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WEIGHTED OSTROWSKI, OSTROWSKI-GRÜSS AND OSTROWSKI-ČEBYŠEV TYPE INEQUALITIES ON TIME SCALES

ADNAN TUNA, YONG JIANG, AND WENJUN LIU

ABSTRACT. In this paper we derive weighted Ostrowski, Ostrowski-Grüss and Ostrowski-Čebyšev type inequalities on time scales. We also give some other interesting inequalities on time scales as special cases.

1. Introduction

In 1938, Ostrowski derived a formula to estimate the absolute deviation of a differentiable function from its integral mean [24], the so-called Ostrowski inequality, which can also be shown by using Montgomery identity [22]. In 1997, by combining Montgomery identity and Grüss's integral inequality, Dragomir and Wang [10] proved the following Ostrowski-Grüss type integral inequality, which is a connection between Ostrowski's inequality and Grüss's inequality.

Theorem 1. *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) such that there exist constants $\gamma, \Gamma \in \mathbb{R}$, with $\gamma \leq f'(x) \leq \Gamma$, $x \in [a,b]$. Then we have*

$$\left| f(x) - \frac{f(b)-f(a)}{b-a}\left(x - \frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \frac{1}{4}(b-a)(\Gamma - \gamma), \tag{1.1}$$

for all $x \in [a,b]$.

In [9], Dragomir and Barnett pointed out a new estimation of the left membership of (1.1) as follows.

Theorem 2. *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and twice differentiable on (a,b), whose second derivative $f'' : (a,b) \to \mathbb{R}$ is bounded on (a,b). Then we have*

$$\left| f(x) - \frac{f(b)-f(a)}{b-a}\left(x - \frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \frac{M}{2}\left\{ \left[\frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} + \frac{1}{4} \right]^2 + \frac{1}{12} \right\}(b-a)^2 \tag{1.2}$$

for all $x \in [a,b]$, where $M = \sup_{a<t<b} |f''(t)| < \infty$.

Recently, Ahmad et. al [2] developed some new Ostrowski and Čebyšev type inequalities involving two functions, by using an identity of Dragomir and Barnett proved in [9]. In [28], Tseng, Hwang and Dragomir established the following generalizations of weighted Ostrowski type inequalities for mappings of bounded variation.

Theorem 3. *Let us have $0 \leq \alpha \leq 1$, $g : [a,b] \to [0,\infty)$ continuous and positive on (a,b) and let $h : [a,b] \to \mathbb{R}$ be differentiable such that $h'(t) = g(t)$ on $[a,b]$. Let $c = h^{-1}\left(\left(1 - \frac{\alpha}{2}\right)h(a) + \frac{\alpha}{2}h(b)\right)$ and $d = h^{-1}\left(\frac{\alpha}{2}h(a) + \left(1 - \frac{\alpha}{2}\right)h(b)\right)$. Let $f : [a,b] \to \mathbb{R}$ be a mapping of bounded variation. Then, for all $x \in [c,d]$, we have*

$$\left| \int_a^b f(t)g(t)dt - \left[(1-\alpha)f(x) + \alpha\frac{f(a)+f(b)}{2}\right]\int_a^b g(t)dt \right| \leq K \bigvee_a^b (f), \tag{1.3}$$

where

$$K := \begin{cases} \dfrac{1-\alpha}{2}\displaystyle\int_a^b g(t)dt + \left| h(x) - \dfrac{h(a)+h(b)}{2} \right|, & 0 \leq \alpha \leq \frac{1}{2}, \\ \max\left\{ \dfrac{1-\alpha}{2}\displaystyle\int_a^b g(t)dt + \left| h(x) - \dfrac{h(a)+h(b)}{2} \right|, \dfrac{\alpha}{2}\displaystyle\int_a^b g(t)dt \right\}, & \frac{1}{2} < \alpha < \frac{2}{3}, \\ \dfrac{\alpha}{2}\displaystyle\int_a^b g(t)dt, & \frac{2}{3} \leq \alpha \leq 1 \end{cases} \tag{1.4}$$

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and $\bigvee_{a}^{b}(f)$ *denotes the total varfiation of* f *on the interval* $[a,b]$. *In (3.1), the constant* $\frac{1-\alpha}{2}$ *for* $0 \leq \alpha \leq \frac{1}{2}$ *and the constant* $\frac{\alpha}{2}$ *for* $\frac{2}{3} \leq \alpha \leq 1$ *are the best possible.*

The development of the theory of time scales was initiated by Hilger [11] in 1988 as a theory capable to contain both difference and differential calculus in a consistent way. Since then, many authors have studied the theory of certain integral inequalities on time scales (see [3, 6, 7, 8, 12, 14, 15, 16, 17, 18, 19, 20, 23, 25, 26, 27, 29, 31]). For examples, by using the Montgomery identity on time scales, Bohner and Matthews [7] established the following Ostrowski inequality on time scales which unify discrete, continuous and many other cases.

Theorem 4. *Let* $a, b, s, t \in \mathbb{T}$, $a < b$ *and* $f : [a,b] \to \mathbb{R}$ *be differentiable. Then*

$$\left| f(t) - \frac{1}{b-a} \int_a^b f(\sigma(s)) \triangle s \right| \leq \frac{M}{b-a} [h_2(t,a) + h_2(t,b)], \tag{1.5}$$

where $M = \sup_{a<t<b} \left| f^{\triangle}(t) \right| < \infty$. *This inequality is sharp in the sense that the right-hand side of (1.5) cannot be replaced by a smaller one.*

Recently, Karpuz and Özkan [12] generalized Ostrowski's inequality and Montgomery's identity on arbitrary time scales by the means of generalized polynomials on time scales. By introducing a parameter, Liu, Ngô and Chen [20] also extended a generalization of the above inequality on time scales. In [16], Liu and Ngô derived a inequality of Ostrowski-Grüss type on time scales by using the Grüss inequality on time scales. Then, Ngô and Liu [23] gave a sharp Grüss type inequality on time scales and applied it to the sharp Ostrowski-Grüss inequality on time scales. Motivated by the ideas of [7, 20, 23, 30], Tuna and Daghan [29] studied generalizations of Ostrowski and Ostrowski-Grüss type inequality on time scales. More recently, Liu, Tuna and Jiang [21] derived a weighted Montgomery identity on time scales and then established weighted Ostrowski type, Trapezoid type, Grüss type and Ostrowski-Grüss like inequalities on time scales, respectively.

Motivated by the above research, the purpose of this paper is to obtain weighted Ostrowski, Ostrowski-Grüss and Ostrowski-Čebyšev type inequalities on time scales. We also give some other interesting inequalities on time scales as special cases.

This paper is organized as follows. In Section 2, we briefly present the general definitions and theorems related to the time scales calculus. The weighted Ostrowski, Ostrowski-Grüss and Ostrowski-Čebyšev type inequalities on time scales are derived in Section 3.

2. Time Scales Essentials

In this section we briefly introduce the time scales theory. For further details and proofs we refer the reader to Hilger's Ph.D. thesis [11], the books [4, 5, 13], and the survey [1].

Definition 1. *A time scale* $\mathbb{T}$ *is an arbitrary nonempty closed subset of* $\mathbb{R}$.

We assume throughout that $\mathbb{T}$ has the topology that is inherited from the standard topology on $\mathbb{R}$. It also assumed throughout that in $\mathbb{T}$ the interval $[a,b]_{\mathbb{T}}$ means the set $\{t \in \mathbb{T} : s < t\}$ for the points $a < b$ in $\mathbb{T}$. Since a time scale may not be connected, we need the following concept of jump operators.

Definition 2. *For* $t \in \mathbb{T}$, *we define the forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ *by* $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$, *while the backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ *is defined by* $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$.

The jump operators σ and ρ allow the classification of points in $\mathbb{T}$ as follows.

Definition 3. *If* $\sigma(t) > t$, *then we say that* t *is right-scattered, while if* $\rho(t) < t$ *then we say that* t *is left-scattered. Points that are right-scattered and left-scattered at the same time are called isolated. If* $\sigma(t) = t$, *the* t *is called right-dense, and if* $\rho(t) = t$ *then* t *is called left-dense, Points that both right-dense and left-dense are called dense.*

Definition 4. *The mapping* $\mu : \mathbb{T} \to \mathbb{R}^+$ *defined by* $\mu(t) = \sigma(t) - t$ *is called the graininess function. The set* $\mathbb{T}^k$ *is defined as follows: if* $\mathbb{T}$ *has a left-scattered maximum* m, *then* $\mathbb{T}^k = \mathbb{T} - \{m\}$; *otherwise,* $\mathbb{T}^k = \mathbb{T}$.

If $\mathbb{T} = \mathbb{R}$, then $\mu(t) = 0$, and when $\mathbb{T} = \mathbb{Z}$, we have $\mu(t) = 1$.

Definition 5. *Let* $f : \mathbb{T} \to \mathbb{R}$. f *is called differentiable at* $t \in \mathbb{T}^k$, *with (delta) derivative* $f^{\triangle}(t) \in \mathbb{R}$, *if given* $\varepsilon > 0$ *there exists a neighborhood* U *of* t *such that,*

$$\left| f(\sigma(t)) - f(s) - f^{\triangle}(t)[\sigma(t) - s] \right| \leq \varepsilon \left| \sigma(t) - s \right|, \quad \forall\ s \in U.$$

If $\mathbb{T} = \mathbb{R}$, then $f^{\triangle}(t) = \frac{df(t)}{dt}$, and if $\mathbb{T} = \mathbb{Z}$, then $f^{\triangle}(t) = f(t+1) - f(t)$.

Theorem 5. *Assume $f, g : \mathbb{T} \to \mathbb{R}$ are differfentiable at $t \in \mathbb{T}^k$. Then the prfoduct $fg : \mathbb{T} \to \mathbb{R}$ is differfentiable at t with*

$$(fg)^{\triangle}(t) = f^{\triangle}(t)g(t) + f(\sigma(t))g^{\triangle}(t) = f(t)g^{\triangle}(t) + f^{\triangle}(t)g(\sigma(t)).$$

Definition 6. *The function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous (denote $f \in C_{rd}(\mathbb{T}, \mathbb{R})$), if it is continuous at all right-dense points $t \in \mathbb{T}$ and its left-sided limits exist at all left-dense points $t \in \mathbb{T}$.*

Definition 7. *Let $f \in C_{rd}(\mathbb{T},\mathbb{R})$. Then $g : \mathbb{T} \to \mathbb{R}$ is called the antiderivative of f on $\mathbb{T}$ if it satisfies $g^{\triangle}(t) = f(t)$ for any $t \in \mathbb{T}^k$. In this case, we defined*

$$\int_a^t f(s)\triangle s = g(t) - g(a), \quad t \in \mathbb{T}.$$

Theorem 6. *Let f, g be rd-continuous, $a, b, c \in \mathbb{T}$ and $\alpha, \beta \in \mathbb{R}$. Then*
(1) $\int_a^b [f(t) + g(t)] \triangle t = \int_a^b f(t)\triangle t + \int_a^b g(t)\triangle t,$
(2) $\int_a^b f(t)\triangle t = -\int_b^a f(t)\triangle t,$
(3) $\int_a^b f(t)\triangle t = \int_a^c f(t)\triangle t + \int_c^b f(t)\triangle t,$
(4) $\int_a^b f(t)g^{\triangle}(t)\triangle t = (fg)(b) - (fg)(a) - \int_a^b f^{\triangle}(t)g(\sigma(t))\triangle t,$

Theorem 7. *If f is $\triangle$-integrable on $[a, b]$, then so is $|f|$,and*

$$\left|\int_a^b f(t)\triangle t\right| \le \int_a^b |f(t)| \triangle t.$$

Definition 8. *Let $g_k : \mathbb{T}^2 \to \mathbb{R}$, $k \in \mathbb{N}_0$ be defined by*

$$g_0(t, s) = 1 \quad \text{for all} \quad s, t \in \mathbb{T}$$

and then recursively by

$$g_{k+1}(t, s) = \int_s^t g_k(\sigma(\tau), s)\triangle s \quad \text{for all} \quad s, t \in \mathbb{T}$$

3. Main Results

Theorem 8. *Let $0 \le k \le 1$, $g : [a, b] \to [0, \infty)$ be rd continuous and positive and $h : [a, b] \to \mathbb{R}$ be differentiable such that $h^{\triangle}(t) = g(t)$ on $[a, b]$. Let $a, b, t, x \in \mathbb{T}$, $a < b$ and $f : [a, b] \to \mathbb{R}$ be twice differentiable. Then for all $x \in [a, b]$, we have*

$$\Bigg|(1-\alpha)^2 f(x) + \alpha(1-\alpha)\frac{f(a)+f(b)}{2} + \alpha\frac{f^{\triangle}(a)+f^{\triangle}(b)}{2\int_a^b g(t)\triangle t}\left(\int_a^b W(x,t)\triangle t\right)$$
$$-\frac{1}{\left(\int_a^b g(t)\triangle t\right)^2}\left(\int_a^b W(x,t)\triangle t\right)\left(\int_a^b g(t)f^{\triangle}(\sigma(t))\Delta t\right) - \frac{1-\alpha}{\int_a^b g(t)\triangle t}\int_a^b g(t)f(\sigma(t))\Delta t\Bigg|$$
$$\le \frac{M}{\left(\int_a^b g(t)\triangle t\right)^2}\int_a^b\int_a^b |W(x,t)|\,|W(t,s)|\triangle s\triangle t, \tag{3.1}$$

where $M = \sup_{a<t<b} \left|f^{\triangle\triangle}(t)\right| < \infty$ and

$$W(x,t) = \begin{cases} h(t) - ((1-\frac{\alpha}{2})h(a) + \frac{\alpha}{2}h(b)), & a \le t < x, \\ h(t) - (\frac{\alpha}{2}h(a) + (1-\frac{\alpha}{2})h(b)), & x \le t \le b. \end{cases} \tag{3.2}$$

Proof. Using item (4) of Theorem 6 and (3.2), we have (see also [21])

$$(1-\alpha)^2 f(x) = -\alpha(1-\alpha)\frac{f(a)+f(b)}{2} + \frac{1-\alpha}{\int_a^b g(t)\triangle t}\int_a^b g(t)f(\sigma(t))\Delta t + \frac{1-\alpha}{\int_a^b g(t)\triangle t}\int_a^b W(x,t)f^{\triangle}(t)\triangle t, \tag{3.3}$$

and

$$(1-\alpha)f^{\triangle}(t) = -\alpha\frac{f^{\triangle}(a)+f^{\triangle}(b)}{2} + \frac{1}{\int_a^b g(t)\triangle t}\int_a^b g(s)f^{\triangle}(\sigma(s))\Delta s + \frac{1}{\int_a^b g(t)\triangle t}\int_a^b W(t,s)f^{\triangle\triangle}(s)\triangle s. \tag{3.4}$$

Substituting $(1-\alpha)f^{\triangle}(t)$ in the right hand side of (3.3), we obtain

$$
\begin{aligned}
(1-\alpha)^2 f(x) =& -\alpha(1-\alpha)\frac{f(a)+f(b)}{2} - \alpha\frac{f^{\triangle}(a)+f^{\triangle}(b)}{2\int_a^b g(t)\triangle t}\left(\int_a^b W(x,t)\triangle t\right) \\
&+ \frac{1}{\left(\int_a^b g(t)\triangle t\right)^2}\left(\int_a^b W(x,t)\triangle t\right)\left(\int_a^b g(t)f^{\triangle}(\sigma(t))\Delta t\right) + \frac{1-\alpha}{\int_a^b g(t)\triangle t}\int_a^b g(t)f(\sigma(t))\Delta t \\
&+ \frac{1}{\left(\int_a^b g(t)\triangle t\right)^2}\int_a^b\int_a^b W(x,t)W(t,s)f^{\triangle\triangle}(s)\triangle s\triangle t. \qquad (3.5)
\end{aligned}
$$

From (3.5) and using the properties of modulus, the inequality (3.1) is proved. □

Corollary 1. *In the case of* $\mathbb{T}=\mathbb{R}$ *in Theorem 8, we have*

$$
\begin{aligned}
&\Bigg|(1-\alpha)^2 f(x) + \alpha(1-\alpha)\frac{f(a)+f(b)}{2} + \alpha\frac{f'(a)+f'(b)}{2\int_a^b g(t)dt}\left(\int_a^b W(x,t)dt\right) \\
&-\frac{1}{\left(\int_a^b g(t)dt\right)^2}\left(\int_a^b W(x,t)dt\right)\left(\int_a^b g(t)f'(t)dt\right) - \frac{1-\alpha}{\int_a^b g(t)dt}\int_a^b g(t)f(t)dt\Bigg| \\
\leq& \frac{M}{\left(\int_a^b g(t)dt\right)^2}\int_a^b\int_a^b |W(x,t)|\,|W(t,s)|\,dsdt,
\end{aligned}
$$

where $g(t)=h'(t)$ *on* $[a,b]$, $M=\sup_{a<t<b}|f''(t)|<\infty$ *and*

$$
W(x,t)=\begin{cases} h(t)-((1-\frac{\alpha}{2})h(a)+\frac{\alpha}{2}h(b)), & a\leq t<x, \\ h(t)-(\frac{\alpha}{2}h(a)+(1-\frac{\alpha}{2})h(b)), & x\leq t\leq b. \end{cases}
$$

Corollary 2. *In the case of* $\mathbb{T}=\mathbb{Z}$ *in Theorem 8, we have*

$$
\begin{aligned}
&\Bigg|(1-\alpha)^2 f(x) + \alpha(1-\alpha)\frac{f(a)+f(b)}{2} + \alpha\frac{\triangle f(a)+\triangle f(b)}{2\sum_{t=a}^{b-1} g(t)}\left(\sum_{t=a}^{b-1} W(x,t)\right) \\
&-\frac{1}{\left(\sum_{t=a}^{b-1} g(t)\right)^2}\left(\sum_{t=a}^{b-1} W(x,t)\right)\left(\sum_{t=a}^{b-1} g(t)\triangle f(t+1)\right) - \frac{1-\alpha}{\sum_{t=a}^{b-1} g(t)}\sum_{t=a}^{b-1} g(t)f(t+1)\Bigg| \\
\leq& \frac{M}{\left(\sum_{t=a}^{b-1} g(t)\right)^2}\sum_{t=a}^{b-1}\sum_{s=a}^{b-1}|W(x,t)|\,|W(t,s)|\,,
\end{aligned}
$$

where $g(t)=h(t+1)-h(t)$ *on* $[a,b-1]$, $M=\sup_{a<t<b}\left|\triangle^2 f(t)\right|<\infty$ *and*

$$
W(x,t)=\begin{cases} h(t)-((1-\frac{\alpha}{2})h(a)+\frac{\alpha}{2}h(b)), & a\leq t<x-1, \\ h(t)-(\frac{\alpha}{2}h(a)+(1-\frac{\alpha}{2})h(b)), & x\leq t\leq b-1. \end{cases}
$$

Corollary 3. *Let $f:[a,b]\to\mathbb{R}$, $q>1$, $a=q^m$ and $b=q^n$ with $m<n$ in Theorem 8, Then we have*

$$\begin{aligned}&\left|(1-\alpha)^2 f(x)+\alpha(1-\alpha)\frac{f(q^m)+f(q^n)}{2}+\alpha\frac{f^{\triangle}(q^m)+f^{\triangle}(q^n)}{2\int_{q^m}^{q^n}g(t)\triangle t}\left(\sum_{k=m}^{n-1}W(x,q^k)\right)\right.\\&\left.-\frac{1}{\left(\int_{q^m}^{q^n}g(t)\triangle t\right)^2}\left(\sum_{k=m}^{n-1}W(x,q^k)\right)\left(\int_{q^m}^{q^n}g(t)f^{\triangle}(\sigma(t))\Delta t\right)-\frac{1-\alpha}{\int_{q^m}^{q^n}g(t)\triangle t}\int_{q^m}^{q^n}g(t)f(\sigma(t))\Delta t\right|\\&\leq\frac{M}{\left(\int_{q^m}^{q^n}g(t)\triangle t\right)^2}\sum_{k=m}^{n-1}\sum_{s=m}^{n-1}\left|W(x,q^k)\right|\left|W(q^k,q^s)\right|,\end{aligned}$$

where $g(t)=\frac{h(qt)-h(t)}{(q-1)t}$ *on* $[q^m,q^n]$, $M=\sup_{q^m<t<q^n}\left|\frac{f(q^2t)-(q+1)f(qt)+qf(t)}{q(q-1)^2t^2}\right|<\infty$ *and*

$$W(x,t)=\begin{cases}h(q^k)-((1-\frac{\alpha}{2})h(q^m)+\frac{\alpha}{2}h(q^n)), & q^m\leq q^k<x,\\ h(q^k)-(\frac{\alpha}{2}h(q^m)+(1-\frac{\alpha}{2})h(q^n)), & x\leq q^k\leq q^n.\end{cases}$$

Corollary 4. *In the case of $h(t)=t$ in Theorem 8, we have*

$$\begin{aligned}&\left|(1-\alpha)^2 f(x)+\alpha(1-\alpha)\frac{f(a)+f(b)}{2}-\frac{1-\alpha}{b-a}\int_a^b f(\sigma(t))\Delta t\right.\\&+\alpha\frac{f^{\triangle}(a)+f^{\triangle}(b)}{2(b-a)}\left[-h_2\left(a,\left(1-\frac{\alpha}{2}\right)a+\frac{\alpha}{2}b\right)+h_2\left(x,\left(1-\frac{\alpha}{2}\right)a+\frac{\alpha}{2}b\right)\right.\\&\left.-h_2\left(x,\frac{\alpha}{2}a+\left(1-\frac{\alpha}{2}\right)b\right)+h_2\left(b,\frac{\alpha}{2}a+\left(1-\frac{\alpha}{2}\right)b\right)\right]\\&-\frac{1}{b-a}\left[-h_2\left(a,\left(1-\frac{\alpha}{2}\right)a+\frac{\alpha}{2}b\right)+h_2\left(x,\left(1-\frac{\alpha}{2}\right)a+\frac{\alpha}{2}b\right)\right.\\&\left.\left.-h_2\left(x,\frac{\alpha}{2}a+\left(1-\frac{\alpha}{2}\right)b\right)+h_2\left(b,\frac{\alpha}{2}a+\left(1-\frac{\alpha}{2}\right)b\right)\right]\frac{f(\sigma(b))-f(\sigma(a))}{b-a}\right|\\&\leq\frac{M}{(b-a)^2}\int_a^b|W(x,t)|\left[h_2\left(a,\left(1-\frac{\alpha}{2}\right)a+\frac{\alpha}{2}b\right)+h_2\left(t,\left(1-\frac{\alpha}{2}\right)a+\frac{\alpha}{2}b\right)\right.\\&\left.+h_2\left(t,\frac{\alpha}{2}a+\left(1-\frac{\alpha}{2}\right)b\right)+h_2\left(b,\frac{\alpha}{2}a+\left(1-\frac{\alpha}{2}\right)b\right)\right]\triangle t\end{aligned}$$

for all $\alpha\in[0,1]$ such that $a+\alpha\frac{b-a}{2}, b-\alpha\frac{b-a}{2}\in\mathbb{T}$ and $t\in\left[a+\alpha\frac{b-a}{2}, b-\alpha\frac{b-a}{2}\right]\cap\mathbb{T}$, where $M=\sup_{a<t<b}\left|f^{\triangle\triangle}(t)\right|<\infty$ and

$$W(x,t)=\begin{cases}t-(\left(1-\frac{\alpha}{2}\right)a+\frac{\alpha}{2}b), & a\leq t<x,\\ t-(\frac{\alpha}{2}a+\left(1-\frac{\alpha}{2}\right)b), & x\leq t\leq b.\end{cases}$$

Corollary 5. *In the case of $\alpha=0$ in Corollary 4, we have*

$$\begin{aligned}&\left|f(x)-\frac{f(\sigma(b)-f(\sigma(a))}{b-a}\left(\frac{h_2(x,a)-h_2(x,b)}{b-a}\right)-\frac{1}{b-a}\int_a^b f(\sigma(t))\Delta t\right|\\&\leq\frac{M}{(b-a)^2}\int_a^b|S(x,t)|\left[h_2(t,a)+h_2(t,b)\right]\triangle t,\end{aligned}$$

where $M=\sup_{a<t<b}\left|f^{\triangle\triangle}(t)\right|<\infty$ *and*

$$S(x,t)=\begin{cases}t-a, & a\leq t<x,\\ t-b, & x\leq t\leq b.\end{cases}$$

Corollary 6. *In the case of* $\mathbb{T}=\mathbb{R}$ *in Corollary 4, we have*

$$\begin{aligned}
&\left|(1-\alpha)^2 f(x)+\alpha(1-\alpha)\frac{f(a)+f(b)}{2}-\frac{1-\alpha}{b-a}\int_a^b f(t)dt\right.\\
&\left.+(1-\alpha)\left[\alpha\frac{f'(a)+f'(b)}{2}-\frac{f(b)-f(a)}{b-a}\right]\left(x-\frac{a+b}{2}\right)\right|\\
\leq&\frac{M}{(b-a)^2}\left\{\frac{1}{96}\alpha^2(a-b)^4\left(13\alpha^2-16\alpha+12\right)\right.\\
&+\frac{1}{192}\left[2(x-a)+\alpha(a-b)\right]^2\left[(b-a)^2(13\alpha^2-14\alpha+4)\right.\\
&\left.-2(b-a)(b+a-2x)\alpha+4(x-a)^2+8(b-x)^2\right]\\
&+\frac{1}{192}\left[2(b-x)+\alpha(a-b)\right]^2\left[(b-a)^2(13\alpha^2-14\alpha+4)\right.\\
&\left.\left.+2(b-a)(b+a-2x)\alpha+8(x-a)^2+4(b-x)^2\right]\right\},
\end{aligned}$$

where $M=\sup_{a<t<b}|f''(t)|<\infty$.

Remark 1. *In the case of* $\alpha=0$ *in Corollary 6, we get (1.2) in Theorem 2.*

Corollary 7. *In the case of* $\mathbb{T}=\mathbb{Z}$ *in Corollary 4, we have*

$$\begin{aligned}
&\left|(1-\alpha)^2 f(x)+\alpha(1-\alpha)\frac{f(a)+f(b)}{2}-\frac{1-\alpha}{b-a}\sum_{t=a}^{b-1}f(t+1)\right.\\
&\left.+\left[\alpha\frac{\triangle f(a)+\triangle f(b)}{2}-\frac{f(b+1)-f(a+1)}{b-a}\right]\left((1-\alpha)(x-\frac{a+b}{2})-\frac{1}{2}\right)\right|\\
\leq&\frac{M}{(b-a)^2}\left\{\left(a^4+b^4\right)\left(\frac{13}{48}\alpha^4-\frac{7}{12}\alpha^3+\frac{3}{4}\alpha^2-\frac{5}{12}\alpha+\frac{1}{6}\right)\right.\\
&+\left(a^3+b^3\right)\left[\left(-\frac{1}{2}\alpha^2+\frac{1}{2}\alpha-\frac{1}{2}\right)x+\frac{1}{4}\alpha^2-\frac{1}{4}\alpha+\frac{1}{3}\right]\\
&+\left(a^2+b^2\right)\left[\left(\frac{1}{2}\alpha^2-\frac{1}{2}\alpha+1\right)x^2+\left(-\frac{1}{2}\alpha^2+\frac{1}{2}\alpha-\frac{3}{2}\right)x-\frac{1}{12}\alpha^2+\frac{1}{2}\alpha+\frac{1}{3}\right]\\
&+(a+b)\left(-x^3+\frac{5}{2}x^2-\frac{3}{2}x+\frac{1}{6}\right)+\left(a^2b+b^2a\right)\left[\left(\frac{1}{2}\alpha^2-\frac{1}{2}\alpha-\frac{1}{2}\right)x\right.\\
&\left.-\frac{1}{4}\alpha^2+\frac{1}{4}\alpha+\frac{1}{2}\right]+\left(a^3b+ab^3\right)\left(-\frac{13}{12}\alpha^4+\frac{7}{3}\alpha^3-\frac{5}{2}\alpha^2+\frac{7}{6}\alpha-\frac{1}{6}\right)\\
&+ab\left[\left(-\alpha^2+\alpha+1\right)x^2+\left(\alpha^2-\alpha-2\right)x+\frac{1}{6}\alpha^2-\alpha+\frac{5}{6}\right]\\
&\left.+a^2b^2\left(\frac{13}{8}\alpha^4-\frac{7}{2}\alpha^3+\frac{7}{2}\alpha^2-\frac{3}{2}\alpha+\frac{1}{2}\right)+\frac{1}{2}x^4-\frac{5}{3}x^3+\frac{3}{2}x^2-\frac{1}{3}x\right\},
\end{aligned}$$

where $M=\sup_{a<t<b}\left|\triangle^2 f(t)\right|<\infty$.

Corollary 8. *In the case of* $\alpha=0$ *in Corollary 7, we have*

$$\begin{aligned}
&\left|f(x)-\frac{f(b+1)-f(a+1)}{b-a}\left(x-\frac{b+a}{2}-\frac{1}{2}\right)-\frac{1}{b-a}\sum_{t=a}^{b-1}f(t+1)\right|\\
\leq&\frac{M}{12(b-a)^2}\left\{\left[(x-a)^2+a-x\right]\left[(b-a)^2+(x-a)^2+2(b-x)^2+5b+2a-7x+2\right]\right.\\
&\left.+\left[(b-x)^2+b-x\right]\left[(b-a)^2+(b-x)^2+2(x-a)^2+5a+2b-7x+2\right]\right\},
\end{aligned}$$

where $M=\sup_{a<t<b}\left|\triangle^2 f(t)\right|<\infty$.

Theorem 9. *Let $0 \leq k \leq 1$, $g : [a,b] \to [0,\infty)$ be rd continuous and positive and $h : [a,b] \to \mathbb{R}$ be differentiable such that $h^{\triangle}(t) = g(t)$ on $[a,b]$. Let $a,b,t,x \in \mathbb{T}$, $a < b$ and $f : [a,b] \to \mathbb{R}$ be differentiable. Then for all $x \in [a,b]$, we have*

$$\begin{aligned} &\left| \frac{1}{b-a}\left[(1-\alpha)f(x) + \alpha\frac{f(a)+f(b)}{2}\right]\int_a^b g(t)\Delta t \right. \\ &\left. -\frac{f(b)-f(a)}{(b-a)^2}\int_a^b W(x,t)\triangle t - \frac{1}{b-a}\int_a^b g(t)f(\sigma(t))\Delta t \right| \\ &\leq \left[\frac{1}{b-a}\int_a^b W^2(x,t)\triangle t - \left(\frac{1}{b-a}\int_a^b W(x,t)\triangle t\right)^2\right]^{\frac{1}{2}} \\ &\times \left[\frac{1}{b-a}\int_a^b \left(f^{\triangle}(t)\right)^2 \triangle t - \left(\frac{f(b)-f(a)}{b-a}\right)^2\right]^{\frac{1}{2}}, \end{aligned} \tag{3.6}$$

where

$$W(x,t) = \begin{cases} h(t) - \left(\left(1-\frac{\alpha}{2}\right)h(a) + \frac{\alpha}{2}h(b)\right), & a \leq t < x, \\ h(t) - \left(\frac{\alpha}{2}h(a) + \left(1-\frac{\alpha}{2}\right)h(b)\right), & x \leq t \leq b. \end{cases}$$

Proof. We have

$$\begin{aligned} &\frac{1}{b-a}\int_a^b W(x,t)f^{\triangle}(t)\triangle t - \left(\frac{1}{b-a}\int_a^b W(x,t)\triangle t\right)\left(\frac{1}{b-a}\int_a^b f^{\triangle}(t)\triangle t\right) \\ =&\frac{1}{2(b-a)^2}\int_a^b\int_a^b (W(x,t)-W(x,s))\left(f^{\triangle}(t)-f^{\triangle}(s)\right)\triangle t\triangle s. \end{aligned} \tag{3.7}$$

We also have

$$\int_a^b W(x,t)f^{\triangle}(t)\Delta t = \left[(1-\alpha)f(x) + \alpha\frac{f(a)+f(b)}{2}\right]\int_a^b g(t)\Delta t - \int_a^b g(t)f(\sigma(t))\Delta t \tag{3.8}$$

and

$$\frac{1}{b-a}\int_a^b f^{\triangle}(t)\triangle t = \frac{f(b)-f(a)}{b-a}. \tag{3.9}$$

Using the Cauchy-Schwartz inequality, we may write

$$\begin{aligned} &\left|\frac{1}{2(b-a)^2}\int_a^b\int_a^b (W(x,t)-W(x,s))\left(f^{\triangle}(t)-f^{\triangle}(s)\right)\triangle t\triangle s\right| \\ &\leq \left(\frac{1}{2(b-a)^2}\int_a^b\int_a^b (W(x,t)-W(x,s))^2\triangle t\triangle s\right)^{\frac{1}{2}} \\ &\times \left(\frac{1}{2(b-a)^2}\int_a^b\int_a^b \left(f^{\triangle}(t)-f^{\triangle}(s)\right)^2\triangle t\triangle s\right)^{\frac{1}{2}}. \end{aligned} \tag{3.10}$$

However

$$\int_a^b\int_a^b (W(x,t)-W(x,s))^2\triangle t\triangle s = \frac{1}{b-a}\int_a^b W^2(x,t)\triangle t - \left(\frac{1}{b-a}\int_a^b W(x,t)\triangle t\right)^2,$$

and

$$\int_a^b\int_a^b \left(f^{\triangle}(t)-f^{\triangle}(s)\right)^2\triangle t\triangle s = \frac{1}{b-a}\int_a^b \left(f^{\triangle}(t)\right)^2\triangle t - \left(\frac{1}{b-a}\int_a^b f^{\triangle}(t)\triangle t\right)^2.$$

Using (3.7)-(3.12), we obtain the inequality (3.6). □

Corollary 9. *In the case of $\mathbb{T}=\mathbb{R}$ in Theorem 9, we have*

$$\left|\frac{1}{b-a}\left[(1-\alpha)f(x)+\alpha\frac{f(a)+f(b)}{2}\right]\int_a^b g(t)dt-\frac{f(b)-f(a)}{(b-a)^2}\int_a^b W(x,t)dt-\frac{1}{b-a}\int_a^b g(t)f(t)dt\right|$$

$$\leq\left[\frac{1}{b-a}\int_a^b W^2(x,t)dt-\left(\frac{1}{b-a}\int_a^b W(x,t)dt\right)^2\right]^{\frac{1}{2}}\left[\frac{1}{b-a}\int_a^b (f'(t))^2dt-\left(\frac{f(b)-f(a)}{b-a}\right)^2\right]^{\frac{1}{2}},$$

where $g(t)=h'(t)$ on $[a,b]$ and

$$W(x,t)=\begin{cases} h(t)-\left(\left(1-\frac{\alpha}{2}\right)h(a)+\frac{\alpha}{2}h(b)\right), & a\leq t<x,\\ h(t)-\left(\frac{\alpha}{2}h(a)+\left(1-\frac{\alpha}{2}\right)h(b)\right), & x\leq t\leq b.\end{cases}$$

Corollary 10. *In the case of $\mathbb{T}=\mathbb{Z}$ in Theorem 9, we have*

$$\left|\frac{1}{b-a}\left[(1-\alpha)f(x)+\alpha\frac{f(a)+f(b)}{2}\right]\sum_{t=a}^{b-1}g(t)-\frac{f(b)-f(a)}{(b-a)^2}\sum_{t=a}^{b-1}W(x,t)-\frac{1}{b-a}\sum_{t=a}^{b-1}g(t)f(t+1)\right|$$

$$\leq\left[\frac{1}{b-a}\sum_{t=a}^{b-1}W^2(x,t)-\left(\frac{1}{b-a}\sum_{t=a}^{b-1}W(x,t)\right)^2\right]^{\frac{1}{2}}\left[\frac{1}{b-a}\sum_{t=a}^{b-1}(\triangle f(t))^2-\left(\frac{f(b)-f(a)}{b-a}\right)^2\right]^{\frac{1}{2}},$$

where $g(t)=h(t+1)-h(t)$ on $[a,b-1]$ and

$$W(x,t)=\begin{cases} h(t)-\left(\left(1-\frac{\alpha}{2}\right)h(a)+\frac{\alpha}{2}h(b)\right), & a\leq t<x-1,\\ h(t)-\left(\frac{\alpha}{2}h(a)+\left(1-\frac{\alpha}{2}\right)h(b)\right), & x\leq t\leq b-1.\end{cases}$$

Corollary 11. *Let $f:[a,b]\to\mathbb{R}$, $q>1$, $a=q^m$ and $b=q^n$ with $m<n$ in Theorem 9, Then we have*

$$\left|\frac{1}{q^n-q^m}\left[(1-\alpha)f(x)+\alpha\frac{f(q^m)+f(q^n)}{2}\right]\int_{q^m}^{q^n}g(t)\Delta t\right.$$
$$\left.-\frac{f(q^n)-f(q^m)}{(q^n-q^m)^2}\sum_{k=m}^{n-1}W(x,q^k)-\frac{1}{q^n-q^m}\int_{q^m}^{q^n}g(t)f(\sigma(t))\Delta t\right|$$
$$\leq\left[\frac{1}{q^n-q^m}\sum_{k=m}^{n-1}W^2(x,q^k)-\left(\frac{1}{q^n-q^m}\sum_{k=m}^{n-1}W(x,q^k)\right)^2\right]^{\frac{1}{2}}$$
$$\times\left[\frac{1}{q^n-q^m}\int_{q^m}^{q^n}\left(f^{\triangle}(t)\right)^2\triangle t-\left(\frac{f(q^n)-f(q^m)}{q^n-q^m}\right)^2\right]^{\frac{1}{2}},$$

where $g(t)=\frac{h(qt)-h(t)}{(q-1)t}$ on $[q^m,q^n]$ and

$$W(x,t)=\begin{cases} h(q^k)-\left(\left(1-\frac{\alpha}{2}\right)h(q^m)+\frac{\alpha}{2}h(q^n)\right), & q^m\leq q^k<x,\\ h(q^k)-\left(\frac{\alpha}{2}h(q^n)+\left(1-\frac{\alpha}{2}\right)h(q^n)\right), & x\leq q^k\leq q^n.\end{cases}$$

Corollary 12. *In the case of $h(t)=t$ in Theorem 9, we have*

$$\left|\left[(1-\alpha)f(x)+\alpha\frac{f(a)+f(b)}{2}\right]-\frac{f(b)-f(a)}{(b-a)^2}\left[-h_2\left(a,\left(1-\frac{\alpha}{2}\right)a+\frac{\alpha}{2}b\right)+h_2\left(x,\left(1-\frac{\alpha}{2}\right)a+\frac{\alpha}{2}b\right)\right.\right.$$
$$\left.\left.-h_2\left(x,\frac{\alpha}{2}a+\left(1-\frac{\alpha}{2}\right)b\right)+h_2\left(b,\frac{\alpha}{2}a+\left(1-\frac{\alpha}{2}\right)b\right)\right]-\frac{1}{b-a}\int_a^b f(\sigma(t))\Delta t\right|$$
$$\leq\left\{\frac{1}{b-a}\int_a^b W^2(x,t)\triangle t-\frac{1}{(b-a)^2}\left[-h_2\left(a,\left(1-\frac{\alpha}{2}\right)a+\frac{\alpha}{2}b\right)+h_2\left(x,\left(1-\frac{\alpha}{2}\right)a+\frac{\alpha}{2}b\right)\right.\right.$$
$$\left.\left.-h_2\left(x,\frac{\alpha}{2}a+\left(1-\frac{\alpha}{2}\right)b\right)+h_2\left(b,\frac{\alpha}{2}a+\left(1-\frac{\alpha}{2}\right)b\right)\right]^2\right\}^{\frac{1}{2}}\left[\frac{1}{b-a}\int_a^b\left(f^{\triangle}(t)\right)^2\triangle t-\left(\frac{f(b)-f(a)}{b-a}\right)^2\right]^{\frac{1}{2}},$$

for all $\alpha\in[0,1]$ such that $a+\alpha\frac{b-a}{2},b-\alpha\frac{b-a}{2}\in\mathbb{T}$ and $t\in\left[a+\alpha\frac{b-a}{2},b-\alpha\frac{b-a}{2}\right]\cap\mathbb{T}$, where

$$W(x,t)=\begin{cases} t-\left(\left(1-\frac{\alpha}{2}\right)a+\frac{\alpha}{2}b\right), & a\leq t<x,\\ t-\left(\frac{\alpha}{2}a+\left(1-\frac{\alpha}{2}\right)b\right), & x\leq t\leq b.\end{cases}$$

Corollary 13. *In the case of* $\alpha = 0$ *in Corollary 12, we have*

$$\left|f(x) - \frac{f(b)-f(a)}{b-a}\frac{h_2(x,a)-h_2(x,b)}{b-a} - \frac{1}{b-a}\int_a^b f(\sigma(t))\Delta t\right|$$
$$\leq \left[\frac{1}{b-a}\int_a^b S^2(x,t)\triangle t - \left(\frac{h_2(x,a)-h_2(x,b)}{b-a}\right)^2\right]^{\frac{1}{2}} \left[\frac{1}{b-a}\int_a^b \left(f^{\triangle}(t)\right)^2 \triangle t - \left(\frac{f(b)-f(a)}{b-a}\right)^2\right]^{\frac{1}{2}},$$

where

$$S(x,t) = \begin{cases} t-a, & a \leq t < x, \\ t-b, & x \leq t \leq b. \end{cases}$$

Corollary 14. *In the case of* $\mathbb{T} = \mathbb{R}$ *in Corollary 12, we have*

$$\left|\left[(1-\alpha)f(x) + \alpha\frac{f(a)+f(b)}{2}\right] - (1-\alpha)\frac{f(b)-f(a)}{b-a}\left(x - \frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)dt\right|$$
$$\leq \left[\left(\alpha^2-\alpha\right)(b-x)(x-a) + \frac{1}{12}(b-a)^2\right]^{\frac{1}{2}} \left[\frac{1}{b-a}\int_a^b \left(f'(t)\right)^2 dt - \left(\frac{f(b)-f(a)}{b-a}\right)^2\right]^{\frac{1}{2}}.$$

Remark 2. *In the case of* $\alpha = 0$ *in Corollary 14, we have*

$$\left|f(x) - \frac{f(b)-f(a)}{b-a}\left(x - \frac{b+a}{2}\right) - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{b-a}{2\sqrt{3}}\left[\frac{1}{b-a}\int_a^b \left(f'(t)\right)^2 dt - \left(\frac{f(b)-f(a)}{b-a}\right)^2\right]^{\frac{1}{2}}.$$

Corollary 15. *In the case of* $\mathbb{T} = \mathbb{Z}$ *in Corollary 12, we have*

$$\left|\left[(1-\alpha)f(x) + \alpha\frac{f(a)+f(b)}{2}\right] - \frac{f(b)-f(a)}{b-a}\left[(1-\alpha)\left(x - \frac{b+a}{2}\right) - \frac{1}{2}\right] - \frac{1}{b-a}\sum_{t=a}^{b-1} f(t+1)\right|$$
$$\leq \left[\left(\alpha^2-\alpha\right)(b-x)(x-a) + \frac{1}{12}(b-a)^2 - \frac{1}{12}\right]^{\frac{1}{2}} \left[\frac{1}{b-a}\sum_{t=a}^{b-1}(\triangle f(t))^2 - \left(\frac{f(b)-f(a)}{b-a}\right)^2\right]^{\frac{1}{2}}.$$

Remark 3. *In the case of* $\alpha = 0$ *in Corollary 15, we have*

$$\left|f(x) - \frac{f(b)-f(a)}{b-a}\left(x - \frac{b+a}{2} - \frac{1}{2}\right) - \frac{1}{b-a}\sum_{t=a}^{b-1} f(t+1)\right|$$
$$\leq \frac{1}{2\sqrt{3}}\left[(b-a)^2 - 1\right]^{\frac{1}{2}} \left[\frac{1}{b-a}\sum_{t=a}^{b-1}(\triangle f(t))^2 - \left(\frac{f(b)-f(a)}{b-a}\right)^2\right]^{\frac{1}{2}}.$$

Theorem 10. *Let* $0 \leq k \leq 1$, $g : [a,b] \to [0,\infty)$ *be rd continuous and positive and* $h : [a,b] \to \mathbb{R}$ *be differentiable such that* $h^{\triangle}(t) = g(t)$ *on* $[a,b]$. *Let* $a, b, t, x \in \mathbb{T}$, $a < b$ *and* $p, r : [a,b] \to \mathbb{R}$ *be twice differentiable. Then for all* $x \in [a,b]$, *we have*

$$\left|(1-\alpha)^2 p(x)r(x) + \frac{\alpha}{4\int_a^b g(t)\triangle t}\left(\int_a^b W(x,t)\triangle t\right)\left[r(x)\left(p^{\triangle}(a)+p^{\triangle}(b)\right) + p(x)\left(r^{\triangle}(a)+r^{\triangle}(b)\right)\right]\right.$$
$$- \frac{1-\alpha}{2\int_a^b g(t)\triangle t}\left(r(x)\int_a^b g(t)p(\sigma(t))\Delta t + p(x)\int_a^b g(t)r(\sigma(t))\Delta t\right)$$
$$+ \frac{\alpha(1-\alpha)}{2}\left(r(x)\frac{p(a)+p(b)}{2} + p(x)\frac{r(a)+r(b)}{2}\right)$$
$$\left.- \frac{1}{2\left(\int_a^b g(t)\triangle t\right)^2}\left(\int_a^b W(x,t)\triangle t\right)\left(r(x)\int_a^b g(t)p^{\triangle}(\sigma(t))\Delta t + p(x)\int_a^b g(t)r^{\triangle}(\sigma(t))\Delta t\right)\right|$$

$$\leq \frac{1}{2\left(\int_a^b g(t)\triangle t\right)^2}\left(|r(x)|M + |p(x)|N\right)\int_a^b\int_a^b |W(x,t)||W(t,s)|\triangle s\triangle t, \tag{3.11}$$

where $M = \sup_{a<t<b}\left|p^{\triangle\triangle}(t)\right| < \infty$, $N = \sup_{a<t<b}\left|r^{\triangle\triangle}(t)\right| < \infty$, *and*

$$W(x,t) = \begin{cases} h(t) - \left(\left(1-\frac{\alpha}{2}\right)h(a) + \frac{\alpha}{2}h(b)\right), & a \leq t < x, \\ h(t) - \left(\frac{\alpha}{2}h(a) + \left(1-\frac{\alpha}{2}\right)h(b)\right), & x \leq t \leq b. \end{cases}$$

Proof. We have

$$(1-\alpha)^2 p(x)+\alpha\frac{p^{\triangle}(a)+p^{\triangle}(b)}{2\int_a^b g(t)\triangle t}\left(\int_a^b W(x,t)\triangle t\right)+\alpha(1-\alpha)\frac{p(a)+p(b)}{2}$$
$$-\frac{1}{\left(\int_a^b g(t)\triangle t\right)^2}\left(\int_a^b W(x,t)\triangle t\right)\left(\int_a^b g(t)p^{\triangle}(\sigma(t))\Delta t\right)-\frac{1-\alpha}{\int_a^b g(t)\triangle t}\int_a^b g(t)p(\sigma(t))\Delta t$$
$$=\frac{1}{\left(\int_a^b g(t)\triangle t\right)^2}\int_a^b\int_a^b W(x,t)W(t,s)p^{\triangle\triangle}(s)\triangle s\triangle t \tag{3.12}$$

and

$$(1-\alpha)^2 r(x)+\alpha\frac{r^{\triangle}(a)+r^{\triangle}(b)}{2\int_a^b g(t)\triangle t}\left(\int_a^b W(x,t)\triangle t\right)+\alpha(1-\alpha)\frac{r(a)+r(b)}{2}$$
$$-\frac{1}{\left(\int_a^b g(t)\triangle t\right)^2}\left(\int_a^b W(x,t)\triangle t\right)\left(\int_a^b g(t)r^{\triangle}(\sigma(t))\Delta t\right)-\frac{1-\alpha}{\int_a^b g(t)\triangle t}\int_a^b g(t)r(\sigma(t))\Delta t$$
$$=\frac{1}{\left(\int_a^b g(t)\triangle t\right)^2}\int_a^b\int_a^b W(x,t)W(t,s)r^{\triangle\triangle}(s)\triangle s\triangle t \tag{3.13}$$

Multiplying both sides of (3.12) and (3.13) by $r(x)$ and $p(x)$ respectively, adding the resulting identities and rewriting, we have:

$$2(1-\alpha)^2 p(x)r(x)+\frac{\alpha}{2\int_a^b g(t)\triangle t}\left(\int_a^b W(x,t)\triangle t\right)\left[r(x)\left(p^{\triangle}(a)+p^{\triangle}(b)\right)+p(x)\left(r^{\triangle}(a)+r^{\triangle}(b)\right)\right]$$
$$-\frac{1-\alpha}{\int_a^b g(t)\triangle t}\left(r(x)\int_a^b g(t)p(\sigma(t))\Delta t+p(x)\int_a^b g(t)r(\sigma(t))\Delta t\right)$$
$$+\alpha(1-\alpha)\left(r(x)\frac{p(a)+p(b)}{2}+p(x)\frac{r(a)+r(b)}{2}\right)$$
$$-\frac{1}{\left(\int_a^b g(t)\triangle t\right)^2}\left(\int_a^b W(x,t)\triangle t\right)\left(r(x)\int_a^b g(t)p^{\triangle}(\sigma(t))\Delta t+p(x)\int_a^b g(t)r^{\triangle}(\sigma(t))\Delta t\right)$$
$$=\frac{1}{\left(\int_a^b g(t)\triangle t\right)^2}\left(r(x)\int_a^b\int_a^b W(x,t)W(t,s)p^{\triangle\triangle}(s)\triangle s\triangle t+p(x)\int_a^b\int_a^b W(x,t)W(t,s)r^{\triangle\triangle}(s)\triangle s\triangle t\right).$$

Using properties of the modulus, we get (3.11). □

Corollary 16. *In the case of $\mathbb{T}=\mathbb{R}$ in Theorem 10, we have*

$$\Bigg|(1-\alpha)^2 p(x)r(x)+\frac{\alpha}{4\int_a^b g(t)dt}\left(\int_a^b W(x,t)dt\right)\left[r(x)\left(p'(a)+p'(b)\right)+p(x)\left(r'(a)+r'(b)\right)\right]$$
$$-\frac{1-\alpha}{2\int_a^b g(t)dt}\left(r(x)\int_a^b g(t)p(t)dt+p(x)\int_a^b g(t)r(t)dt\right)+\frac{\alpha(1-\alpha)}{2}\left(r(x)\frac{p(a)+p(b)}{2}+p(x)\frac{r(a)+r(b)}{2}\right)$$
$$-\frac{1}{2\left(\int_a^b g(t)dt\right)^2}\left(\int_a^b W(x,t)dt\right)\left(r(x)\int_a^b g(t)p'(t)dt+p(x)\int_a^b g(t)r'(t)dt\right)\Bigg|$$
$$\leq\frac{1}{2\left(\int_a^b g(t)dt\right)^2}\left(|r(x)|M+|p(x)|N\right)\int_a^b\int_a^b |W(x,t)||W(t,s)|dsdt,$$

where $g(t)=h'(t)$ on $[a,b]$ and $M=\sup_{a<t<b}|p''(t)|<\infty$, $N=\sup_{a<t<b}|r''(t)|<\infty$, and

$$W(x,t)=\begin{cases} h(t)-\left(\left(1-\frac{\alpha}{2}\right)h(a)+\frac{\alpha}{2}h(b)\right), & a\leq t<x,\\ h(t)-\left(\frac{\alpha}{2}h(a)+\left(1-\frac{\alpha}{2}\right)h(b)\right), & x\leq t\leq b.\end{cases}$$

Corollary 17. *In the case of* $\mathbb{T}=\mathbb{Z}$ *in Theorem 10, we have*

$$\begin{aligned}&\Bigg|(1-\alpha)^2 p(x)r(x)+\frac{\alpha}{4\sum\limits_{t=a}^{b-1} g(t)}\left(\sum_{t=a}^{b-1} W(x,t)\right)\left[r(x)\left(\triangle p(a)+\triangle p(b)\right)+p(x)\left(\triangle r(a)+\triangle r(b)\right)\right]\\&-\frac{1-\alpha}{2\sum\limits_{t=a}^{b-1} g(t)}\left(r(x)\sum_{t=a}^{b-1} g(t)p(t+1)+p(x)\sum_{t=a}^{b-1} g(t)r(t+1)\right)\\&+\frac{\alpha(1-\alpha)}{2}\left(r(x)\frac{p(a)+p(b)}{2}+p(x)\frac{r(a)+r(b)}{2}\right)\\&-\frac{1}{2\left(\sum\limits_{t=a}^{b-1} g(t)\right)^2}\left(\sum_{t=a}^{b-1} W(x,t)\right)\left(r(x)\sum_{t=a}^{b-1} g(t)\triangle p(t+1)+p(x)\sum_{t=a}^{b-1} g(t)\triangle r(t+1)\right)\Bigg|\\\le&\frac{1}{2\left(\sum\limits_{t=a}^{b-1} g(t)\right)^2}\left(\left|r(x)\right|M+\left|p(x)\right|N\right)\sum_{t=a}^{b-1}\sum_{s=a}^{b-1}\left|W(x,t)\right|\left|W(t,s)\right|,\end{aligned}$$

where $g(t)=h(t+1)-h(t)$ *on* $[a,b-1]$ *and* $M=\sup\limits_{a<t<b}\left|\triangle^2 p(t)\right|<\infty$, $N=\sup\limits_{a<t<b}\left|\triangle^2 r(t)\right|<\infty$, *and*

$$W(x,t)=\begin{cases} h(t)-(\left(1-\frac{\alpha}{2}\right)h(a)+\frac{\alpha}{2}h(b)), & a\le t<x-1,\\ h(t)-(\frac{\alpha}{2}h(a)+\left(1-\frac{\alpha}{2}\right)h(b)), & x\le t\le b-1.\end{cases}$$

Corollary 18. *Let* $f:[a,b]\to\mathbb{R}$, $q>1$, $a=q^m$ *and* $b=q^n$ *with* $m<n$ *in Theorem 9, Then we have*

$$\begin{aligned}&\Bigg|(1-\alpha)^2 p(x)r(x)+\frac{\alpha}{4\int_{q^m}^{q^n} g(t)\triangle t}\left(\sum_{k=m}^{n-1} W(x,q^k)\right)\left[r(x)\left(p^{\triangle}(q^m)+p^{\triangle}(q^n)\right)+p(x)\left(r^{\triangle}(q^m)+r^{\triangle}(q^n)\right)\right]\\&-\frac{1-\alpha}{2\int_{q^m}^{q^n} g(t)\triangle t}\left(r(x)\int_{q^m}^{q^n} g(t)p(\sigma(t))\Delta t+p(x)\int_{q^m}^{q^n} g(t)r(\sigma(t))\Delta t\right)\\&+\frac{\alpha(1-\alpha)}{2}\left(r(x)\frac{p(q^m)+p(q^n)}{2}+p(x)\frac{r(q^m)+r(q^n)}{2}\right)\\&-\frac{1}{2\left(\int_a^b g(t)\triangle t\right)^2}\left(\sum_{k=m}^{n-1} W(x,q^k)\right)\left(r(x)\int_{q^m}^{q^n} g(t)p^{\triangle}(\sigma(t))\Delta t+p(x)\int_{q^m}^{q^n} g(t)r^{\triangle}(\sigma(t))\Delta t\right)\Bigg|\\\le&\frac{1}{2\left(\int_{q^m}^{q^n} g(t)\triangle t\right)^2}\left(\left|r(x)\right|M+\left|p(x)\right|N\right)\sum_{k=m}^{n-1}\sum_{s=m}^{n-1}\left|W(x,q^k)\right|\left|W(q^k,q^s)\right|,\end{aligned}$$

where $g(t)=\frac{h(qt)-h(t)}{(q-1)t}$ *on* $[q^m,q^n]$,

$$M=\sup_{q^m<t<q^n}\left|\frac{p(q^2t)-(q+1)p(qt)+pf(t)}{q(q-1)^2t^2}\right|<\infty,\quad N=\sup_{q^m<t<q^n}\left|\frac{r(q^2t)-(q+1)r(qt)+qr(t)}{q(q-1)^2t^2}\right|<\infty$$

and

$$W(x,t)=\begin{cases} h(q^k)-(\left(1-\frac{\alpha}{2}\right)h(q^m)+\frac{\alpha}{2}h(q^n)), & q^m\le q^k<x,\\ h(q^k)-(\frac{\alpha}{2}h(q^n)+\left(1-\frac{\alpha}{2}\right)h(q^n)), & x\le q^k\le q^n.\end{cases}$$

Corollary 19. *In the case of $h(t)=t$ in Theorem 10, we have*

$$\Bigg|(1-\alpha)^2 p(x)r(x)+\frac{\alpha}{4(b-a)}\left[-h_2\left(a,\left(1-\frac{\alpha}{2}\right)a+\frac{\alpha}{2}b\right)+h_2\left(x,\left(1-\frac{\alpha}{2}\right)a+\frac{\alpha}{2}b\right)\right.$$
$$\left.-h_2\left(x,\frac{\alpha}{2}a+\left(1-\frac{\alpha}{2}\right)b\right)+h_2\left(b,\frac{\alpha}{2}a+\left(1-\frac{\alpha}{2}\right)b\right)\right]\left[r(x)\left(p^{\triangle}(a)+p^{\triangle}(b)\right)+p(x)\left(r^{\triangle}(a)+r^{\triangle}(b)\right)\right]$$
$$-\frac{1-\alpha}{2(b-a)}\left(r(x)\int_a^b p(\sigma(t))\Delta t+p(x)\int_a^b r(\sigma(t))\Delta t\right)+\frac{\alpha(1-\alpha)}{2}\left(r(x)\frac{p(a)+p(b)}{2}+p(x)\frac{r(a)+r(b)}{2}\right)$$
$$-\frac{1}{2(b-a)^2}\left[-h_2\left(a,\left(1-\frac{\alpha}{2}\right)a+\frac{\alpha}{2}b\right)+h_2\left(x,\left(1-\frac{\alpha}{2}\right)a+\frac{\alpha}{2}b\right)-h_2\left(x,\frac{\alpha}{2}a+\left(1-\frac{\alpha}{2}\right)b\right)\right.$$
$$\left.+h_2\left(b,\frac{\alpha}{2}a+\left(1-\frac{\alpha}{2}\right)b\right)\right]\left[r(x)\left(p(\sigma(b))-p(\sigma(a))\right)+p(x)\left(r(\sigma(b))-r(\sigma(a))\right)\right]\Bigg|$$
$$\leq\frac{1}{2(b-a)^2}\left(|r(x)|M+|p(x)|N\right)\int_a^b|W(x,t)|\left[h_2\left(a,\left(1-\frac{\alpha}{2}\right)a+\frac{\alpha}{2}b\right)\right.$$
$$\left.+h_2\left(t,\left(1-\frac{\alpha}{2}\right)a+\frac{\alpha}{2}b\right)+h_2\left(t,\frac{\alpha}{2}a+\left(1-\frac{\alpha}{2}\right)b\right)+h_2\left(b,\frac{\alpha}{2}a+\left(1-\frac{\alpha}{2}\right)b\right)\right]\triangle t,$$

for all $\alpha\in[0,1]$ such that $a+\alpha\frac{b-a}{2}, b-\alpha\frac{b-a}{2}\in\mathbb{T}$ and $t\in\left[a+\alpha\frac{b-a}{2}, b-\alpha\frac{b-a}{2}\right]\cap\mathbb{T}$, where $M=\sup_{a<t<b}\left|p^{\triangle\triangle}(t)\right|<\infty$, $N=\sup_{a<t<b}\left|r^{\triangle\triangle}(t)\right|<\infty$, and

$$W(x,t)=\begin{cases} t-\left(\left(1-\frac{\alpha}{2}\right)a+\frac{\alpha}{2}b\right), & a\leq t<x,\\ t-\left(\frac{\alpha}{2}a+\left(1-\frac{\alpha}{2}\right)b\right), & x\leq t\leq b.\end{cases}$$

Corollary 20. *In the case of $\alpha=0$ in Corollary 19, we have*

$$\Bigg|p(x)r(x)-\frac{1}{2(b-a)}\left(r(x)\int_a^b p(\sigma(t))\Delta t+p(x)\int_a^b r(\sigma(t))\Delta t\right)$$
$$-\frac{1}{2(b-a)}\left(r(x)\int_a^b p(\sigma(t))\Delta t+p(x)\int_a^b r(\sigma(t))\Delta t\right)$$
$$-\frac{1}{2(b-a)}\left(\frac{h_2(x,a)-h_2(x,b)}{b-a}\right)\left[r(x)\left(p(\sigma(b))-p(\sigma(a))\right)+p(x)\left(r(\sigma(b))-r(\sigma(a))\right)\right]\Bigg|$$
$$\leq\frac{1}{2(b-a)^2}\left(|r(x)|M+|p(x)|N\right)\int_a^b|S(x,t)|\left[h_2(t,a)+h_2(t,b)\right]\triangle t,$$

where $M=\sup_{a<t<b}\left|p^{\triangle\triangle}(t)\right|<\infty$, $N=\sup_{a<t<b}\left|r^{\triangle\triangle}(t)\right|<\infty$, and

$$S(x,t)=\begin{cases} t-a, & a\leq t<x,\\ t-b, & x\leq t\leq b.\end{cases}$$

Corollary 21. *In the case of $\mathbb{T}=\mathbb{R}$ in Corollary 19, we have*

$$\Bigg|(1-\alpha)^2p(x)r(x)+\frac{\alpha(1-\alpha)}{4}\left(x-\frac{a+b}{2}\right)\left[r(x)\left(p'(a)+p'(b)\right)+p(x)\left(r'(a)+q'(b)\right)\right]$$
$$-\frac{1-\alpha}{2(b-a)}\left(r(x)\int_a^b p(t)dt+p(x)\int_a^b r(t)dt\right)+\frac{\alpha(1-\alpha)}{2}\left(r(x)\frac{p(a)+p(b)}{2}+p(x)\frac{r(a)+r(b)}{2}\right)$$
$$-\frac{1-\alpha}{2(b-a)}\left(x-\frac{a+b}{2}\right)\left[r(x)\left(p(b)-p(a)\right)+p(x)\left(r(b)-r(a)\right)\right]\Bigg|$$
$$\leq\frac{1}{2(b-a)^2}\left(|r(x)|M+|p(x)|N\right)\left\{\frac{1}{96}\alpha^2(a-b)^4\left(13\alpha^2-16\alpha+12\right)\right.$$
$$+\frac{1}{192}\left[2(x-a)+\alpha(a-b)\right]^2\left[(b-a)^2(13\alpha^2-14\alpha+4)-2(b-a)(b+a-2x)\alpha+4(x-a)^2+8(b-x)^2\right]$$
$$\left.+\frac{1}{192}\left[2(b-x)+\alpha(a-b)\right]^2\left[(b-a)^2(13\alpha^2-14\alpha+4)+2(b-a)(b+a-2x)\alpha+8(x-a)^2+4(b-x)^2\right]\right\}$$

where $M=\sup_{a<t<b}|p''(t)|<\infty$, $N=\sup_{a<t<b}|r''(t)|<\infty$.

Remark 4. *In the case of* $\alpha = 0$ *in Corollary 21, we have*

$$\begin{aligned}&\Bigg|p(x)r(x) - \frac{1}{2(b-a)}\left(r(x)\int_a^b p(t)dt + p(x)\int_a^b r(t)dt\right)\\&-\frac{1-\alpha}{2(b-a)}\left(x-\frac{a+b}{2}\right)\left[r(x)\left(p(b)-p(a)\right)+p(x)\left(r(b)-r(a)\right)\right]\Bigg|\\&\le\frac{\left(|r(x)|M+|p(x)|N\right)}{4}\left\{\left(\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}+\frac{1}{4}\right)^2+\frac{1}{12}\right\}(b-a)^2,\end{aligned}$$

where $M = \sup_{a<t<b}|p''(t)| < \infty$, $N = \sup_{a<t<b}|r''(t)| < \infty$. *This is the result given in* [2, Theorem 5].

Corollary 22. *In the case of* $\mathbb{T} = \mathbb{Z}$ *in Corollary 19, we have*

$$\begin{aligned}&\Bigg|(1-\alpha)^2 p(x)r(x) + \frac{\alpha}{4}\left((1-\alpha)\left(x-\frac{a+b}{2}\right)-\frac{1}{2}\right)\left[r(x)\left(\triangle p(a)+\triangle p(b)\right)+p(x)\left(\triangle r(a)+\triangle r(b)\right)\right]\\&-\frac{1-\alpha}{2(b-a)}\left(r(x)\sum_{t=a}^{b-1}p(t+1)+p(x)\sum_{t=a}^{b-1}r(t+1)\right)+\frac{\alpha(1-\alpha)}{2}\left(r(x)\frac{p(a)+p(b)}{2}+p(x)\frac{r(a)+r(b)}{2}\right)\\&-\frac{1}{2(b-a)}\left((1-\alpha)\left(x-\frac{a+b}{2}\right)-\frac{1}{2}\right)\left[r(x)\left(p(b+1)-p(a+1)\right)+p(x)\left(r(b+1)-r(a+1)\right)\right]\Bigg|\\&\le\frac{1}{2(b-a)^2}\left(|r(x)|M+|p(x)|N\right)\Bigg[\left(a^4+b^4\right)\left(\frac{13}{48}\alpha^4-\frac{7}{12}\alpha^3+\frac{3}{4}\alpha^2-\frac{5}{12}\alpha+\frac{1}{6}\right)\\&+\left(a^3+b^3\right)\left(\left(-\frac{1}{2}\alpha^2+\frac{1}{2}\alpha-\frac{1}{2}\right)x+\frac{1}{4}\alpha^2-\frac{1}{4}\alpha+\frac{1}{3}\right)\\&+\left(a^2+b^2\right)\left(\left(\frac{1}{2}\alpha^2-\frac{1}{2}\alpha+1\right)x^2+\left(-\frac{1}{2}\alpha^2+\frac{1}{2}\alpha-\frac{3}{2}\right)x-\frac{1}{12}\alpha^2+\frac{1}{2}\alpha+\frac{1}{3}\right)\\&+(a+b)\left(-x^3+\frac{5}{2}x^2-\frac{3}{2}x+\frac{1}{6}\right)+\left(a^2b+b^2a\right)\Bigg(\left(\frac{1}{2}\alpha^2-\frac{1}{2}\alpha-\frac{1}{2}\right)x\\&-\frac{1}{4}\alpha^2+\frac{1}{4}\alpha+\frac{1}{2}\Bigg)+\left(a^3b+ab^3\right)\left(-\frac{13}{12}\alpha^4+\frac{7}{3}\alpha^3-\frac{5}{2}\alpha^2+\frac{7}{6}\alpha-\frac{1}{6}\right)\\&+ab\left(\left(-\alpha^2+\alpha+1\right)x^2+\left(\alpha^2-\alpha-2\right)x+\frac{1}{6}\alpha^2-\alpha+\frac{5}{6}\right)\\&+a^2b^2\left(\frac{13}{8}\alpha^4-\frac{7}{2}\alpha^3+\frac{7}{2}\alpha^2-\frac{3}{2}\alpha+\frac{1}{2}\right)+\frac{1}{2}x^4-\frac{5}{3}x^3+\frac{3}{2}x^2-\frac{1}{3}x\Bigg],\end{aligned}$$

where $M = \sup_{a<t<b}\left|\triangle^2 p(t)\right| < \infty$ *and* $N = \sup_{a<t<b}\left|\triangle^2 r(t)\right| < \infty$.

Remark 5. *In the case of* $\alpha = 0$ *in Corollary 22, we have*

$$\begin{aligned}&\Bigg|p(x)r(x)-\frac{1}{2(b-a)}\left(r(x)\sum_{t=a}^{b-1}p(t+1)+p(x)\sum_{t=a}^{b-1}r(t+1)\right)\\&-\frac{1}{2(b-a)}\left(x-\frac{a+b}{2}-\frac{1}{2}\right)\left[r(x)\left(p(b+1)-p(a+1)\right)+p(x)\left(r(b+1)-r(a+1)\right)\right]\Bigg|\\&\le\frac{1}{24(b-a)^2}\left(|r(x)|M+|p(x)|N\right)\left\{\left[(x-a)^2+a-x\right]\left[(b-a)^2+(x-a)^2+2(b-x)^2+5b+2a-7x+2\right]\right.\\&\left.+\left[(b-x)^2+b-x\right]\left[(b-a)^2+(b-x)^2+2(x-a)^2+5a+2b-7x+2\right]\right\},\end{aligned}$$

where $M = \sup_{a<t<b}\left|\triangle^2 p(t)\right| < \infty$ *and* $N = \sup_{a<t<b}\left|\triangle^2 r(t)\right| < \infty$.

Theorem 11. *Let* $0 \le k \le 1$, $g : [a, b] \to [0, \infty)$ *be rd continuous and positive and* $h : [a, b] \to \mathbb{R}$ *be differentiable such that* $h^{\triangle}(t) = g(t)$ *on* $[a, b]$. *Let* $a, b, t, x \in \mathbb{T}$, $a < b$ *and* $p, r : [a, b] \to \mathbb{R}$ *be twice differentiable. Then for all*

$x \in [a,b]$, we have

$$
\begin{aligned}
&\Bigg|(1-\alpha)^4 p(x)r(x) + \frac{\alpha^2}{4\left(\int_a^b g(t)\triangle t\right)^2}\left(\int_a^b W(x,t)\triangle t\right)^2 \left(p^{\triangle}(a)+p^{\triangle}(b)\right)\left(r^{\triangle}(a)+r^{\triangle}(b)\right)\\
&+\alpha(1-\alpha)^2\left(\int_a^b W(x,t)\triangle t\right)\left(p(x)\frac{r^{\triangle}(a)+r^{\triangle}(b)}{2\int_a^b g(t)\triangle t}+r(x)\frac{p^{\triangle}(a)+p^{\triangle}(b)}{2\int_a^b g(t)\triangle t}\right)\\
&-\frac{(1-\alpha)^2}{\left(\int_a^b g(t)\triangle t\right)^2}\left(\int_a^b W(x,t)\triangle t\right)\left(p(x)\int_a^b g(t)r^{\triangle}(\sigma(t))\Delta t + r(x)\int_a^b g(t)p^{\triangle}(\sigma(t))\Delta t\right)\\
&-\frac{(1-\alpha)^3}{\int_a^b g(t)\triangle t}\left(p(x)\int_a^b g(t)r(\sigma(t))\Delta t + r(x)\int_a^b g(t)p(\sigma(t))\Delta t\right)\\
&+\alpha(1-\alpha)^3\left(p(x)\frac{r(a)+r(b)}{2}+r(x)\frac{p(a)+p(b)}{2}\right)\\
&+\frac{\alpha^2(1-\alpha)}{4\int_a^b g(t)\triangle t}\left(\int_a^b W(x,t)\triangle t\right)\left(\left(p^{\triangle}(a)+p^{\triangle}(b)\right)(r(a)+r(b)) + \left(r^{\triangle}(a)+r^{\triangle}(b)\right)(p(a)+p(b))\right)\\
&-\frac{\alpha}{2\left(\int_a^b g(t)\triangle t\right)^3}\left(\int_a^b W(x,t)\triangle t\right)^2\left(\left(p^{\triangle}(a)+p^{\triangle}(b)\right)\left(\int_a^b g(t)r^{\triangle}(\sigma(t))\Delta t\right)\right.\\
&\left.+\left(r^{\triangle}(a)+r^{\triangle}(b)\right)\left(\int_a^b g(t)p^{\triangle}(\sigma(t))\Delta t\right)\right)\\
&-\frac{\alpha(1-\alpha)}{2\left(\int_a^b g(t)\triangle t\right)^2}\left(\int_a^b W(x,t)\triangle t\right)\left(\left(p^{\triangle}(a)+p^{\triangle}(b)\right)\int_a^b g(t)r(\sigma(t))\Delta t\right.\\
&\left.+\left(r^{\triangle}(a)+r^{\triangle}(b)\right)\int_a^b g(t)p(\sigma(t))\Delta t\right)\\
&-\frac{\alpha(1-\alpha)}{\left(\int_a^b g(t)\triangle t\right)^2}\left(\int_a^b W(x,t)\triangle t\right)\left(\frac{p(a)+p(b)}{2}\int_a^b g(t)r^{\triangle}(\sigma(t))\Delta t + \frac{r(a)+r(b)}{2}\int_a^b g(t)p^{\triangle}(\sigma(t))\Delta t\right)\\
&+\frac{\alpha^2(1-\alpha)^2}{4}(p(a)+p(b))(r(a)+r(b)) + \frac{(1-\alpha)^2}{\left(\int_a^b g(t)\triangle t\right)^2}\int_a^b g(t)p(\sigma(t))\Delta t\int_a^b g(t)r(\sigma(t))\Delta t\\
&-\frac{\alpha(1-\alpha)^2}{\int_a^b g(t)\triangle t}\left(\frac{p(a)+p(b)}{2}\int_a^b g(t)r(\sigma(t))\Delta t + \frac{r(a)+r(b)}{2}\int_a^b g(t)p(\sigma(t))\Delta t\right)\\
&+\frac{1}{\left(\int_a^b g(t)\triangle t\right)^4}\left(\int_a^b W(x,t)\triangle t\right)^2\left(\int_a^b g(t)p^{\triangle}(\sigma(t))\Delta t\right)\left(\int_a^b g(t)r^{\triangle}(\sigma(t))\Delta t\right)\\
&+\frac{1-\alpha}{\left(\int_a^b g(t)\triangle t\right)^3}\left(\int_a^b W(x,t)\triangle t\right)\left(\int_a^b g(t)p^{\triangle}(\sigma(t))\Delta t\int_a^b g(t)r(\sigma(t))\Delta t\right.\\
&\left.+\int_a^b g(t)r^{\triangle}(\sigma(t))\Delta t\int_a^b g(t)p(\sigma(t))\Delta t\right)\Bigg|
\end{aligned}
$$

$$
\le \frac{MN}{\left(\int_a^b g(t)\triangle t\right)^4}\left(\int_a^b\int_a^b |W(x,t)|\,|W(t,s)|\triangle s\triangle t\right)^2, \tag{3.14}
$$

where $M = \sup_{a<t<b}\left|p^{\triangle\triangle}(t)\right| < \infty$, $N = \sup_{a<t<b}\left|r^{\triangle\triangle}(t)\right| < \infty$, *and*

$$
W(x,t) = \begin{cases} h(t) - \left(\left(1-\frac{\alpha}{2}\right)h(a) + \frac{\alpha}{2}h(b)\right), & a \le t < x,\\ h(t) - \left(\frac{\alpha}{2}h(a) + \left(1-\frac{\alpha}{2}\right)h(b)\right), & x \le t \le b.\end{cases}
$$

Proof. Multiplying the left and right sides of the identities (3.12) and (3.13), we get

$$
\begin{aligned}
&(1-\alpha)^4 p(x)r(x) + \frac{\alpha^2}{4\left(\int_a^b g(t)\triangle t\right)^2}\left(\int_a^b W(x,t)\triangle t\right)^2 \left(p^{\triangle}(a)+p^{\triangle}(b)\right)\left(r^{\triangle}(a)+r^{\triangle}(b)\right)\\
&+\alpha(1-\alpha)^2\left(\int_a^b W(x,t)\triangle t\right)\left(p(x)\frac{r^{\triangle}(a)+r^{\triangle}(b)}{2\int_a^b g(t)\triangle t}+r(x)\frac{p^{\triangle}(a)+p^{\triangle}(b)}{2\int_a^b g(t)\triangle t}\right)\\
&-\frac{(1-\alpha)^2}{\left(\int_a^b g(t)\triangle t\right)^2}\left(\int_a^b W(x,t)\triangle t\right)\left(p(x)\int_a^b g(t)r^{\triangle}(\sigma(t))\Delta t + r(x)\int_a^b g(t)p^{\triangle}(\sigma(t))\Delta t\right)\\
&-\frac{(1-\alpha)^3}{\int_a^b g(t)\triangle t}\left(p(x)\int_a^b g(t)r(\sigma(t))\Delta t + r(x)\int_a^b g(t)p(\sigma(t))\Delta t\right)+\alpha(1-\alpha)^3\left(p(x)\frac{r(a)+r(b)}{2}+r(x)\frac{p(a)+p(b)}{2}\right)\\
&+\frac{\alpha^2(1-\alpha)}{4\int_a^b g(t)\triangle t}\left(\int_a^b W(x,t)\triangle t\right)\left(\left(p^{\triangle}(a)+p^{\triangle}(b)\right)(r(a)+r(b)) + \left(r^{\triangle}(a)+r^{\triangle}(b)\right)(p(a)+p(b))\right)\\
&-\frac{\alpha}{2\left(\int_a^b g(t)\triangle t\right)^3}\left(\int_a^b W(x,t)\triangle t\right)^2\left(\left(p^{\triangle}(a)+p^{\triangle}(b)\right)\left(\int_a^b g(t)r^{\triangle}(\sigma(t))\Delta t\right)\right.\\
&\left.+\left(r^{\triangle}(a)+r^{\triangle}(b)\right)\left(\int_a^b g(t)p^{\triangle}(\sigma(t))\Delta t\right)\right)\\
&-\frac{\alpha\,(1-\alpha)}{2\left(\int_a^b g(t)\triangle t\right)^2}\left(\int_a^b W(x,t)\triangle t\right)\left(\left(p^{\triangle}(a)+p^{\triangle}(b)\right)\int_a^b g(t)r(\sigma(t))\Delta t\right.\\
&\left.+\left(r^{\triangle}(a)+r^{\triangle}(b)\right)\int_a^b g(t)p(\sigma(t))\Delta t\right)\\
&-\frac{\alpha(1-\alpha)}{\left(\int_a^b g(t)\triangle t\right)^2}\left(\int_a^b W(x,t)\triangle t\right)\left(\frac{p(a)+p(b)}{2}\int_a^b g(t)r^{\triangle}(\sigma(t))\Delta t+\frac{r(a)+r(b)}{2}\int_a^b g(t)p^{\triangle}(\sigma(t))\Delta t\right)\\
&+\frac{\alpha^2(1-\alpha)^2}{4}(p(a)+p(b))(r(a)+r(b))+\frac{(1-\alpha)^2}{\left(\int_a^b g(t)\triangle t\right)^2}\int_a^b g(t)p(\sigma(t))\Delta t\int_a^b g(t)r(\sigma(t))\Delta t\\
&-\frac{\alpha(1-\alpha)^2}{\int_a^b g(t)\triangle t}\left(\frac{p(a)+p(b)}{2}\int_a^b g(t)r(\sigma(t))\Delta t+\frac{r(a)+r(b)}{2}\int_a^b g(t)p(\sigma(t))\Delta t\right)\\
&+\frac{1}{\left(\int_a^b g(t)\triangle t\right)^4}\left(\int_a^b W(x,t)\triangle t\right)^2\left(\int_a^b g(t)p^{\triangle}(\sigma(t))\Delta t\right)\left(\int_a^b g(t)r^{\triangle}(\sigma(t))\Delta t\right)\\
&+\frac{1-\alpha}{\left(\int_a^b g(t)\triangle t\right)^3}\left(\int_a^b W(x,t)\triangle t\right)\left(\int_a^b g(t)p^{\triangle}(\sigma(t))\Delta t\int_a^b g(t)r(\sigma(t))\Delta t\right.\\
&\left.+\int_a^b g(t)r^{\triangle}(\sigma(t))\Delta t\int_a^b g(t)p(\sigma(t))\Delta t\right)\\
=&\frac{1}{\left(\int_a^b g(t)\triangle t\right)^4}\left(\int_a^b\int_a^b W(x,t)W(t,s)p^{\triangle\triangle}(s)\triangle s\triangle t\right)\left(\int_a^b\int_a^b W(x,t)W(t,s)r^{\triangle\triangle}(s)\triangle s\triangle t\right).
\end{aligned}
$$

Using properties of the modulus, we get (3.14). □

Corollary 23. *In the case of* $\mathbb{T}=\mathbb{R}$ *in Theorem 11, we have*

$$\begin{aligned}
&\Bigg|(1-\alpha)^4 p(x)r(x) + \frac{\alpha^2}{4\left(\int_a^b g(t)dt\right)^2}\left(\int_a^b W(x,t)dt\right)^2 \left(p'(a)+p'(b)\right)\left(r'(a)+r'(b)\right) \\
&+\alpha(1-\alpha)^2\left(\int_a^b W(x,t)dt\right)\left(p(x)\frac{r'(a)+r'(b)}{2\int_a^b g(t)dt}+r(x)\frac{p'(a)+p'(b)}{2\int_a^b g(t)dt}\right) \\
&-\frac{(1-\alpha)^2}{\left(\int_a^b g(t)dt\right)^2}\left(\int_a^b W(x,t)dt\right)\left(p(x)\int_a^b g(t)r'(t)dt + r(x)\int_a^b g(t)p'(t)dt\right) \\
&-\frac{(1-\alpha)^3}{\int_a^b g(t)dt}\left(p(x)\int_a^b g(t)r(t)dt + r(x)\int_a^b g(t)p(t)dt\right) \\
&+\alpha(1-\alpha)^3\left(p(x)\frac{r(a)+r(b)}{2}+r(x)\frac{p(a)+p(b)}{2}\right) \\
&+\frac{\alpha^2(1-\alpha)}{4\int_a^b g(t)dt}\left(\int_a^b W(x,t)dt\right)\left(\left(p'(a)+p'(b)\right)\left(r(a)+r(b)\right) + \left(r'(a)+r'(b)\right)\left(p(a)+p(b)\right)\right) \\
&-\frac{\alpha}{2\left(\int_a^b g(t)dt\right)^3}\left(\int_a^b W(x,t)dt\right)^2\left(\left(p'(a)+p'(b)\right)\left(\int_a^b g(t)r'(t)dt\right)+\left(r'(a)+r'(b)\right)\left(\int_a^b g(t)p'(t)dt\right)\right) \\
&-\frac{\alpha\,(1-\alpha)}{2\left(\int_a^b g(t)dt\right)^2}\left(\int_a^b W(x,t)dt\right)\left(\left(p'(a)+p'(b)\right)\int_a^b g(t)r(t)dt + \left(r'(a)+r'(b)\right)\int_a^b g(t)p(t)dt\right) \\
&+\frac{\alpha^2(1-\alpha)^2}{4}\left(p(a)+p(b)\right)\left(r(a)+r(b)\right) \\
&-\frac{\alpha(1-\alpha)}{\left(\int_a^b g(t)dt\right)^2}\left(\int_a^b W(x,t)dt\right)\left(\frac{p(a)+p(b)}{2}\int_a^b g(t)r'(t)dt + \frac{r(a)+r(b)}{2}\int_a^b g(t)p'(t)dt\right) \\
&+\frac{(1-\alpha)^2}{\left(\int_a^b g(t)dt\right)^2}\int_a^b g(t)p(t)dt\int_a^b g(t)r(t)dt \\
&-\frac{\alpha(1-\alpha)^2}{\int_a^b g(t)dt}\left(\frac{p(a)+p(b)}{2}\int_a^b g(t)r(t)dt+\frac{r(a)+r(b)}{2}\int_a^b g(t)p(t)dt\right) \\
&+\frac{1}{\left(\int_a^b g(t)dt\right)^4}\left(\int_a^b W(x,t)dt\right)^2\left(\int_a^b g(t)p'(t)dt\right)\left(\int_a^b g(t)r'(t)dt\right) \\
&+\frac{1-\alpha}{\left(\int_a^b g(t)dt\right)^3}\left(\int_a^b W(x,t)dt\right)\left(\int_a^b g(t)p'(t)dt\int_a^b g(t)r(t)dt + \int_a^b g(t)r'(t)dt\int_a^b g(t)p(t)dt\right)\Bigg| \\
\leq&\frac{MN}{\left(\int_a^b g(t)dt\right)^4}\left(\int_a^b\int_a^b |W(x,t)|\,|W(t,s)|\,dsdt\right)^2,
\end{aligned}$$

where $g(t)=h'(t)$ *on* $[a,b]$ *and* $M=\sup_{a<t<b}|p''(t)|<\infty$, $N=\sup_{a<t<b}|r''(t)|<\infty$, *and*

$$W(x,t)=\begin{cases} h(t)-\left(\left(1-\frac{\alpha}{2}\right)h(a)+\frac{\alpha}{2}h(b)\right), & a\le t<x, \\ h(t)-\left(\frac{\alpha}{2}h(a)+\left(1-\frac{\alpha}{2}\right)h(b)\right), & x\le t\le b. \end{cases}$$

Corollary 24. *In the case of $\mathbb{T}=\mathbb{Z}$ in Theorem 11, we have*

$$
\begin{aligned}
&\Bigg|(1-\alpha)^4 p(x)r(x)+\frac{\alpha^2}{4\left(\sum\limits_{t=a}^{b-1}g(t)\right)^2}\left(\sum_{t=a}^{b-1}W(x,t)\right)^2(\triangle p(a)+\triangle p(b))(\triangle r(a)+\triangle r(b))\\
&+\alpha(1-\alpha)^2\left(\sum_{t=a}^{b-1}W(x,t)\right)\left(p(x)\frac{\triangle r(a)+\triangle r(b)}{2\sum\limits_{t=a}^{b-1}g(t)}+r(x)\frac{\triangle p(a)+\triangle p(b)}{2\sum\limits_{t=a}^{b-1}g(t)}\right)\\
&-\frac{(1-\alpha)^2}{\left(\sum\limits_{t=a}^{b-1}g(t)\right)^2}\left(\sum_{t=a}^{b-1}W(x,t)\right)\left(p(x)\sum_{t=a}^{b-1}g(t)\triangle r(t+1)+r(x)\sum_{t=a}^{b-1}g(t)\triangle p(t+1)\right)\\
&-\frac{(1-\alpha)^3}{\sum\limits_{t=a}^{b-1}g(t)}\left(p(x)\sum_{t=a}^{b-1}g(t)r(t+1)+r(x)\sum_{t=a}^{b-1}g(t)p(t+1)\right)+\alpha(1-\alpha)^3\left(p(x)\frac{r(a)+r(b)}{2}+r(x)\frac{p(a)+p(b)}{2}\right)\\
&+\frac{\alpha^2(1-\alpha)}{4\sum\limits_{t=a}^{b-1}g(t)}\left(\sum_{t=a}^{b-1}W(x,t)\right)((\triangle p(a)+\triangle p(b))(r(a)+r(b))+(\triangle r(a)+\triangle r(b))(p(a)+p(b)))\\
&-\frac{\alpha}{2\left(\sum\limits_{t=a}^{b-1}g(t)\right)^3}\left(\sum_{t=a}^{b-1}W(x,t)\right)^2\left((\triangle p(a)+\triangle p(b))\left(\sum_{t=a}^{b-1}g(t)\triangle r(t+1)\right)+(\triangle r(a)+\triangle r(b))\left(\sum_{t=a}^{b-1}g(t)\triangle p(t+1)\right)\right)\\
&-\frac{\alpha\,(1-\alpha)}{2\left(\sum\limits_{t=a}^{b-1}g(t)\right)^2}\left(\sum_{t=a}^{b-1}W(x,t)\right)\left((\triangle p(a)+\triangle p(b))\sum_{t=a}^{b-1}g(t)r(t+1)+(\triangle r(a)+\triangle r(b))\sum_{t=a}^{b-1}g(t)p(t+1)\right)\\
&-\frac{\alpha(1-\alpha)}{\left(\sum\limits_{t=a}^{b-1}g(t)\right)^2}\left(\sum_{t=a}^{b-1}W(x,t)\right)\left(\frac{p(a)+p(b)}{2}\sum_{t=a}^{b-1}g(t)\triangle r(t+1)+\frac{r(a)+r(b)}{2}\sum_{t=a}^{b-1}g(t)\triangle p(t+1)\right)\\
&+\frac{\alpha^2(1-\alpha)^2}{4}(p(a)+p(b))(r(a)+r(b))+\frac{(1-\alpha)^2}{\left(\int_a^b g(t)dt\right)^2}\sum_{t=a}^{b-1}g(t)p(t+1)\sum_{t=a}^{b-1}g(t)r(t+1)\\
&-\frac{\alpha(1-\alpha)^2}{\sum\limits_{t=a}^{b-1}g(t)}\left(\frac{p(a)+p(b)}{2}\sum_{t=a}^{b-1}g(t)r(t+1)+\frac{r(a)+r(b)}{2}\sum_{t=a}^{b-1}g(t)p(t+1)\right)\\
&+\frac{1}{\left(\sum\limits_{t=a}^{b-1}g(t)\right)^4}\left(\sum_{t=a}^{b-1}W(x,t)\right)^2\left(\sum_{t=a}^{b-1}g(t)\triangle p(t+1)\right)\left(\sum_{t=a}^{b-1}g(t)\triangle r(t+1)\right)\\
&+\frac{1-\alpha}{\left(\sum\limits_{t=a}^{b-1}g(t)\right)^3}\left(\sum_{t=a}^{b-1}W(x,t)\right)\left(\sum_{t=a}^{b-1}g(t)\triangle p(t+1)\sum_{t=a}^{b-1}g(t)r(t+1)+\sum_{t=a}^{b-1}g(t)\triangle r(t+1)\sum_{t=a}^{b-1}g(t)p(t+1)\right)\Bigg|\\
\le&\frac{MN}{\left(\sum\limits_{t=a}^{b-1}g(t)\right)^4}\left(\sum_{t=a}^{b-1}\sum_{s=a}^{b-1}|W(x,t)|\,|W(t,s)|\right)^2,
\end{aligned}
$$

where $g(t)=h(t+1)-h(t)$ *on* $[a,b-1]$ *and* $M=\sup\limits_{a<t<b}\left|\triangle^2 p(t)\right|<\infty$, $N=\sup\limits_{a<t<b}\left|\triangle^2 r(t)\right|<\infty$, *and*

$$
W(x,t)=\begin{cases} h(t)-(\left(1-\frac{\alpha}{2}\right)h(a)+\frac{\alpha}{2}h(b)), & a\le t<x-1,\\ h(t)-(\frac{\alpha}{2}h(a)+\left(1-\frac{\alpha}{2}\right)h(b)), & x\le t\le b-1.\end{cases}
$$

Corollary 25. *In the case of* $h(t)=t$ *in Theorem 11, we have*

$$\Bigg|(1-\alpha)^4 p(x)r(x)+\frac{\alpha^2}{4(b-a)^2}\left[-h_2\left(a,\left(1-\frac{\alpha}{2}\right)a+\frac{\alpha}{2}b\right)+h_2\left(x,\left(1-\frac{\alpha}{2}\right)a+\frac{\alpha}{2}b\right)\right.$$
$$\left.-h_2\left(x,\frac{\alpha}{2}a+\left(1-\frac{\alpha}{2}\right)b\right)+h_2\left(b,\frac{\alpha}{2}a+\left(1-\frac{\alpha}{2}\right)b\right)\right]^2\left(p^{\triangle}(a)+p^{\triangle}(b)\right)\left(r^{\triangle}(a)+r^{\triangle}(b)\right)$$
$$+\alpha(1-\alpha)^2\left[-h_2\left(a,\left(1-\frac{\alpha}{2}\right)a+\frac{\alpha}{2}b\right)+h_2\left(x,\left(1-\frac{\alpha}{2}\right)a+\frac{\alpha}{2}b\right)\right.$$
$$\left.-h_2\left(x,\frac{\alpha}{2}a+\left(1-\frac{\alpha}{2}\right)b\right)+h_2\left(b,\frac{\alpha}{2}a+\left(1-\frac{\alpha}{2}\right)b\right)\right]\left(p(x)\frac{r^{\triangle}(a)+r^{\triangle}(b)}{2(b-a)}+r(x)\frac{p^{\triangle}(a)+p^{\triangle}(b)}{2(b-a)}\right)$$
$$-\frac{(1-\alpha)^2}{(b-a)^2}\left[-h_2\left(a,\left(1-\frac{\alpha}{2}\right)a+\frac{\alpha}{2}b\right)+h_2\left(x,\left(1-\frac{\alpha}{2}\right)a+\frac{\alpha}{2}b\right)\right.$$
$$\left.-h_2\left(x,\frac{\alpha}{2}a+\left(1-\frac{\alpha}{2}\right)b\right)+h_2\left(b,\frac{\alpha}{2}a+\left(1-\frac{\alpha}{2}\right)b\right)\right]\left(p(x)\left(r(\sigma(b))-r(\sigma(a)\right)+r(x)\left(p(\sigma(b))-p(\sigma(a))\right)\right)$$
$$-\frac{(1-\alpha)^3}{b-a}\left(p(x)\int_a^b r(\sigma(t))\Delta t+r(x)\int_a^b p(\sigma(t))\Delta t\right)+\alpha(1-\alpha)^3\left(p(x)\frac{r(a)+r(b)}{2}+r(x)\frac{p(a)+p(b)}{2}\right)$$
$$+\frac{\alpha^2(1-\alpha)}{4(b-a)}\left[-h_2\left(a,\left(1-\frac{\alpha}{2}\right)a+\frac{\alpha}{2}b\right)+h_2\left(x,\left(1-\frac{\alpha}{2}\right)a+\frac{\alpha}{2}b\right)\right.$$
$$\left.-h_2\left(x,\frac{\alpha}{2}a+\left(1-\frac{\alpha}{2}\right)b\right)+h_2\left(b,\frac{\alpha}{2}a+\left(1-\frac{\alpha}{2}\right)b\right)\right]\left(\left(p^{\triangle}(a)+p^{\triangle}(b)\right)\left(r(a)+r(b)\right)+\left(r^{\triangle}(a)+r^{\triangle}(b)\right)\left(p(a)+p(b)\right)\right)$$
$$-\frac{\alpha}{2(b-a)^3}\left(\left(p^{\triangle}(a)+p^{\triangle}(b)\right)\left(r(\sigma(b))-r(\sigma(a))\right)\right.$$
$$\left.+\left(r^{\triangle}(a)+r^{\triangle}(b)\right)\left(p(\sigma(b))-p(\sigma(a))\right)\right)\left[-h_2\left(a,\left(1-\frac{\alpha}{2}\right)a+\frac{\alpha}{2}b\right)+h_2\left(x,\left(1-\frac{\alpha}{2}\right)a+\frac{\alpha}{2}b\right)\right.$$
$$\left.-h_2\left(x,\frac{\alpha}{2}a+\left(1-\frac{\alpha}{2}\right)b\right)+h_2\left(b,\frac{\alpha}{2}a+\left(1-\frac{\alpha}{2}\right)b\right)\right]^2$$
$$-\frac{\alpha(1-\alpha)}{2(b-a)^2}\left[-h_2\left(a,\left(1-\frac{\alpha}{2}\right)a+\frac{\alpha}{2}b\right)+h_2\left(x,\left(1-\frac{\alpha}{2}\right)a+\frac{\alpha}{2}b\right)-h_2\left(x,\frac{\alpha}{2}a+\left(1-\frac{\alpha}{2}\right)b\right)+h_2\left(b,\frac{\alpha}{2}a+\left(1-\frac{\alpha}{2}\right)b\right)\right]$$
$$\left(\left(p^{\triangle}(a)+p^{\triangle}(b)\right)\int_a^b r(\sigma(t))\Delta t+\left(r^{\triangle}(a)+r^{\triangle}(b)\right)\int_a^b p(\sigma(t))\Delta t\right)+\frac{\alpha^2(1-\alpha)^2}{4}\left(p(a)+p(b)\right)\left(r(a)+r(b)\right)$$
$$-\frac{\alpha(1-\alpha)}{(b-a)^2}\left[-h_2\left(a,\left(1-\frac{\alpha}{2}\right)a+\frac{\alpha}{2}b\right)+h_2\left(x,\left(1-\frac{\alpha}{2}\right)a+\frac{\alpha}{2}b\right)-h_2\left(x,\frac{\alpha}{2}a+\left(1-\frac{\alpha}{2}\right)b\right)+h_2\left(b,\frac{\alpha}{2}a+\left(1-\frac{\alpha}{2}\right)b\right)\right]$$
$$\left(\frac{p(a)+p(b)}{2}\left(r(\sigma(b))-r(\sigma(a))\right)+\frac{r(a)+r(b)}{2}\left(p(\sigma(b))-p(\sigma(a))\right)\right)+\frac{(1-\alpha)^2}{(b-a)^2}\int_a^b p(\sigma(t))\Delta t\int_a^b r(\sigma(t))\Delta t$$
$$-\frac{\alpha(1-\alpha)^2}{b-a}\left(\frac{p(a)+p(b)}{2}\int_a^b r(\sigma(t))\Delta t+\frac{r(a)+r(b)}{2}\int_a^b p(\sigma(t))\Delta t\right)$$
$$+\frac{1}{(b-a)^4}\left[-h_2\left(a,\left(1-\frac{\alpha}{2}\right)a+\frac{\alpha}{2}b\right)+h_2\left(x,\left(1-\frac{\alpha}{2}\right)a+\frac{\alpha}{2}b\right)\right.$$
$$\left.-h_2\left(x,\frac{\alpha}{2}a+\left(1-\frac{\alpha}{2}\right)b\right)+h_2\left(b,\frac{\alpha}{2}a+\left(1-\frac{\alpha}{2}\right)b\right)\right]^2\left(\left(p(\sigma(b))-p(\sigma(a))\right)\right)\left(\left(r(\sigma(b))-r(\sigma(a))\right)\right)$$
$$+\frac{1-\alpha}{(b-a)^3}\left[-h_2\left(a,\left(1-\frac{\alpha}{2}\right)a+\frac{\alpha}{2}b\right)+h_2\left(x,\left(1-\frac{\alpha}{2}\right)a+\frac{\alpha}{2}b\right)-h_2\left(x,\frac{\alpha}{2}a+\left(1-\frac{\alpha}{2}\right)b\right)\right.$$
$$\left.+h_2\left(b,\frac{\alpha}{2}a+\left(1-\frac{\alpha}{2}\right)b\right)\right]\left(\left(p(\sigma(b))-p(\sigma(a))\right)\int_a^b r(\sigma(t))\Delta t+\left(r(\sigma(b))-r(\sigma(a))\right)\int_a^b p(\sigma(t))\Delta t\right)\Bigg|$$
$$\leq\frac{MN}{(b-a)^4}\left(\int_a^b |W(x,t)|\left[h_2\left(a,\left(1-\frac{\alpha}{2}\right)a+\frac{\alpha}{2}b\right)+h_2\left(t,\left(1-\frac{\alpha}{2}\right)a+\frac{\alpha}{2}b\right)\right.\right.$$
$$\left.\left.+h_2\left(t,\frac{\alpha}{2}a+\left(1-\frac{\alpha}{2}\right)b\right)+h_2\left(b,\frac{\alpha}{2}a+\left(1-\frac{\alpha}{2}\right)b\right)\right]\triangle t\right)^2$$

for all $\alpha\in[0,1]$ *such that* $a+\alpha\frac{b-a}{2}, b-\alpha\frac{b-a}{2}\in\mathbb{T}$ *and* $t\in\left[a+\alpha\frac{b-a}{2}, b-\alpha\frac{b-a}{2}\right]\cap\mathbb{T}$, *where* $M=\sup_{a<t<b}\left|p^{\triangle\triangle}(t)\right|<\infty$, $N=\sup_{a<t<b}\left|r^{\triangle\triangle}(t)\right|<\infty$, *and*

$$W(x,t)=\begin{cases} t-\left(\left(1-\frac{\alpha}{2}\right)a+\frac{\alpha}{2}b\right), & a\le t<x,\\ t-\left(\frac{\alpha}{2}a+\left(1-\frac{\alpha}{2}\right)b\right), & x\le t\le b.\end{cases}$$

Corollary 26. *In the case of* $\alpha = 0$ *in Corollary 25, we have*

$$\Bigg|p(x)r(x) - \frac{1}{b-a}\left[p(x)\left(r(\sigma(b)) - r(\sigma(a))\right) + r(x)\left(p(\sigma(b)) - p(\sigma(a))\right)\right]\frac{h_2(x,a) - h_2(x,b)}{b-a}$$
$$- \frac{1}{b-a}\left(p(x)\int_a^b r(\sigma(t))\Delta t + r(x)\int_a^b p(\sigma(t))\Delta t\right) + \frac{1}{(b-a)^2}\int_a^b p(\sigma(t))\Delta t\int_a^b r(\sigma(t))\Delta t$$
$$+ \frac{1}{(b-a)^2}\left[\left(\frac{h_2(x,a) - h_2(x,b)}{b-a}\right)^2\left(\left(p(\sigma(b)) - p(\sigma(a))\right)\right)\left(\left(r(\sigma(b)) - r(\sigma(a))\right)\right)\right]$$
$$+ \frac{1}{(b-a)^2}\left[\frac{h_2(x,a) - h_2(x,b)}{b-a}\left(p(\sigma(b)) - p(\sigma(a))\right)\int_a^b r(\sigma(t))\Delta t + \left(r(\sigma(b)) - r(\sigma(a))\right)\int_a^b p(\sigma(t))\Delta t\right]\Bigg|$$
$$\leq \frac{MN}{(b-a)^4}\left(\int_a^b\int_a^b |S(x,t)|\left(h_2(t,a) + h_2(t,b)\right)\triangle t\right)^2,$$

where $M = \sup_{a<t<b}\left|p^{\triangle\triangle}(t)\right| < \infty$, $N = \sup_{a<t<b}\left|r^{\triangle\triangle}(t)\right| < \infty$, *and*

$$S(x,t) = \begin{cases} t-a, & a \leq t < x, \\ t-b, & x \leq t \leq b. \end{cases}$$

Corollary 27. *In the case of* $\mathbb{T} = \mathbb{R}$ *in Corollary 25, we have*

$$\Bigg|(1-\alpha)^4 p(x)r(x) + \frac{(1-\alpha)^2\alpha^2}{4}\left(x - \frac{a+b}{2}\right)^2\left(p'(a) + p'(b)\right)\left(r'(a) + r'(b)\right)$$
$$+ \alpha(1-\alpha)^3\left(x - \frac{a+b}{2}\right)\left(p(x)\frac{r'(a) + r'(b)}{2} + r(x)\frac{p'(a) + p'(b)}{2}\right)$$
$$- \frac{(1-\alpha)^3}{b-a}\left(x - \frac{a+b}{2}\right)\left(p(x)\left(r(b) - r(a)\right) + r(x)\left(p(b) - p(a)\right)\right)$$
$$- \frac{(1-\alpha)^3}{b-a}\left(p(x)\int_a^b r(t)dt + r(x)\int_a^b p(t)dt\right) + \alpha(1-\alpha)^3\left(p(x)\frac{r(a) + r(b)}{2} + r(x)\frac{p(a) + p(b)}{2}\right)$$
$$+ \frac{\alpha^2(1-\alpha)^2}{4}\left(x - \frac{a+b}{2}\right)\left(\left(p'(a) + p'(b)\right)\left(r(a) + r(b)\right) + \left(r'(a) + r'(b)\right)\left(p(a) + p(b)\right)\right)$$
$$- \frac{\alpha(1-\alpha)^2}{2(b-a)}\left(\left(p'(a) + p'(b)\right)\left(r(b) - r(a)\right) + \left(r'(a) + r'(b)\right)\left(p(b) - p(a)\right)\right)\left(x - \frac{a+b}{2}\right)^2$$
$$- \frac{\alpha(1-\alpha)^2}{2(b-a)}\left(x - \frac{a+b}{2}\right)\left(\left(p'(a) + p'(b)\right)\int_a^b r(t)dt + \left(r'(a) + r'(b)\right)\int_a^b p(t)dt\right)$$
$$- \frac{\alpha(1-\alpha)^2}{b-a}\left(x - \frac{a+b}{2}\right)\left(\frac{p(a) + p(b)}{2}\left(r(b) - r(a)\right) + \frac{r(a) + r(b)}{2}\left(p(b) - p(a)\right)\right)$$
$$+ \frac{\alpha^2(1-\alpha)^2}{4}\left(p(a) + p(b)\right)\left(r(a) + r(b)\right) + \frac{(1-\alpha)^2}{(b-a)^2}\left(\int_a^b p(t)dt\right)\left(\int_a^b r(t)dt\right)$$
$$- \frac{\alpha(1-\alpha)^2}{b-a}\left(\frac{p(a) + p(b)}{2}\int_a^b r(t)dt + \frac{r(a) + r(b)}{2}\int_a^b p(t)dt\right)$$
$$+ \frac{(1-\alpha)^2}{(b-a)^2}\left(x - \frac{a+b}{2}\right)^2\left(p(b) - p(a)\right)\left(r(b) - r(a)\right)$$
$$+ \frac{(1-\alpha)^2}{(b-a)^2}\left(x - \frac{a+b}{2}\right)\left(\left(p(b) - p(a)\right)\int_a^b r(t)dt + \left(r(b) - r(a)\right)\int_a^b p(t)dt\right)\Bigg|$$
$$\leq \frac{MN}{(b-a)^4}\Bigg\{\frac{1}{96}\alpha^2(a-b)^4\left(13\alpha^2 - 16\alpha + 12\right) + \frac{1}{192}\left[2(x-a) + \alpha(a-b)\right]^2\big[(b-a)^2(13\alpha^2 - 14\alpha + 4)$$
$$-2(b-a)(b+a-2x)\alpha + 4(x-a)^2 + 8(b-x)^2\big] + \frac{1}{192}\left[2(b-x) + \alpha(a-b)\right]^2\big[(b-a)^2(13\alpha^2 - 14\alpha + 4)$$
$$+2(b-a)(b+a-2x)\alpha + 8(x-a)^2 + 4(b-x)^2\big]\Bigg\}^2,$$

where $M = \sup_{a<t<b}|p''(t)| < \infty$, $N = \sup_{a<t<b}|r''(t)| < \infty$.

Corollary 28. *In the case of* $\mathbb{T}=\mathbb{Z}$ *in Corollary 25, we have*

$$\begin{aligned}
&\Bigg|(1-\alpha)^4 p(x)r(x)+\frac{\alpha^2}{4}\left((1-\alpha)\left(x-\frac{a+b}{2}\right)-\frac{1}{2}\right)^2(\triangle p(a)+\triangle p(b))(\triangle r(a)+\triangle r(b))\\
&+\alpha(1-\alpha)^2\left((1-\alpha)\left(x-\frac{a+b}{2}\right)-\frac{1}{2}\right)\left(p(x)\frac{\triangle r(a)+\triangle r(b)}{2}+r(x)\frac{\triangle p(a)+\triangle p(b)}{2}\right)\\
&-\frac{(1-\alpha)^2}{b-a}\left((1-\alpha)\left(x-\frac{a+b}{2}\right)-\frac{1}{2}\right)(p(x)(r(b+1)-r(a+1))\\
&+r(x)(p(b+1)-p(a+1)))-\frac{(1-\alpha)^3}{b-a}\left(p(x)\sum_{t=a}^{b-1}r(t+1)+r(x)\sum_{t=a}^{b-1}p(t+1)\right)\\
&+\alpha(1-\alpha)^3\left(p(x)\frac{r(a)+r(b)}{2}+r(x)\frac{p(a)+p(b)}{2}\right)\\
&+\frac{\alpha^2(1-\alpha)}{4}\left((1-\alpha)\left(x-\frac{a+b}{2}\right)-\frac{1}{2}\right)((\triangle p(a)+\triangle p(b))(r(a)+r(b))\\
&+(\triangle r(a)+\triangle r(b))(p(a)+p(b)))-\frac{\alpha}{2(b-a)}((\triangle p(a)+\triangle p(b))(r(b+1)-r(a+1))\\
&+(\triangle r(a)+\triangle r(b))(p(b+1)-p(a+1)))\left((1-\alpha)\left(x-\frac{a+b}{2}\right)-\frac{1}{2}\right)^2\\
&-\frac{\alpha(1-\alpha)}{2(b-a)}\left((1-\alpha)\left(x-\frac{a+b}{2}\right)-\frac{1}{2}\right)\left((\triangle p(a)+\triangle p(b))\sum_{t=a}^{b-1}r(t+1)\right.\\
&\left.+(\triangle r(a)+\triangle r(b))\sum_{t=a}^{b-1}p(t+1)\right)+\frac{\alpha^2(1-\alpha)^2}{4}(p(a)+p(b))(r(a)+r(b))\\
&-\frac{\alpha(1-\alpha)}{b-a}\left((1-\alpha)\left(x-\frac{a+b}{2}\right)-\frac{1}{2}\right)\left(\frac{p(a)+p(b)}{2}(r(b+1)-r(a+1))\right.\\
&\left.+\frac{r(a)+r(b)}{2}(p(b+1)-p(a+1))\right)+\frac{(1-\alpha)^2}{(b-a)^2}\left(\sum_{t=a}^{b-1}p(t+1)\right)\left(\sum_{t=a}^{b-1}r(t+1)\right)\\
&-\frac{\alpha(1-\alpha)^2}{b-a}\left(\frac{p(a)+p(b)}{2}\sum_{t=a}^{b-1}r(t+1)+\frac{r(a)+r(b)}{2}\sum_{t=a}^{b-1}p(t+1)\right)\\
&+\frac{1}{(b-a)^2}\left((1-\alpha)\left(x-\frac{a+b}{2}\right)-\frac{1}{2}\right)^2((p(b+1)-p(a+1)))((r(b+1)-r(a+1)))\\
&+\frac{1-\alpha}{(b-a)^2}\left((1-\alpha)\left(x-\frac{a+b}{2}\right)-\frac{1}{2}\right)\left((p(b+1)-p(a+1))\sum_{t=a}^{b-1}r(t+1)+(r(b+1)-r(a+1))\sum_{t=a}^{b-1}p(t+1)\right)\Bigg|\\
\le&\frac{MN}{(b-a)^4}\Bigg\{\left(a^4+b^4\right)\left(\frac{13}{48}\alpha^4-\frac{7}{12}\alpha^3+\frac{3}{4}\alpha^2-\frac{5}{12}\alpha+\frac{1}{6}\right)+\left(a^3+b^3\right)\left[\left(-\frac{1}{2}\alpha^2+\frac{1}{2}\alpha-\frac{1}{2}\right)x+\frac{1}{4}\alpha^2-\frac{1}{4}\alpha+\frac{1}{3}\right]\\
&+\left(a^2+b^2\right)\left[\left(\frac{1}{2}\alpha^2-\frac{1}{2}\alpha+1\right)x^2+\left(-\frac{1}{2}\alpha^2+\frac{1}{2}\alpha-\frac{3}{2}\right)x-\frac{1}{12}\alpha^2+\frac{1}{2}\alpha+\frac{1}{3}\right]\\
&+(a+b)\left(-x^3+\frac{5}{2}x^2-\frac{3}{2}x+\frac{1}{6}\right)+\left(a^2b+b^2a\right)\left[\left(\frac{1}{2}\alpha^2-\frac{1}{2}\alpha-\frac{1}{2}\right)x-\frac{1}{4}\alpha^2+\frac{1}{4}\alpha+\frac{1}{2}\right]\\
&+\left(a^3b+ab^3\right)\left(-\frac{13}{12}\alpha^4+\frac{7}{3}\alpha^3-\frac{5}{2}\alpha^2+\frac{7}{6}\alpha-\frac{1}{6}\right)\\
&+ab\left[\left(-\alpha^2+\alpha+1\right)x^2+\left(\alpha^2-\alpha-2\right)x+\frac{1}{6}\alpha^2-\alpha+\frac{5}{6}\right]\\
&+a^2b^2\left(\frac{13}{8}\alpha^4-\frac{7}{2}\alpha^3+\frac{7}{2}\alpha^2-\frac{3}{2}\alpha+\frac{1}{2}\right)+\frac{1}{2}x^4-\frac{5}{3}x^3+\frac{3}{2}x^2-\frac{1}{3}x\Bigg\}^2,
\end{aligned}$$

where $M=\sup_{a<t<b}\left|\triangle^2 p(t)\right|<\infty$ *and* $N=\sup_{a<t<b}\left|\triangle^2 r(t)\right|<\infty$.

Remark 6. *In the case of* $\alpha = 0$ *in Corollary 27, we have*

$$\Bigg| p(x)r(x) - \frac{1}{b-a}\left(x - \frac{a+b}{2}\right)\left(p(x)\left(r(b)-r(a)\right) + r(x)\left(p(b)-p(a)\right)\right)$$
$$- \frac{1}{b-a}\left(p(x)\int_a^b r(t)dt + r(x)\int_a^b p(t)dt\right) + \frac{1}{(b-a)^2}\int_a^b p(t)dt\int_a^b r(t)dt$$
$$+ \frac{1}{(b-a)^2}\left(x - \frac{a+b}{2}\right)^2\left(p(b)-p(a)\right)\left(r(b)-r(a)\right)$$
$$+ \frac{1}{(b-a)^2}\left(x - \frac{a+b}{2}\right)\left(\left(p(b)-p(a)\right)\int_a^b r(t)dt + \left(r(b)-r(a)\right)\int_a^b p(t)dt\right)\Bigg|$$
$$\le \frac{MN}{4}\left\{\left(\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2} + \frac{1}{4}\right)^2 + \frac{1}{12}\right\}^2 (b-a)^4,$$

where $M = \sup_{a<t<b} |p''(t)| < \infty$, $N = \sup_{a<t<b} |r''(t)| < \infty$. *This is the result given in* [2, Theorem 4].

Remark 7. *In the case of* $\alpha = 0$ *in Corollary 28, we have*

$$\Bigg| p(x)r(x) - \frac{1}{b-a}\left(x - \frac{a+b}{2} - \frac{1}{2}\right)\left(p(x)\left(r(b+1)-r(a+1)\right) + r(x)\left(p(b+1)-p(a+1)\right)\right)$$
$$- \frac{1}{b-a}\left(p(x)\sum_{t=a}^{b-1} r(t+1) + r(x)\sum_{t=a}^{b-1} p(t+1)\right) + \frac{1}{(b-a)^2}\left(\sum_{t=a}^{b-1} p(t+1)\right)\left(\sum_{t=a}^{b-1} r(t+1)\right)$$
$$+ \frac{1}{(b-a)^2}\left(x - \frac{a+b}{2} - \frac{1}{2}\right)^2\left(p(b+1)-p(a+1)\right)\left(r(b+1)-r(a+1)\right)$$
$$+ \frac{1}{(b-a)^2}\left(x - \frac{a+b}{2} - \frac{1}{2}\right)\left(\left(p(b+1)-p(a+1)\right)\sum_{t=a}^{b-1} r(t+1) + \left(r(b+1)-r(a+1)\right)\sum_{t=a}^{b-1} p(t+1)\right)\Bigg|$$
$$\le \frac{MN}{(b-a)^4}\left\{\frac{1}{12}\left[(x-a)^2 + a - x\right]\left[(b-a)^2 + (x-a)^2 + 2(b-x)^2 + 5b + 2a - 7x + 2\right]\right.$$
$$\left. + \frac{1}{12}\left[(b-x)^2 + b - x\right]\left[(b-a)^2 + (b-x)^2 + 2(x-a)^2 + 5a + 2b - 7x + 2\right]\right\}^2,$$

where $M = \sup_{a<t<b} \left|\triangle^2 p(t)\right| < \infty$ *and* $N = \sup_{a<t<b} \left|\triangle^2 r(t)\right| < \infty$.

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(A. Tuna) Department of Mathematics, Faculty of Science and Arts, University of Niğde, Merkez 51240, Niğde, Turkey

E-mail address: atuna@nigde.edu.tr

(Y. Jiang) College of Mathematics and Physics, Nanjing University of Information Science and Technology, Nanjing 210044, China

E-mail address: jiang@nuist.edu.cn

(W. J. Liu) College of Mathematics and Physics, Nanjing University of Information Science and Technology, Nanjing 210044, China

E-mail address: wjliu@nuist.edu.cn