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ON WEIGHTED OSTROWSKI TYPE, TRAPEZOID TYPE, GRÜSS TYPE AND OSTROWSKI-GRÜSS LIKE INEQUALITIES ON TIME SCALES

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Abstract. In this paper we first derive a weighted Montgomery identity on time scales and then establish weighted Ostrowski type, Trapezoid type, Grüss type and Ostrowski-Grüss like inequalities on time scales, respectively. These results not only provide a generalization of the known results, but also give some other interesting inequalities on time scales as special cases.

1. Introduction

The development of the theory of time scales was initiated by Hilger [10] in 1988 as a theory capable to contain both difference and differential calculus in a consistent way. Since then, many authors have studied certain integral inequalities on time scales (see [2, 8, 11, 13, 14, 15, 16, 17, 18, 19, 20, 22, 24, 25, 28, 29, 30, 31, 33, 34]). In [3], Bohner and Matthews derived the Montgomery identity on time scales and established the following Ostrowski inequality on time scales, which unifies and extends corresponding discrete [9], continuous [21, 23] and other cases.

Theorem 1. *Let $a, b, s, t \in \mathbb{T}$, $a < b$ and $f : [a, b] \to \mathbb{R}$ be differentiable. Then*

$$\left| f(t) - \frac{1}{b-a} \int_a^b f(\sigma(s)) \triangle s \right| \leq \frac{M}{b-a} (h_2(t, a) + h_2(t, b)), \tag{1.1}$$

where $M = \sup_{a<t<b} \left| f^{\triangle}(t) \right| < \infty$. This inequality is sharp in the sense that the right-hand side of (1.1) cannot be replaced by a smaller one.

By introducing a parameter, Liu, Ngô and Chen [19] extended a generalization of the above inequality on time scales. Karpuz and Özkan [11] generalized Ostrowski's inequality and Montgomery's identity on arbitrary time scales by the means of generalized polynomials on time scales. Ngô and Liu [22] gave a sharp Grüss type inequality on time scales and then applied it to the sharp Ostrowski-Grüss inequality on time scales. Motivated by the work of [3, 19, 22, 35], Tuna and Daghan [33] studied generalizations of Ostrowski and Ostrowski-Grüss type inequality on time scales.

Recently, Tseng, Hwang and Dragomir [32] established the following generalizations of weighted Ostrowski type inequalities for mappings of bounded variation.

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Theorem 2. *Let us have* $0 \leq \alpha \leq 1$, $g : [a,b] \to [0,\infty)$ *continuous and positive on* (a,b) *and let* $h : [a,b] \to \mathbb{R}$ *be differentiable such that* $h'(t) = g(t)$ *on* $[a,b]$. *Let* $c = h^{-1}\left(\left(1-\frac{\alpha}{2}\right)h(a) + \frac{\alpha}{2}h(b)\right)$ *and* $d = h^{-1}\left(\frac{\alpha}{2}h(a) + \left(1-\frac{\alpha}{2}\right)h(b)\right)$. *Let* $f : [a,b] \to \mathbb{R}$ *be a mapping of bounded variation. Then, for all* $x \in [c,d]$, *we have*

$$\left|\int_a^b f(t)g(t)dt - \left[(1-\alpha)f(x) + \alpha\frac{f(a)+f(b)}{2}\right]\int_a^b g(t)dt\right| \leq K\bigvee_a^b(f), \tag{1.2}$$

where

$$K := \begin{cases} \dfrac{1-\alpha}{2}\displaystyle\int_a^b g(t)dt + \left|h(x) - \dfrac{h(a)+h(b)}{2}\right|, & 0 \leq \alpha \leq \frac{1}{2}, \\ \max\left\{\dfrac{1-\alpha}{2}\displaystyle\int_a^b g(t)dt + \left|h(x) - \dfrac{h(a)+h(b)}{2}\right|, \dfrac{\alpha}{2}\displaystyle\int_a^b g(t)dt\right\}, & \frac{1}{2} < \alpha < \frac{2}{3}, \\ \dfrac{\alpha}{2}\displaystyle\int_a^b g(t)dt, & \frac{2}{3} \leq \alpha \leq 1 \end{cases}$$

and $\bigvee_a^b(f)$ *denotes the total varfiation of* f *on the interval* $[a,b]$. *In (1.2), the constant* $\frac{1-\alpha}{2}$ *for* $0 \leq \alpha \leq \frac{1}{2}$ *and the constant* $\frac{\alpha}{2}$ *for* $\frac{2}{3} \leq \alpha \leq 1$ *are the best possible.*

Motivated by the above researches, the purpose of this paper is to obtain some weighted Ostrowski type, Trapezoid type, Grüss type and Ostrowski-Grüss like inequalities with a parameter on time scales based on a weighted Montgomery identity on time scales. These results not only provide a generalization of the known results, but also give some other interesting inequalities on time scales as special cases.

This paper is organized as follows. In Section 2, we briefly present the general definitions and theorems related to the time scales calculus. The weighted Montgomery identity, weighted Ostrowski type inequality, weighted Trapezoid type inequality, weighted Grüss type inequality and Ostrowski-Grüss like inequality on time scales are derived in Subsection 3.1, 3.2, 3.3, 3.4 and 3.5, respectively.

2. Time Scales Essentials

In this section we briefly introduce the time scales theory. For further details and proofs we refer the reader to Hilger's Ph.D. thesis [10], the books [4, 5, 12], and the survey [1].

Definition 1. *A time scale* $\mathbb{T}$ *is an arbitrary nonempty closed subset of* $\mathbb{R}$.

We assume throughout that $\mathbb{T}$ has the topology that is inherited from the standard topology on $\mathbb{R}$. It also assumed throughout that in $\mathbb{T}$ the interval $[a,b]_{\mathbb{T}}$ means the set $\{t \in \mathbb{T} : s < t\}$ for the points $a < b$ in $\mathbb{T}$. Since a time scale may not be connected, we need the following concept of jump operators.

Definition 2. *For* $t \in \mathbb{T}$, *we define the forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ *by* $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$, *while the backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ *is defined by* $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$.

The jump operators σ and ρ allow the classification of points in $\mathbb{T}$ as follows.

Definition 3. *If $\sigma(t) > t$, then we say that t is right-scattered, while if $\rho(t) < t$ then we say that t is left-scattered. Points that are right-scattered and left-scattered at the same time are called isolated. If $\sigma(t) = t$, the t is called right-dense, and if $\rho(t) = t$ then t is called left-dense, Points that both right-dense and left-dense are called dense.*

Definition 4. *The mapping $\mu : \mathbb{T} \to \mathbb{R}^+$ defined by $\mu(t) = \sigma(t) - t$ is called the graininess function. The set $\mathbb{T}^k$ is defined as follows: if $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T} - \{m\}$; otherwise, $\mathbb{T}^k = \mathbb{T}$.*

If $\mathbb{T} = \mathbb{R}$, then $\mu(t) = 0$, and when $\mathbb{T} = \mathbb{Z}$, we have $\mu(t) = 1$.

Definition 5. *Let $f : \mathbb{T} \to \mathbb{R}$. f is called differentiable at $t \in \mathbb{T}^k$, with (delta) derivative $f^{\triangle}(t) \in \mathbb{R}$, if given $\varepsilon > 0$ there exists a neighborhood U of t such that,*

$$\left|f(\sigma(t)) - f(s) - f^{\triangle}(t)[\sigma(t) - s]\right| \leq \varepsilon \left|\sigma(t) - s\right|, \quad \forall\ s \in U.$$

If $\mathbb{T} = \mathbb{R}$, then $f^{\triangle}(t) = \frac{df(t)}{dt}$, and if $\mathbb{T} = \mathbb{Z}$, then $f^{\triangle}(t) = f(t+1) - f(t)$.

Theorem 3. *Assume $f, g : \mathbb{T} \to \mathbb{R}$ are differfentiable at $t \in \mathbb{T}^k$. Then the prfoduct $fg : \mathbb{T} \to \mathbb{R}$ is differfentiable at t with*

$$(fg)^{\triangle}(t) = f^{\triangle}(t)g(t) + f(\sigma(t))g^{\triangle}(t) = f(t)g^{\triangle}(t) + f^{\triangle}(t)g(\sigma(t)).$$

Definition 6. *The function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous (denote $f \in C_{rd}(\mathbb{T}, \mathbb{R})$), if it is continuous at all right-dense points $t \in \mathbb{T}$ and its left-sided limits exist at all left-dense points $t \in \mathbb{T}$.*

Definition 7. *Let $f \in C_{rd}(\mathbb{T},\mathbb{R})$. Then $g : \mathbb{T} \to \mathbb{R}$ is called the antiderivative of f on $\mathbb{T}$ if it satisfies $g^{\triangle}(t) = f(t)$ for any $t \in \mathbb{T}^k$. In this case, we defined*

$$\int_a^t f(s) \triangle s = g(t) - g(a), \quad t \in \mathbb{T}.$$

Theorem 4. *Let f, g be rd-continuous, $a, b, c \in \mathbb{T}$ and $\alpha, \beta \in \mathbb{R}$. Then*
(1) $\int_a^b [f(t) + g(t)] \triangle t = \int_a^b f(t) \triangle t + \int_a^b g(t) \triangle t,$
(2) $\int_a^b f(t) \triangle t = -\int_b^a f(t) \triangle t,$
(3) $\int_a^b f(t) \triangle t = \int_a^c f(t) \triangle t + \int_c^b f(t) \triangle t,$
(4) $\int_a^b f(t)g^{\triangle}(t) \triangle t = (fg)(b) - (fg)(a) - \int_a^b f^{\triangle}(t)g(\sigma(t)) \triangle t,$

Theorem 5. *If f is $\triangle$-integrable on $[a, b]$, then so is $|f|$,and*

$$\left|\int_a^b f(t) \triangle t\right| \leq \int_a^b |f(t)| \triangle t.$$

Definition 8. *Let $g_k : \mathbb{T}^2 \to \mathbb{R}$, $k \in \mathbb{N}_0$ be defined by*

$$g_0(t, s) = 1 \quad \textit{for all} \quad s, t \in \mathbb{T}$$

and then recursively by

$$g_{k+1}(t, s) = \int_s^t g_k(\sigma(\tau), s) \triangle s \quad \textit{for all} \quad s, t \in \mathbb{T}$$

In what follows, we assume that $\mathbb{T}$ is a time scale and $[a, b]$ denotes $[a, b] \cap \mathbb{T}$.

3. Main Results

In this section we first derive a weighted Montgomery identity on time scales and then establish weighted Ostrowski type, Trapezoid type, Grüss type and Ostrowski-Grüss like inequalities on time scales, respectively.

3.1. **Weighted Montgomery identity on time scales.**

Lemma 1. *Let* $0 \le \alpha \le 1$, $g : [a,b] \to [0,\infty)$ *be rd continuous and positive and* $h : [a,b] \to \mathbb{R}$ *be differentiable such that* $h^{\triangle}(t) = g(t)$ *on* $[a,b]$. *Let* $a, b, s, t \in \mathbb{T}$, $a < b$, $c = h^{-1}\left(\left(1-\frac{\alpha}{2}\right)h(a) + \frac{\alpha}{2}h(b)\right)$, $d = h^{-1}\left(\frac{\alpha}{2}h(a) + \left(1-\frac{\alpha}{2}\right)h(b)\right)$ *and* $f : [a,b] \to \mathbb{R}$ *be differentiable. Then for all* $s \in [c,d]$, *we have*

$$\int_a^b W(t,s) f^{\Delta}(s)\Delta s$$
$$= \left[(1-\alpha)f(t) + \alpha\frac{f(a)+f(b)}{2}\right]\int_a^b g(t)\Delta t - \int_a^b g(s)f(\sigma(s))\Delta s \tag{3.1}$$

where

$$W(t,s) = \begin{cases} h(s) - \left(\left(1-\frac{\alpha}{2}\right)h(a) + \frac{\alpha}{2}h(b)\right), & a \le s < t, \\ h(s) - \left(\frac{\alpha}{2}h(a) + \left(1-\frac{\alpha}{2}\right)h(b)\right), & t \le s \le b. \end{cases}$$

Proof. Using item (4) of Theorem 4, we have

$$\int_a^t \left[h(s) - \left(\left(1-\frac{\alpha}{2}\right)h(a) + \frac{\alpha}{2}h(b)\right)\right] f^{\Delta}(s)\Delta s$$
$$= \left[h(t) - \left(\left(1-\frac{\alpha}{2}\right)h(a) + \frac{\alpha}{2}h(b)\right)\right] f(x) - \frac{\alpha}{2}\left(h(a)-h(b)\right)f(a) - \int_a^t g(s)f(\sigma(s))\Delta s$$

and

$$\int_a^t \left[h(s) - \left(\frac{\alpha}{2}h(t) + \left(1-\frac{\alpha}{2}\right)h(b)\right)\right] f^{\Delta}(s)\Delta s$$
$$= \frac{\alpha}{2}\left(h(a)-h(b)\right)f(b) - \left[h(t) - \left(\frac{\alpha}{2}h(a) + \left(1-\frac{\alpha}{2}\right)h(b)\right)\right] f(x) - \int_t^b g(s)f(\sigma(s))\Delta s.$$

Therefore, the equality (3.1) is proved by adding the above two identities. □

Corollary 1. *If we take* $\mathbb{T} = \mathbb{R}$ *in Lemma 1, we have*

$$\int_a^b W(t,s)f'(s)ds = \left[(1-\alpha)f(t) + \alpha\frac{f(a)+f(b)}{2}\right]\int_a^b g(t)dt - \int_a^b g(s)f(s)ds,$$

where $g(t) = h'(t)$ *on* $[a,b]$ *and*

$$W(t,s) = \begin{cases} h(s) - \left(\left(1-\frac{\alpha}{2}\right)h(a) + \frac{\alpha}{2}h(b)\right), & a \le s < t, \\ h(s) - \left(\frac{\alpha}{2}h(a) + \left(1-\frac{\alpha}{2}\right)h(b)\right), & t \le s \le b. \end{cases}$$

This is the result given in [32, equation (2.2)].

Corollary 2. *If we take* $\mathbb{T} = \mathbb{Z}$ *in Lemma 1, we have*

$$\sum_{s=a}^{b-1} W(t,s)\Delta f(s) = \left[(1-\alpha)f(t) + \alpha\frac{f(a)+f(b)}{2}\right]\sum_{t=a}^{b-1} g(t) - \sum_{t=a}^{b-1} g(t)f(t+1),$$

where $g(t) = h(t+1) - h(t)$ *on* $[a, b-1]$ *and*

$$W(t,s) = \begin{cases} h(s) - \left(\left(1-\frac{\alpha}{2}\right)h(a) + \frac{\alpha}{2}h(b)\right), & a \le s < t-1, \\ h(s) - \left(\frac{\alpha}{2}h(a) + \left(1-\frac{\alpha}{2}\right)h(b)\right), & t \le s \le b-1. \end{cases}$$

Corollary 3. *Let $f : [a, b] \to \mathbb{R}$, $q > 1$, $a = q^m$ and $b = q^n$ with $m < n$ in Lemma 1. Then we have*

$$\sum_{k=m}^{n-1} W(t, q^k) \frac{f(q^{k+1}) - f(q^k)}{(q-1)q^k}$$
$$= \left\{(1-\alpha)f(t) + \alpha \frac{f(a)+f(b)}{2}\right\} \int_{q^m}^{q^n} g(t) d_q t - \int_{q^m}^{q^n} g(t) f(qt) d_q t,$$

where

$$W(t,s) = \begin{cases} h(q^k) - (\left(1-\frac{\alpha}{2}\right) h(q^m) + \frac{\alpha}{2} h(q^n)), & q^m \le q^k < t, \\ h(q^k) - (\frac{\alpha}{2} h(q^m) + \left(1-\frac{\alpha}{2}\right) h(q^n)), & t \le q^k \le q^n. \end{cases}$$

Corollary 4. *If we take $h(t) = t$ in Lemma 1, we have*

$$(1-\alpha)f(t) = -\alpha \frac{f(a)+f(b)}{2} + \frac{1}{b-a} \int_a^b W(t,s) f^{\Delta}(s) \Delta s - \frac{1}{b-a} \int_a^b f(\sigma(s)) \Delta s,$$

where

$$W(t,s) = \begin{cases} s - (\left(1-\frac{\alpha}{2}\right) a + \frac{\alpha}{2} b), & a \le s < t, \\ s - (\frac{\alpha}{2} a + \left(1-\frac{\alpha}{2}\right) b), & t \le s \le b. \end{cases}$$

This is the result given in [19, Lemma 1]*, where the continuous, discrete and quantum calculus cases are also presented.*

Corollary 5. *If we take $\alpha = 0$ in Corollary 4, we have*

$$f(t) = \frac{1}{b-a} \int_a^b S(t,s)) f^{\Delta}(s) \Delta s + \frac{1}{b-a} \int_a^b f(\sigma(s)) \Delta s, \tag{3.2}$$

where

$$S(t,s) = \begin{cases} s-a, & a \le s < t, \\ s-b, & t \le s \le b. \end{cases}$$

This is the Montgomery identity on time scales established in [3, Lemma 3.1].

3.2. Weighted Ostrowski type inequality on time scales.

Theorem 6. *Let $0 \le \alpha \le 1$, $g : [a, b] \to [0, \infty)$ be rd continuous and positive and $h : [a, b] \to \mathbb{R}$ be differentiable such that $h^{\triangle}(t) = g(t)$ on $[a, b]$. Let $a, b, s, t \in \mathbb{T}$, $a < b$, $c = h^{-1}\left(\left(1-\frac{\alpha}{2}\right) h(a) + \frac{\alpha}{2} h(b)\right)$, $d = h^{-1}\left(\frac{\alpha}{2} h(a) + \left(1-\frac{\alpha}{2}\right) h(b)\right)$ and $f : [a, b] \to \mathbb{R}$ be differentiable. Then for all $s \in [c, d]$, we have*

$$\left| \left[(1-\alpha)f(t) + \alpha \frac{f(a)+f(b)}{2}\right] \int_a^b g(t) \Delta t - \int_a^b g(s) f(\sigma(s)) \Delta s \right|$$
$$\le M \int_a^b |W(t,s)| \triangle s, \tag{3.3}$$

where

$$W(t,s) = \begin{cases} h(s) - (\left(1-\frac{\alpha}{2}\right) h(a) + \frac{\alpha}{2} h(b)), & a \le s < t, \\ h(s) - (\frac{\alpha}{2} h(a) + \left(1-\frac{\alpha}{2}\right) h(b)), & t \le s \le b. \end{cases}$$

and

$$M = \sup_{a<t<b} \left| f^{\Delta}(t) \right| < \infty.$$

Proof. We can easily obtain the proof of Theorem 6 from Lemma 1. □

Corollary 6. *If we take $\mathbb{T}=\mathbb{R}$ in Theorem 6, we have*

$$\left|\left[(1-\alpha)f(t)+\alpha\frac{f(a)+f(b)}{2}\right]\int_a^b g(t)dt-\int_a^b g(s)f(s)ds\right|\leq M\int_a^b |W(t,s)|\,ds,$$

where $g(t)=h'(t)$ on $[a,b]$,

$$W(t,s)=\begin{cases} h(s)-\left(\left(1-\frac{\alpha}{2}\right)h(a)+\frac{\alpha}{2}h(b)\right), & a\leq s<t,\\ h(s)-\left(\frac{\alpha}{2}h(a)+\left(1-\frac{\alpha}{2}\right)h(b)\right), & t\leq s\leq b.\end{cases}$$

and

$$M=\sup_{a<t<b}|f'(t)|<\infty.$$

Corollary 7. *If we take $\mathbb{T}=\mathbb{Z}$ in Theorem 6, we have*

$$\left|\left[(1-\alpha)f(t)+\alpha\frac{f(a)+f(b)}{2}\right]\sum_{t=a}^{b-1}g(t)-\sum_{t=a}^{b-1}g(t)f(t+1)\right|\leq M\sum_{s=a}^{b-1}|W(t,s)|,$$

where $g(t)=h(t+1)-h(t)$ on $[a,b]$,

$$W(t,s)=\begin{cases} h(s)-\left(\left(1-\frac{\alpha}{2}\right)h(a)+\frac{\alpha}{2}h(b)\right), & a\leq s<t-1,\\ h(s)-\left(\frac{\alpha}{2}h(a)+\left(1-\frac{\alpha}{2}\right)h(b)\right), & t\leq s\leq b-1.\end{cases}$$

and

$$M=\sup_{a<t<b-1}|\triangle f(t)|\ <\infty.$$

Corollary 8. *Let $f:[a,b]\to\mathbb{R}$. $q>1$, $a=q^m$, and $b=q^n$ with $m<n$ in Theorem 6. Then*

$$\left|\left[(1-\alpha)f(t)+\alpha\frac{f(a)+f(b)}{2}\right]\int_{q^m}^{q^n} g(t)d_qt-\int_{q^m}^{q^n} g(t)f(qt)d_qt\right|\leq M\sum_{k=m}^{n-1}\left|W(t,q^k)\right|,$$

where

$$W(t,s)=\begin{cases} h(q^k)-\left(\left(1-\frac{\alpha}{2}\right)h(q^m)+\frac{\alpha}{2}h(q^n)\right), & q^m\leq q^k<t,\\ h(q^k)-\left(\frac{\alpha}{2}h(q^m)+\left(1-\frac{\alpha}{2}\right)h(q^n)\right), & t\leq q^k\leq q^n.\end{cases}$$

and

$$M=\sup_{q^m<t<q^n}\left|\frac{f(q^{k+1})-f(q^k)}{(q-1)q^k}\right|<\infty.$$

Corollary 9. *If we take $h(t)=t$ in Theorem 6, we have*

$$\begin{aligned}&\left|\left[(1-\alpha)f(t)+\alpha\frac{f(a)+f(b)}{2}\right]-\frac{1}{b-a}\int_a^b f(\sigma(s))\Delta s\right|\\ \leq&\frac{M}{b-a}\left[h_2\left(a,a+\alpha\frac{b-a}{2}\right)+h_2\left(t,a+\alpha\frac{b-a}{2}\right)\right.\\ &\left.+h_2\left(t,b-\alpha\frac{b-a}{2}\right)+h_2\left(b,b-\alpha\frac{b-a}{2}\right)\right]\end{aligned}$$

for all $\alpha\in[0,1]$ such that $a+\alpha\frac{b-a}{2}, b-\alpha\frac{b-a}{2}\in\mathbb{T}$ and $t\in\left[a+\alpha\frac{b-a}{2}, b-\alpha\frac{b-a}{2}\right]$ $\cap\mathbb{T}$, where $M=\sup_{a<t<b}\left|f^{\triangle}(t)\right|<\infty$. This is the result given in [19, Theorem 4], *where the continuous, discrete and quantum calculus cases are also presented.*

Remark 1. *If we take $k=0$ in Corollary 9, we get (1.1).*

3.3. **Weighted Trapezoid type inequality on time scales.**

Theorem 7. *Let* $0 \leq \alpha \leq 1$, $g : [a,b] \to [0,\infty)$ *be rd continuous and positive and* $h : [a,b] \to \mathbb{R}$ *be differentiable such that* $h^{\triangle}(t) = g(t)$ *on* $[a,b]$. *Let* $a, b, s, t \in \mathbb{T}$, $a < b$, $c = h^{-1}\left(\left(1-\frac{\alpha}{2}\right)h(a) + \frac{\alpha}{2}h(b)\right)$, $d = h^{-1}\left(\frac{\alpha}{2}h(a) + \left(1-\frac{\alpha}{2}\right)h(b)\right)$ *and* $f : [a,b] \to \mathbb{R}$ *be differentiable. Then for all* $s \in [c,d]$, *we have*

$$\begin{aligned} &\Bigg|(2-\alpha)\frac{f(a)+f(b)}{2} + \alpha\frac{f(\sigma(a))+f(\sigma(b))}{2} \\ &-\frac{1}{\int_a^b g(t)\triangle t}\int_a^b g(s)\left[f(\sigma(s))+f(\sigma^2(s))\right]\Delta s\Bigg| \\ &\leq \frac{M(M+M^{\sigma})}{|f(b)-f(a)|\int_a^b g(t)\triangle t}\int_a^b\left(\int_a^b |W(t,s)|\,\Delta s\right)\triangle t, \end{aligned} \tag{3.4}$$

where

$$W(t,s) = \begin{cases} h(s) - (\left(1-\frac{\alpha}{2}\right)h(a) + \frac{\alpha}{2}h(b)), & a \leq s < t, \\ h(s) - (\frac{\alpha}{2}h(a) + \left(1-\frac{\alpha}{2}\right)h(b)), & t \leq s \leq b. \end{cases}$$

and

$$M = \sup_{a<t<b}\left|f^{\Delta}(t)\right| < \infty \quad \text{and} \quad M^{\sigma} = \sup_{a<t<b}\left|f^{\Delta}(\sigma(t))\right| < \infty.$$

Proof. We have

$$\begin{aligned} (1-\alpha)f(t) =& -\alpha\frac{f(a)+f(b)}{2} + \frac{1}{\int_a^b g(t)\Delta t}\int_a^b g(s)f(\sigma(s))\Delta s \\ &+ \frac{1}{\int_a^b g(t)\Delta t}\int_a^b W(t,s)f^{\Delta}(s)\Delta s \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} (1-\alpha)f(\sigma(t)) =& -\alpha\frac{f(\sigma(a))+f(\sigma(b))}{2} + \frac{1}{\int_a^b g(t)\Delta t}\int_a^b g(s)f(\sigma^2(s))\Delta s \\ &+ \frac{1}{\int_a^b g(t)\Delta t}\int_a^b W(t,s)f^{\Delta}(\sigma(s))\Delta s. \end{aligned} \tag{3.6}$$

Adding (3.5) and (3.6), we get

$$\begin{aligned} (1-\alpha)\left(f(t)+f(\sigma(t))\right) =& -\alpha\left(\frac{f(a)+f(b)}{2} + \frac{f(\sigma(a))+f(\sigma(b))}{2}\right) \\ &+ \frac{1}{\int_a^b g(t)\triangle t}\int_a^b g(s)\left(f(\sigma(s))+f(\sigma^2(s))\right)\Delta s \\ &+ \frac{1}{\int_a^b g(t)\triangle t}\int_a^b W(t,s))\left(f^{\Delta}(s)+f^{\Delta}(\sigma(s))\right)\Delta s. \end{aligned} \tag{3.7}$$

Multiplying (3.7) by $f^{\Delta}(t)$, using Theorem 3 and integrating the resulting identity on $[a,b]$, we have

$$\begin{aligned}
&(1-\alpha)(f^2(b)-f^2(a))\\
=&-\alpha\left(\frac{f(a)+f(b)}{2}+\frac{f(\sigma(a))+f(\sigma(b))}{2}\right)(f(b)-f(a))\\
&+\frac{f(b)-f(a)}{\int_a^b g(t)\triangle t}\int_a^b g(s)\left[f(\sigma(s))+f(\sigma^2(s))\right]\Delta s\\
&+\frac{1}{\int_a^b g(t)\triangle t}\int_a^b f^{\Delta}(t)\left(\int_a^b W(t,s)\left(f^{\Delta}(s)+f^{\Delta}(\sigma(s))\right)\Delta s\right)\triangle t. \qquad (3.8)
\end{aligned}$$

From (3.8), we obtain

$$\begin{aligned}
&\left|(2-\alpha)\frac{f(a)+f(b)}{2}+\alpha\frac{f(\sigma(a))+f(\sigma(b))}{2}\right.\\
&\left.-\frac{1}{\int_a^b g(t)\triangle t}\int_a^b g(s)\left[f(\sigma(s))+f(\sigma^2(s))\right]\Delta s\right|\\
\leq&\frac{M(M+M^{\sigma})}{|f(b)-f(a)|\int_a^b g(t)\triangle t}\int_a^b\left(\int_a^b |W(t,s)|\,\Delta s\right)\triangle t.
\end{aligned}$$

Therefore, the inequality (3.4) is proved. ☐

Corollary 10. *If we take $\mathbb{T}=\mathbb{R}$ in Theorem 7, we have*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{\int_a^b g(t)dt}\int_a^b g(s)f(s)ds\right|\\
\leq&\frac{M^2}{|f(b)-f(a)|\int_a^b g(t)dt}\int_a^b\left(\int_a^b |W(t,s)|\,ds\right)dt.
\end{aligned}$$

where

$$W(t,s)=\begin{cases} h(s)-\left(\left(1-\frac{\alpha}{2}\right)h(a)+\frac{\alpha}{2}h(b)\right), & a\leq s<t,\\ h(s)-\left(\frac{\alpha}{2}h(a)+\left(1-\frac{\alpha}{2}\right)h(b)\right), & t\leq s\leq b.\end{cases}$$

and

$$M=\sup_{a<t<b}|f'(t)|<\infty.$$

Corollary 11. *If we take $\mathbb{T}=\mathbb{Z}$ in Theorem 7, we have*

$$\begin{aligned}
&\left|(2-\alpha)\frac{f(a)+f(b)}{2}+\alpha\frac{f(a+1)+f(b+1)}{2}\right.\\
&\left.-\frac{1}{\int_a^b g(t)\triangle t}\sum_{t=a}^{b-1}g(t)\left[f(t+1)+f(t+2)\right]\right|\\
\leq&\frac{M(M+M^{\sigma})}{|f(b)-f(a)|\sum\limits_{t=a}^{b-1}g(t)\triangle t}\sum_{t=a}^{b-1}\left(\sum_{s=a}^{b-1}|W(t,s)|\right).
\end{aligned}$$

where

$$W(t,s)=\begin{cases} h(s)-\left(\left(1-\frac{\alpha}{2}\right)h(a)+\frac{\alpha}{2}h(b)\right), & a\leq s<t-1,\\ h(s)-\left(\frac{\alpha}{2}h(a)+\left(1-\frac{\alpha}{2}\right)h(b)\right), & t\leq s\leq b-1.\end{cases}$$

and

$$M=\sup_{a<t<b-1}|\Delta f(t)|<\infty \quad \text{and} \quad M^{\sigma}=\sup_{a<t<b-1}|\Delta f(t+1)|<\infty.$$

Corollary 12. *If we take $h(t)=t$ in Theorem 7, we have*

$$\left|(2-\alpha)\frac{f(a)+f(b)}{2}+\alpha\frac{f(\sigma(a))+f(\sigma(b))}{2}-\frac{1}{b-a}\int_a^b\left[f(\sigma(s))+f(\sigma^2(s))\right]\Delta s\right|$$
$$\leq\frac{M(M+M^{\sigma})}{|f(b)-f(a)|(b-a)}\int_a^b\left[h_2\left(a,a+\alpha\frac{b-a}{2}\right)+h_2\left(t,a+\alpha\frac{b-a}{2}\right)\right.$$
$$\left.+h_2\left(t,b-\alpha\frac{b-a}{2}\right)+h_2\left(b,b-\alpha\frac{b-a}{2}\right)\right]\triangle t,$$

for all $\alpha\in[0,1]$ such that $a+\alpha\frac{b-a}{2},b-\alpha\frac{b-a}{2}\in\mathbb{T}$ and $t\in\left[a+\alpha\frac{b-a}{2},b-\alpha\frac{b-a}{2}\right]$ $\cap\mathbb{T}$, where

$$M=\sup_{a<t<b}\left|f^{\Delta}(t)\right|<\infty \text{ and } N=\sup_{a<t<b}\left|f^{\Delta}(\sigma(t))\right|<\infty.$$

Corollary 13. *If we take $\mathbb{T}=\mathbb{R}$ in Corollary 12, we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(s)ds\right|\leq\frac{M^2(b-a)^2}{|f(b)-f(a)|}\left(\frac{3\alpha^2-3\alpha+2}{6}\right),$$

where

$$M=\sup_{a<t<b}|f'(t)|<\infty.$$

Remark 2. *If we set $\alpha=0$ in Corollary 13, we get exactly* [27, Theorem 1(a_1)].

Corollary 14. *If we take $\mathbb{T}=\mathbb{Z}$ in Corollary 12, we have*

$$\left|(2-\alpha)\frac{f(a)+f(b)}{2}+\alpha\frac{f(a+1)+f(b+1)}{2}-\frac{1}{b-a}\sum_{s=a}^{b-1}[f(s+1)+f(s+2)]\right|$$
$$\leq\frac{M(M+M^{\sigma})}{12|f(b)-f(a)|}\left[\left(3\alpha^2-6\alpha+4\right)(b-a)^2+8\right],$$

where

$$M=\sup_{a<t<b-1}|\Delta f(t)|<\infty \quad \text{and} \quad M^{\sigma}=\sup_{a<t<b-1}|\Delta f(t+1)|<\infty.$$

3.4. **Weighted Grüss type inequalities on time scales.**

Theorem 8. *Let $0\leq\alpha\leq1$, $g:[a,b]\to[0,\infty)$ be rd continuous and positive and $h:[a,b]\to\mathbb{R}$ be differentiable such that $h^{\triangle}(t)=g(t)$ on $[a,b]$. Let $a,b,s,t\in\mathbb{T}$, $a<b$, $c=h^{-1}\left(\left(1-\frac{\alpha}{2}\right)h(a)+\frac{\alpha}{2}h(b)\right)$, $d=h^{-1}\left(\frac{\alpha}{2}h(a)+\left(1-\frac{\alpha}{2}\right)h(b)\right)$ and $f:$*

$[a,b] \to \mathbb{R}$ *be differentiable. Then for all* $s \in [c,d]$, *we have*

$$\begin{aligned}
&\Bigg| 2(1-\alpha)\int_a^b p(t)q(t)\triangle t \\
&+\alpha\left[\frac{p(a)+p(b)}{2}\int_a^b q(t)\Delta t+\frac{q(a)+q(b)}{2}\int_a^b p(t)\Delta t\right] \\
&-\frac{1}{\int_a^b g(t)\triangle t}\left[\left(\int_a^b g(s)p(\sigma(s))\Delta s\right)\left(\int_a^b q(t)\triangle t\right)\right. \\
&\left.+\left(\int_a^b g(s)q(\sigma(s))\Delta s\right)\left(\int_a^b p(t)\triangle t\right)\right]\Bigg| \\
&\le \frac{1}{\int_a^b g(t)\Delta t}\int_a^b \left(P\left|q(t)\right|+Q\left|p(t)\right|\right)\left(\int_a^b \left|W(t,s)\right|\Delta s\right)\triangle t, \qquad (3.9)
\end{aligned}$$

where

$$W(t,s)=\begin{cases} h(s)-\left(\left(1-\frac{\alpha}{2}\right)h(a)+\frac{\alpha}{2}h(b)\right), & a\le s<t, \\ h(s)-\left(\frac{\alpha}{2}h(a)+\left(1-\frac{\alpha}{2}\right)h(b)\right), & t\le s\le b. \end{cases}$$

$$P=\sup_{a<t<b}\left|p^{\Delta}(t)\right|<\infty \quad \textit{and} \quad Q=\sup_{a<t<b}\left|q^{\Delta}(t)\right|<\infty.$$

Proof. We have

$$\begin{aligned}
(1-\alpha)p(t)= &-\alpha\frac{p(a)+p(b)}{2}+\frac{1}{\int_a^b g(t)\Delta t}\int_a^b g(s)p(\sigma(s))\Delta s \\
&+\frac{1}{\int_a^b g(t)\Delta t}\int_a^b W(t,s)p^{\Delta}(s)\Delta s \qquad (3.10)
\end{aligned}$$

and

$$\begin{aligned}
(1-\alpha)q(t)= &-\alpha\frac{q(a)+q(b)}{2}+\frac{1}{\int_a^b g(t)\Delta t}\int_a^b g(s)q(\sigma(s))\Delta s \\
&+\frac{1}{\int_a^b g(t)\Delta t}\int_a^b W(t,s)q^{\Delta}(s)\Delta s \qquad (3.11)
\end{aligned}$$

Multiplying (3.10) by $q(t)$ and (3.11) by $p(t)$, adding and then integrating the result from a to b, we have

$$\begin{aligned}
&2(1-\alpha)\int_a^b p(t)q(t)\triangle t \qquad (3.12) \\
&=-\alpha\left[\frac{p(a)+p(b)}{2}\int_a^b q(t)\Delta t+\frac{q(a)+q(b)}{2}\int_a^b p(t)\Delta t\right] \\
&+\frac{1}{\int_a^b g(t)\triangle t}\left[\left(\int_a^b g(s)p(\sigma(s))\Delta s\right)\left(\int_a^b q(t)\triangle t\right)\right. \\
&\left.+\left(\int_a^b g(s)q(\sigma(s))\Delta s\right)\left(\int_a^b p(t)\triangle t\right)\right]
\end{aligned}$$

$$+\frac{1}{\int_a^b g(t)\triangle t}\left[\int_a^b q(t)\left(\int_a^b S(t,s)p^{\Delta}(s)\Delta s\right)\triangle t\right.$$

$$\left.+\int_a^b p(t)\left(\int_a^b S(t,s)q^{\Delta}(s)\Delta s\right)\triangle t\right] \tag{3.13}$$

From (3.13) and using the properties of modulus, we get the inequality (3.9). □

Corollary 15. *If we take $\mathbb{T}=\mathbb{R}$ in Theorem 8, we have*

$$\left|2(1-\alpha)\int_a^b p(t)q(t)dt\right.$$
$$+\alpha\left[\frac{p(a)+p(b)}{2}\int_a^b q(t)dt+\frac{q(a)+q(b)}{2}\int_a^b p(t)dt\right]$$
$$-\frac{1}{\int_a^b g(t)dt}\left[\left(\int_a^b g(s)p(s)ds\right)\left(\int_a^b q(t)dt\right)\right.$$
$$\left.\left.+\left(\int_a^b g(s)q(s)ds\right)\left(\int_a^b p(t)dt\right)\right|$$
$$\leq\frac{1}{\int_a^b g(t)dt}\int_a^b \left(P\left|q(t)\right|+Q\left|p(t)\right|\right)\left(\int_a^b \left|W(t,s)\right|ds\right)dt$$

where

$$W(t,s)=\begin{cases} h(s)-(\left(1-\frac{\alpha}{2}\right)h(a)+\frac{\alpha}{2}h(b)), & a\leq s<t,\\ h(s)-(\frac{\alpha}{2}h(a)+\left(1-\frac{\alpha}{2}\right)h(b)), & t\leq s\leq b.\end{cases}$$

$$P=\sup_{a<t<b}\left|p'(t)\right|<\infty \quad and \quad Q=\sup_{a<t<b}\left|q'(t)\right|<\infty.$$

Corollary 16. *If we take $\mathbb{T}=\mathbb{Z}$ in Theorem 8, we have*

$$\left|2(1-\alpha)\sum_{t=a}^{b-1}p(t)q(t)+\alpha\left[\frac{p(a)+p(b)}{2}\sum_{t=a}^{b-1}q(t)+\frac{q(a)+q(b)}{2}\sum_{t=a}^{b-1}p(t)\right]\right.$$
$$-\frac{1}{\sum\limits_{t=a}^{b-1}g(t)}\left[\left(\sum_{t=a}^{b-1}g(t)p(t+1)\right)\left(\sum_{t=a}^{b-1}q(t)\right)\right.$$
$$\left.\left.+\left(\sum_{t=a}^{b-1}g(t)q(t+1)\right)\left(\sum_{t=a}^{b-1}p(t)\right)\right|$$
$$\leq\frac{1}{\sum\limits_{t=a}^{b-1}g(t)}\sum_{t=a}^{b-1}\left(P\left|q(t)\right|+Q\left|p(t)\right|\right)\left(\sum_{s=a}^{b-1}\left|W(t,s)\right|\right)$$

where

$$W(t,s)=\begin{cases} h(s)-(\left(1-\frac{\alpha}{2}\right)h(a)+\frac{\alpha}{2}h(b)), & a\leq s<t-1,\\ h(s)-(\frac{\alpha}{2}h(a)+\left(1-\frac{\alpha}{2}\right)h(b)), & t\leq s\leq b-1,\end{cases}$$

$$P=\sup_{a<t<b-1}\left|\triangle p(t)\right|<\infty \quad and \quad Q=\sup_{a<t<b-1}\left|\Delta q(t)\right|<\infty.$$

Corollary 17. *If we take $h(t) = t$ in Theorem 8, we have*

$$\begin{aligned}&\Bigg|2(1-\alpha)\int_a^b p(t)q(t)\triangle t\\&+\alpha\left[\frac{p(a)+p(b)}{2}\int_a^b q(t)\Delta t+\frac{q(a)+q(b)}{2}\int_a^b p(t)\Delta t\right]\\&-\frac{1}{b-a}\left[\left(\int_a^b p(\sigma(s))\Delta s\right)\left(\int_a^b q(t)\triangle t\right)+\left(\int_a^b q(\sigma(s))\Delta s\right)\left(\int_a^b p(t)\triangle t\right)\right]\Bigg|\\\le&\frac{1}{b-a}\int_a^b \left(P\left|q(t)\right|+Q\left|p(t)\right|\right)\left[h_2\left(a,a+\alpha\frac{b-a}{2}\right)+h_2\left(t,a+\alpha\frac{b-a}{2}\right)\right.\\&\left.+h_2\left(t,b-\alpha\frac{b-a}{2}\right)+h_2\left(b,b-\alpha\frac{b-a}{2}\right)\right]\triangle t\end{aligned}$$

for all $\alpha \in [0,1]$ such that $a+\alpha\frac{b-a}{2}, b-\alpha\frac{b-a}{2} \in \mathbb{T}$ and $t \in \left[a+\alpha\frac{b-a}{2}, b-\alpha\frac{b-a}{2}\right]\cap\mathbb{T}$, where

$$P = \sup_{a<t<b}\left|p^{\Delta}(t)\right| < \infty \quad \text{and} \quad Q = \sup_{a<t<b}\left|q^{\Delta}(t)\right| < \infty.$$

Corollary 18. *If we take $\mathbb{T} = \mathbb{R}$ in Corollary 17, we have*

$$\begin{aligned}&\Bigg|(1-\alpha)\int_a^b p(t)q(t)dt+\frac{\alpha}{2}\left[\frac{p(a)+p(b)}{2}\int_a^b q(t)dt+\frac{q(a)+q(b)}{2}\int_a^b p(t)dt\right]\\&-\frac{1}{b-a}\left(\int_a^b p(s)ds\right)\left(\int_a^b q(t)dt\right)\Bigg|\\\le&\frac{1}{2}\int_a^b \left(P\left|q(t)\right|+Q\left|p(t)\right|\right)\left[\frac{1}{4}(b-a)\big((1-\alpha)^2+\alpha^2\big)+\frac{1}{b-a}\left(t-\frac{a+b}{2}\right)^2\right]dt\end{aligned}$$

where

$$P = \sup_{a<t<b}|p'(t)| < \infty \quad \text{and} \quad Q = \sup_{a<t<b}|q'(t)| < \infty.$$

Remark 3. *If we set $\alpha = 0$ in Corollary 18, we get exactly* [27, Theorem 2(b1)].

Corollary 19. *If we take $\mathbb{T} = \mathbb{Z}$ in Corollary 17, we have*

$$\begin{aligned}&\Bigg|2(1-\alpha)\sum_{t=a}^{b-1} p(t)q(t)\triangle t\\&+\alpha\left[\frac{p(a)+p(b)}{2}\sum_{t=a}^{b-1} q(t)\Delta t+\frac{q(a)+q(b)}{2}\sum_{t=a}^{b-1} p(t)\Delta t\right]\\&-\frac{1}{b-a}\left[\left(\sum_{t=a}^{b-1} p(t+1)\right)\left(\sum_{t=a}^{b-1} q(t)\right)+\left(\sum_{t=a}^{b-1} q(t+1)\right)\left(\sum_{t=a}^{b-1} p(t)\right)\right]\Bigg|\\\le&\frac{1}{b-a}\sum_{t=a}^{b-1}\left(P\left|q(t)\right|+Q\left|p(t)\right|\right)\left[\left(\frac{\alpha^2}{2}-\frac{\alpha}{2}+\frac{1}{4}\right)(b-a)^2+\left(t-\frac{a+b+1}{2}\right)^2-\frac{1}{4}\right]\end{aligned}$$

where

$$P = \sup_{a<t<b-1}|\triangle p(t)| < \infty \quad \text{and} \quad Q = \sup_{a<t<b-1}|\triangle q(t)| < \infty.$$

3.5. **Ostrowski-Grüss like inequalities on time scales.**

Theorem 9. *Let* $0 \leq \alpha \leq 1$, $g : [a,b] \to [0,\infty)$ *be rd continuous and positive and* $h : [a,b] \to \mathbb{R}$ *be differentiable such that* $h^{\triangle}(t) = g(t)$ *on* $[a,b]$. *Let* $a, b, s, t \in \mathbb{T}$, $a < b$, $c = h^{-1}\left(\left(1-\frac{\alpha}{2}\right)h(a) + \frac{\alpha}{2}h(b)\right)$, $d = h^{-1}\left(\frac{\alpha}{2}h(a) + \left(1-\frac{\alpha}{2}\right)h(b)\right)$ *and* $f : [a,b] \to \mathbb{R}$ *be differentiable. Then for all* $s \in [c,d]$, *we have*

$$\begin{aligned}
&\Bigg|2(1-\alpha)p(t)q(t) + \alpha\left[\frac{p(a)+p(b)}{2}q(t) + \frac{q(a)+q(b)}{2}p(t)\right] \\
&- \frac{1}{\int_a^b g(t)\triangle t}\left[q(t)\int_a^b g(s)p(\sigma(s))\Delta s \ + p(t)\int_a^b g(s)q(\sigma(s))\Delta s\right]\Bigg| \\
&\leq \frac{1}{\int_a^b g(t)\Delta t}\left(P\left|q(t)\right| + Q\left|p(t)\right|\right)\int_a^b \left|W(t,s)\right|\Delta s
\end{aligned} \tag{3.14}$$

and

$$\begin{aligned}
&\Bigg|(1-\alpha)^2 p(t)q(t) + \frac{\alpha(1-\alpha)}{2}\left[p(t)(q(a)+q(b)) + q(t)(p(a)+p(b))\right] \\
&+ \frac{\alpha^2}{4}\left(p(a)+p(b)\right)\left(q(a)+q(b)\right) \\
&- \frac{1-\alpha}{\int_a^b g(t)\Delta t}\left[p(t)\int_a^b g(s)q(\sigma(s))\Delta s + q(t)\int_a^b g(s)p(\sigma(s))\Delta s\right] \\
&- \frac{\alpha}{2\int_a^b g(t)\Delta t}\left[(p(a)+p(b))\int_a^b g(s)q(\sigma(s))\Delta s\right. \\
&\left.+(q(a)+q(b))\int_a^b g(s)p(\sigma(s))\Delta s\right] \\
&+\frac{1}{\left(\int_a^b g(t)\Delta t\right)^2}\left(\int_a^b g(s)q(\sigma(s))\Delta s\right)\left(\int_a^b g(s)p(\sigma(s))\Delta s\right)\Bigg| \\
&\leq \frac{PQ}{\left(\int_a^b g(t)\Delta t\right)^2}\left(\int_a^b \left|W(t,s)\right|\Delta s\right)^2
\end{aligned} \tag{3.15}$$

where

$$W(t,s) = \begin{cases} h(s) - (\left(1-\frac{\alpha}{2}\right)h(a) + \frac{\alpha}{2}h(b)), & a \leq s < t, \\ h(s) - (\frac{\alpha}{2}h(a) + \left(1-\frac{\alpha}{2}\right)h(b)), & t \leq s \leq b. \end{cases}$$

$$P = \sup_{a<t<b}\left|p^{\Delta}(t)\right| < \infty \quad \text{and} \quad Q = \sup_{a<t<b}\left|q^{\Delta}(t)\right| < \infty.$$

Proof. Multiplying (3.10) by $q(t)$ and (3.11) by $p(t)$, adding the resulting identities and rewriting we get

$$\begin{aligned}2(1-\alpha)p(t)q(t) =& -\alpha\left[\frac{p(a)+p(b)}{2}q(t)+\frac{q(a)+q(b)}{2}p(t)\right]\\&+\frac{1}{\int_a^b g(t)\triangle t}\left[q(t)\int_a^b g(s)p(\sigma(s))\Delta s+p(t)\int_a^b g(s)q(\sigma(s))\Delta s\right]\\&+\frac{1}{\int_a^b g(t)\Delta t}\left[q(t)\int_a^b W(t,s)p^{\Delta}(s)\Delta s+p(t)\int_a^b W(t,s)q^{\Delta}(s)\Delta s\right]\end{aligned}\tag{3.16}$$

From (3.16) and using the properties of modulus, we get (3.14).

Multiplying the left sides and right sides of (3.10) and (3.11) we get

$$\begin{aligned}&(1-\alpha)^2p(t)q(t)+\frac{\alpha(1-\alpha)}{2}\left[p(t)(q(a)+q(b))+q(t)(p(a)+p(b))\right]\\&+\frac{\alpha^2}{4}\left(p(a)+p(a)\right)\left(q(a)+q(a)\right)\\&-\frac{1-\alpha}{\int_a^b g(t)\Delta t}\left[p(t)\int_a^b g(s)q(\sigma(s))\Delta s+q(t)\int_a^b g(s)p(\sigma(s))\Delta s\right]\\&-\frac{\alpha}{2\int_a^b g(t)\Delta t}\left[(p(a)+p(b))\int_a^b g(s)q(\sigma(s))\Delta s\right.\\&\left.+(q(a)+q(b))\int_a^b g(s)p(\sigma(s))\Delta s\right]\\&+\frac{1}{\left(\int_a^b g(t)\Delta t\right)^2}\left(\int_a^b g(s)q(\sigma(s))\Delta s\right)\left(\int_a^b g(s)p(\sigma(s))\Delta s\right)\\=&\frac{1}{\left(\int_a^b g(t)\Delta t\right)^2}\left(\int_a^b W(t,s)p^{\Delta}(s)\Delta s\right)\left(\int_a^b W(t,s)q^{\Delta}(s)\Delta s\right)\end{aligned}\tag{3.17}$$

From (3.17) and using the properties of modulus, we get (3.15). ☐

Corollary 20. *If we take $\mathbb{T}=\mathbb{R}$ in Theorem 9, we have*

$$\begin{aligned}&\left|2(1-\alpha)p(t)q(t)+\alpha\left[\frac{p(a)+p(b)}{2}q(t)+\frac{q(a)+q(b)}{2}p(t)\right]\right.\\&\left.-\frac{1}{\int_a^b g(t)dt}\left[q(t)\int_a^b g(s)p(s)ds+p(t)\int_a^b g(s)q(s)s\right]\right|\\\le&\frac{1}{\int_a^b g(t)dt}\left(P\left|q(t)\right|+Q\left|p(t)\right|\right)\int_a^b\left|W(t,s)\right|ds\end{aligned}$$

and

$$\Bigg|(1-\alpha)^2 p(t)q(t) + \frac{\alpha(1-\alpha)}{2}\left[p(t)(q(a)+q(b)) + q(t)(p(a)+p(b))\right]$$
$$+\frac{\alpha^2}{4}\left(p(a)+p(b)\right)\left(q(a)+q(b)\right)$$
$$-\frac{1-\alpha}{\int_a^b g(t)dt}\left[p(t)\int_a^b g(s)q(s)ds + q(t)\int_a^b g(s)p(s)ds\right]$$
$$-\frac{\alpha}{2\int_a^b g(t)dt}\left[(p(a)+p(b))\int_a^b g(s)q(s)ds \; + (q(a)+q(b))\int_a^b g(s)p(s)ds\right]$$
$$+\frac{1}{\left(\int_a^b g(t)dt\right)^2}\left(\int_a^b g(s)q(s)ds\right)\left(\int_a^b g(s)p(s)ds\right)\Bigg|$$
$$\leq \frac{PQ}{\left(\int_a^b g(t)dt\right)^2}\left(\int_a^b |W(t,s)|\, ds\right)^2$$

where

$$W(t,s) = \begin{cases} h(s) - \left(\left(1-\frac{\alpha}{2}\right)h(a) + \frac{\alpha}{2}h(b)\right), & a \leq s < t, \\ h(s) - \left(\frac{\alpha}{2}h(a) + \left(1-\frac{\alpha}{2}\right)h(b)\right), & t \leq s \leq b. \end{cases}$$
$$P = \sup_{a<t<b} |p'(t)| < \infty \quad \text{and} \quad Q = \sup_{a<t<b} |q'(t)| < \infty.$$

Corollary 21. *If we take $\mathbb{T} = \mathbb{Z}$ in Theorem 9, we have*

$$\Bigg|2(1-\alpha)p(t)q(t) + \alpha\left[\frac{p(a)+p(b)}{2}q(t) + \frac{q(a)+q(b)}{2}p(t)\right]$$
$$-\frac{1}{\sum\limits_{t=a}^{b-1} g(t)\triangle t}\left[q(t)\sum_{t=a}^{b-1} g(t)p(t+1) + p(t)\sum_{t=a}^{b-1} g(t)q(t+1)\right]\Bigg|$$
$$\leq \frac{1}{\sum\limits_{t=a}^{b-1} g(t)}\left(P\,|q(t)| + Q\,|p(t)|\right)\sum_{s=a}^{b-1}|W(t,s)|$$

and

$$\Bigg|(1-\alpha)^2 p(t)q(t) + \frac{\alpha(1-\alpha)}{2}\left[p(t)(q(a)+q(b)) + q(t)(p(a)+p(b))\right]$$
$$+\frac{\alpha^2}{4}\left(p(a)+p(b)\right)\left(q(a)+q(b)\right)$$
$$-\frac{1-\alpha}{\sum\limits_{t=a}^{b-1} g(t)}\left[p(t)\sum_{t=a}^{b-1} g(t)q(t+1) + q(t)\sum_{t=a}^{b-1} g(t)p(t+1)\right]$$
$$-\frac{\alpha}{2\sum\limits_{t=a}^{b-1} g(t)}\left[(p(a)+p(b))\sum_{t=a}^{b-1} g(t)q(t+1) \; + (q(a)+q(b))\sum_{t=a}^{b-1} g(t)p(t+1)\right]$$

$$+\frac{1}{\left(\sum\limits_{t=a}^{b-1} g(t)\right)^2}\left(\sum_{t=a}^{b-1} g(t)q(t+1)\right)\left(\sum_{t=a}^{b-1} g(t)p(t+1)\right)\Bigg|$$

$$\leq \frac{PQ}{\left(\sum\limits_{t=a}^{b-1} g(t)\right)^2}\left(\sum_{s=a}^{b-1} |W(t,s)|\right)^2$$

where

$$W(t,s)=\begin{cases} h(s)-(\left(1-\frac{\alpha}{2}\right)h(a)+\frac{\alpha}{2}h(b)), & a\leq s<t-1 \\ h(s)-(\frac{\alpha}{2}h(a)+\left(1-\frac{\alpha}{2}\right)h(b)), & t\leq s\leq b-1, \end{cases}$$

$$P=\sup_{a<t<b-1}|\triangle p(t)|<\infty \quad \textit{and} \quad Q=\sup_{a<t<b-1}|\Delta q(t)|<\infty.$$

Corollary 22. *If we take $h(t)=t$ in Theorem 9, we have*

$$\Bigg|2(1-\alpha)p(t)q(t)+\alpha\left[\frac{p(a)+p(b)}{2}q(t)+\frac{q(a)+q(b)}{2}p(t)\right]$$

$$-\frac{1}{b-a}\left[q(t)\int_a^b p(\sigma(s))\Delta s+p(t)\int_a^b q(\sigma(s))\Delta s\right]\Bigg|$$

$$\leq \frac{1}{b-a}\left(P\left|q(t)\right|+Q\left|p(t)\right|\right)\left[h_2\left(a,a+\alpha\frac{b-a}{2}\right)+h_2\left(t,a+\alpha\frac{b-a}{2}\right)\right.$$

$$\left.+h_2\left(t,b-\alpha\frac{b-a}{2}\right)+h_2\left(b,b-\alpha\frac{b-a}{2}\right)\right]$$

and

$$\Bigg|(1-\alpha)^2 p(t)q(t)+\frac{\alpha(1-\alpha)}{2}\left[p(t)(q(a)+q(b))+q(t)(p(a)+p(b))\right]$$

$$+\frac{\alpha^2}{4}\left(p(a)+p(a)\right)\left(q(a)+q(a)\right)$$

$$-\frac{1-\alpha}{b-a}\left[p(t)\int_a^b q(\sigma(s))\Delta s+q(t)\int_a^b p(\sigma(s))\Delta s\right]$$

$$-\frac{\alpha}{2(b-a)}\left[(p(a)+p(b))\int_a^b q(\sigma(s))\Delta s\ +(q(a)+q(b))\int_a^b p(\sigma(s))\Delta s\right]$$

$$+\frac{1}{(b-a)^2}\left(\int_a^b q(\sigma(s))\Delta s\right)\left(\int_a^b p(\sigma(s))\Delta s\right)\Bigg|$$

$$\leq \frac{PQ}{(b-a)^2}\left(\left[h_2\left(a,a+\alpha\frac{b-a}{2}\right)+h_2\left(t,a+\alpha\frac{b-a}{2}\right)\right.\right.$$

$$\left.\left.+h_2\left(t,b-\alpha\frac{b-a}{2}\right)+h_2\left(b,b-\alpha\frac{b-a}{2}\right)\right]\right)^2$$

for all $\alpha\in[0,1]$ such that $a+\alpha\frac{b-a}{2}, b-\alpha\frac{b-a}{2}\in\mathbb{T}$ and $t\in\left[a+\alpha\frac{b-a}{2}, b-\alpha\frac{b-a}{2}\right]$ $\cap\mathbb{T}$, where

$$P=\sup_{a<t<b}\left|p^{\Delta}(t)\right|<\infty \quad \textit{and} \quad Q=\sup_{a<t<b}\left|q^{\Delta}(t)\right|<\infty.$$

Corollary 23. *If we take $\mathbb{T}=\mathbb{R}$ in Corollary 22, we have*

$$\Bigg|2(1-\alpha)p(t)q(t)+\alpha\left[\frac{p(a)+p(b)}{2}q(t)+\frac{q(a)+q(b)}{2}p(t)\right]$$
$$-\frac{1}{b-a}\left[q(t)\int_a^b p(s)ds+p(t)\int_a^b q(s)ds\right]\Bigg|$$
$$\leq(P\,|q(t)|+Q\,|p(t)|)\left[\frac{1}{4}(b-a)\big((1-\alpha)^2+\alpha^2\big)+\frac{1}{b-a}\left(t-\frac{a+b}{2}\right)^2\right]$$

and

$$\Bigg|(1-\alpha)^2p(t)q(t)+\frac{\alpha(1-\alpha)}{2}\left[p(t)(q(a)+q(b))+q(t)(p(a)+p(b))\right]$$
$$+\frac{\alpha^2}{4}\left(p(a)+p(b)\right)\left(q(a)+q(b)\right)-\frac{1-\alpha}{b-a}\left[p(t)\int_a^b q(s)ds+q(t)\int_a^b p(s)ds\right]$$
$$-\frac{\alpha}{2(b-a)}\left[(p(a)+p(b))\int_a^b q(s)ds\ +(q(a)+q(b))\int_a^b p(s)ds\right]$$
$$+\frac{1}{(b-a)^2}\left(\int_a^b q(s)ds\right)\left(\int_a^b p(s)ds\right)\Bigg|$$
$$\leq\frac{PQ}{(b-a)^2}\left[\frac{1}{4}(b-a)\big((1-\alpha)^2+\alpha^2\big)+\frac{1}{b-a}\left(t-\frac{a+b}{2}\right)^2\right]^2$$

where

$$P=\sup_{a<t<b}|p'(t)|<\infty\quad\text{and}\quad Q=\sup_{a<t<b}|q'(t)|<\infty.$$

Corollary 24. *If we take $\mathbb{T}=\mathbb{Z}$ in Corollary 22, we have*

$$\Bigg|2(1-\alpha)p(t)q(t)+\alpha\left[\frac{p(a)+p(b)}{2}q(t)+\frac{q(a)+q(b)}{2}p(t)\right]$$
$$-\frac{1}{b-a}\left[q(t)\sum_{t=a}^{b-1}p(t+1)+p(t)\sum_{t=a}^{b-1}q(t+1)\right]\Bigg|$$
$$\leq\frac{1}{b-a}\left(P\,|q(t)|+Q\,|p(t)|\right)\left[\left(\frac{\alpha^2}{2}-\frac{\alpha}{2}+\frac{1}{4}\right)(b-a)^2+\left(t-\frac{a+b+1}{2}\right)^2-\frac{1}{4}\right]$$

and

$$\Bigg|(1-\alpha)^2p(t)q(t)+\frac{\alpha(1-\alpha)}{2}\left[p(t)(q(a)+q(b))+q(t)(p(a)+p(b))\right]$$
$$+\frac{\alpha^2}{4}\left(p(a)+p(a)\right)\left(q(a)+q(a)\right)-\frac{1-\alpha}{b-a}\left[p(t)\sum_{t=a}^{b-1}q(t+1)+q(t)\sum_{t=a}^{b-1}p(t+1)\right]$$
$$-\frac{\alpha}{2(b-a)}\left[(p(a)+p(b))\sum_{t=a}^{b-1}q(t+1)\ +(q(a)+q(b))\sum_{t=a}^{b-1}p(t+1)\right]$$
$$+\frac{1}{(b-a)^2}\left(\sum_{t=a}^{b-1}p(t+1)\right)\left(\sum_{t=a}^{b-1}q(t+1)\right)\Bigg|$$

$$\leq \frac{PQ}{(b-a)^2}\left[\left(\frac{\alpha^2}{2}-\frac{\alpha}{2}+\frac{1}{4}\right)(b-a)^2+\left(t-\frac{a+b+1}{2}\right)^2-\frac{1}{4}\right]^2$$

where

$$P=\sup_{a<t<b-1}|\triangle p(t)|<\infty \quad \textit{and} \quad Q=\sup_{a<t<b-1}|\triangle q(t)|<\infty.$$

Remark 4. *If we set* $\alpha=0$, $a=0$, $b=n$ *in Corollary 24, we get exactly* [26, Theorem 2.1].

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