\triangle t\right)^2, \qquad (3.11)$$

and

$$\int_a^b\int_a^b \left(f^{\triangle}(t)-f^{\triangle}(s)\right)^2\triangle t\triangle s=\frac{1}{b-a}\int_a^b \left(f^{\triangle}(t)\right)^2\triangle t-\left(\frac{1}{b-a}\int_a^b f^{\triangle}(t)\triangle t\right)^2. \qquad (3.12)$$

Using (3.7)-(3.12), we obtain the inequality (3.6). □

Corollary 8. *In the case of $\mathbb{T}=\mathbb{R}$ in Theorem 9, we have*

$$\left|\frac{1}{b-a}\left\{(1-k)f(x)+k\left[\frac{\int_a^x g(t)dt}{\int_a^b g(t)dt}f(a)+\frac{\int_x^b g(t)dt}{\int_a^b g(t)dt}f(b)\right]\right\}\int_a^b g(t)dt\right.$$
$$\left.-\frac{f(b)-f(a)}{(b-a)^2}\int_a^b S(x,t)dt-\frac{1}{b-a}\int_a^b g(t)f(t)dt\right|$$
$$\leq\left[\frac{1}{b-a}\int_a^b S^2(x,t)dt-\left(\frac{1}{b-a}\int_a^b S(x,t)dt\right)^2\right]^{\frac{1}{2}}\left[\frac{1}{b-a}\int_a^b (f'(t))^2\,dt-\left(\frac{f(b)-f(a)}{b-a}\right)^2\right]^{\frac{1}{2}},$$

where $g(t)=h'(t)$ on $[a,b]$ and

$$S(x,t)=\begin{cases} h(t)-((1-k)h(a)+kh(x)), & a\leq t<x,\\ h(t)-(kh(x)+(1-k)h(b)), & x\leq t\leq b.\end{cases}$$

Corollary 9. *In the case of $\mathbb{T}=\mathbb{Z}$ in Theorem 9, we have*

$$\left|\frac{1}{b-a}\left\{(1-k)f(x)+k\left[\frac{\sum_{t=a}^{x-1}g(t)}{\sum_{t=a}^{b-1}g(t)}f(a)+\frac{\sum_{t=x}^{b-1}g(t)}{\sum_{t=a}^{b-1}g(t)}f(b)\right]\right\}\sum_{t=a}^{b-1}g(t)\right.$$
$$\left.-\frac{f(b)-f(a)}{(b-a)^2}\sum_{t=a}^{b-1}S(x,t)-\frac{1}{b-a}\sum_{t=a}^{b-1}g(t)f(t+1)\right|$$
$$\leq\left[\frac{1}{b-a}\sum_{t=a}^{b-1}S^2(x,t)-\left(\frac{1}{b-a}\sum_{t=a}^{b-1}S(x,t)\right)^2\right]^{\frac{1}{2}}\left[\frac{1}{b-a}\sum_{t=a}^{b-1}(\triangle f(t))^2-\left(\frac{f(b)-f(a)}{b-a}\right)^2\right]^{\frac{1}{2}},$$

where $g(t)=h(t+1)-h(t)$ on $[a,b-1]$ and

$$S(x,t)=\begin{cases} h(t)-((1-k)h(a)+kh(x)), & a\leq t<x-1,\\ h(t)-(kh(x)+(1-k)h(b)), & x\leq t\leq b-1.\end{cases}$$

Corollary 10. *Let $f:[a,b]\to\mathbb{R}$, $q>1$, $a=q^m$ and $b=q^n$ with $m<n$ in Theorem 9, Then we have*

$$\left|\frac{1}{q^n-q^m}\left\{(1-k)f(x)+k\left[\frac{\int_{q^m}^x g(t)\triangle t}{\int_{q^m}^{q^n} g(t)\triangle t}f(q^m)+\frac{\int_x^{q^n} g(t)\triangle t}{\int_{q^m}^{q^n} g(t)\triangle t}f(q^n)\right]\right\}\int_{q^m}^{q^n} g(t)\triangle t\right.$$
$$\left.-\frac{f(q^n)-f(q^m)}{(q^n-q^m)^2}\sum_{k=m}^{n-1}S(x,q^k)-\frac{1}{q^n-q^m}\int_{q^m}^{q^n} g(t)f(\sigma(t))\Delta t\right|$$
$$\leq\left[\frac{1}{q^n-q^m}\sum_{k=m}^{n-1}S^2(x,q^k)-\left(\frac{1}{q^n-q^m}\sum_{k=m}^{n-1}S(x,q^k)\right)^2\right]^{\frac{1}{2}}$$
$$\times\left[\frac{1}{q^n-q^m}\int_{q^m}^{q^n}\left(f^{\triangle}(t)\right)^2\triangle t-\left(\frac{f(q^n)-f(q^m)}{q^n-q^m}\right)^2\right]^{\frac{1}{2}},$$

where $g(t)=\frac{h(qt)-h(t)}{(q-1)t}$ on $[q^m,q^n]$ and

$$S(x,t)=\begin{cases} h(q^k)-((1-k)h(q^m)+kh(x)), & q^m\leq q^k<x,\\ h(q^k)-(kh(x)+(1-k)h(q^n)), & x\leq q^k\leq q^n.\end{cases}$$

Corollary 11. *In the case of* $h(t)=t$ *in Theorem 9, we have*

$$\left|(1-k)f(x)+k\left[\frac{x-a}{b-a}f(a)+\frac{b-x}{b-a}f(b)\right]-\frac{f(b)-f(a)}{(b-a)^2}\times[-h_2(a,(1-k)a+kx)\right.$$

$$\left.+h_2(x,(1-k)a+kx)-h_2(x,kx+(1-k)b)+h_2(b,kx+(1-k)b)]-\frac{1}{b-a}\int_a^b f(\sigma(t))\Delta t\right|$$

$$\leq\left[\frac{1}{b-a}\int_a^b S^2(x,t)\triangle t-\frac{1}{(b-a)^2}\left[-h_2(a,(1-k)a+kx)+h_2(x,(1-k)a+kx)\right.\right.$$

$$\left.\left.-h_2(x,kx+(1-k)b)+h_2(b,kx+(1-k)b)\right]^2\right]^{\frac{1}{2}}\left[\frac{1}{b-a}\int_a^b\left(f^{\triangle}(t)\right)^2\triangle t-\left(\frac{f(b)-f(a)}{b-a}\right)^2\right]^{\frac{1}{2}}$$

for all $k\in[0,1]$ *such that* $(1-k)a+kx$ *and* $kx+(1-k)b$ *are in* $\mathbb{T}$*, where*

$$S(x,t)=\begin{cases} t-((1-k)a+kx), & a\leq t<x,\\ t-(kx+(1-k)b), & x\leq t\leq b.\end{cases}$$

Corollary 12. *In the case of* $k=0$ *in Corollary 11, we have*

$$\left|f(x)-\frac{f(b)-f(a)}{b-a}\frac{h_2(x,a)-h_2(x,b)}{b-a}-\frac{1}{b-a}\int_a^b f(\sigma(t))\Delta t\right|$$

$$\leq\left[\frac{1}{b-a}\int_a^b S^2(x,t)\triangle t-\left(\frac{h_2(x,a)-h_2(x,b)}{b-a}\right)^2\right]^{\frac{1}{2}}\left[\frac{1}{b-a}\int_a^b\left(f^{\triangle}(t)\right)^2\triangle t-\left(\frac{f(b)-f(a)}{b-a}\right)^2\right]^{\frac{1}{2}},$$

where

$$S(x,t)=\begin{cases} t-a, & a\leq t<x,\\ t-b, & x\leq t\leq b.\end{cases}$$

Corollary 13. *In the case of* $\mathbb{T}=\mathbb{R}$ *in Corollary 11, we have*

$$\left|(1-k)f(x)+k\left[\frac{x-a}{b-a}f(a)+\frac{b-x}{b-a}f(b)\right]-(1-2k)\left(x-\frac{a+b}{2}\right)\frac{f(b)-f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(t)dt\right|$$

$$\leq\left[(b-x)(x-a)(k^2-k)+\frac{1}{12}(b-a)^2\right]^{\frac{1}{2}}\left[\frac{1}{b-a}\int_a^b\left(f'(t)\right)^2dt-\left(\frac{f(b)-f(a)}{b-a}\right)^2\right]^{\frac{1}{2}}.$$

Remark 3. *In the case of* $k=0$ *in Corollary 13, we have*

$$\left|f(x)-\frac{f(b)-f(a)}{b-a}\left(x-\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\frac{b-a}{2\sqrt{3}}\left[\frac{1}{b-a}\int_a^b\left(f'(t)\right)^2dt-\left(\frac{f(b)-f(a)}{b-a}\right)^2\right]^{\frac{1}{2}}.$$

This result, for the case of $x=\frac{a+b}{2}$*, was given in* [8, Lemma 1].

Corollary 14. *In the case of* $\mathbb{T}=\mathbb{Z}$ *in Corollary 11, we have*

$$\left|(1-k)f(x)+k\left[\frac{x-a}{b-a}f(a)+\frac{b-x}{b-a}f(b)\right]-\frac{f(b)-f(a)}{b-a}\left[(1-2k)\left(x-\frac{a+b}{2}\right)-\frac{1}{2}\right]\right.$$

$$\left.-\frac{1}{b-a}\sum_{t=a}^{b-1}f(t+1)\right|$$

$$\leq\left[(b-x)(x-a)(k^2-k)+\frac{1}{12}\left((b-a)^2-1\right)\right]^{\frac{1}{2}}\left[\frac{1}{b-a}\sum_{t=a}^{b-1}(\triangle f(t))^2-\left(\frac{f(b)-f(a)}{b-a}\right)^2\right]^{\frac{1}{2}}.$$

Remark 4. *In the case of* $k=0$ *in Corollary 14, we have*

$$\left|f(x)-\frac{f(b)-f(a)}{b-a}\left(x-\frac{a+b}{2}-\frac{1}{2}\right)-\frac{1}{b-a}\sum_{t=a}^{b-1}f(t+1)\right|$$

$$\leq\frac{1}{2\sqrt{3}}\left[(b-a)^2-1\right]^{\frac{1}{2}}\left[\frac{1}{b-a}\sum_{t=a}^{b-1}(\triangle f(t))^2-\left(\frac{f(b)-f(a)}{b-a}\right)^2\right]^{\frac{1}{2}}.$$

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