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ON SOME HADAMARD-TYPE INEQUALITIES FOR PRODUCT OF TWO $h-$CONVEX FUNCTIONS ON THE CO-ORDINATES

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Abstract. In this paper Hadamard-type inequalities for product of $h-$convex functions on the co-ordinates on the rectangle from the plane are established. Obtained results generalize the corresponding to some well-known results given before now.

1. INTRODUCTION

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function and $a, b \in I$ with $a < b$. Then the following double inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

is known as Hadamard's inequality for convex mapping. For particular choice of the function f in (1.1) yields some classical inequalities of means.

Definition 1. *(See [12]) A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to Godunova-Levin function or f is said to belong to the class $Q(I)$ if f is non-negative and for all $x, y \in I$ and for $\alpha \in (0,1)$ we have the inequality:*

$$f(\alpha x + (1-\alpha)y) \leq \frac{f(x)}{\alpha} + \frac{f(y)}{1-\alpha}.$$

The class $Q(I)$ was firstly described in [12] by Godunova-Levin. Some further properties of it can be found in [11], [16] and [17]. Among others, it is noted that non-negative monotone and non-negative convex functions belongs to this class of functions. In [6], Breckner introduced $s-$convex functions as a generalization of convex functions. In [7], he proved the important fact that the set-valued map is $s-$convex only if associated support function is $s-$convex. A number of properties and connections with $s-$convexity in the first sense are discussed in paper [13]. It is clear that $s-$convexity is merely convexity for $s = 1$.

Definition 2. *(See [6]) Let $s \in (0,1]$ be fixed real number. A function $f : [0,\infty) \to [0,\infty)$ is said to be $s-$convex in the second sense, or that f belongs to the class K_s^2, if*

$$f(\alpha x + (1-\alpha)y) \leq \alpha^s f(x) + (1-\alpha)^s f(y)$$

for all $x, y \in [0,\infty)$ and $\alpha \in [0,1]$.

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Definition 3. *(See [11]) A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be $P-$function or that f is said to belong to the class $P(I)$ if f is non-negative and for all $x, y \in I$ and $\alpha \in [0,1]$, if*

$$f(\alpha x + (1-\alpha)y) \leq f(x) + f(y).$$

In [9], Dragomir and Fitzpatrick proved the following variant of Hadamard's inequality which holds for $s-$convex function in the second sense:

Theorem 1. *Suppose that $f : [0,\infty) \to [0,\infty)$ is an $s-$convex function in the second sense, where $s \in (0,1)$ and let $a, b \in [0,\infty)$, $a < b$. If $f \in L_1([a,b])$ then the following inequalities hold:*

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{s+1} \tag{1.2}$$

The constant $k = \frac{1}{s+1}$ is the best possible in the second inequality in (1.2).

In [9], Dragomir and Fitzpatrick also proved the following Hadamard-type inequality which holds for $s-$convex functions in the first sense:

Theorem 2. *Suppose that $f : [0,\infty) \to [0,\infty)$ is an $s-$convex function in the first sense, where $s \in (0,1)$ and let $a, b \in [0,\infty)$. If $f \in L_1([a,b])$ then the following inequalities hold:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+sf(b)}{s+1} \tag{1.3}$$

The above inequalities are sharp.

A modification for convex functions which is also known as co-ordinated convex(concave) functions was introduced by Dragomir in [8] as following:

Let us now consider a bidimensional interval $\Delta =: [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$. A mapping $f : \Delta \to \mathbb{R}$ is said to be convex on Δ if the following inequality:

$$f(\alpha x + (1-\alpha)z, \alpha y + (1-\alpha)w) \leq \alpha f(x,y) + (1-\alpha)f(z,w)$$

holds, for all $(x,y),(z,w) \in \Delta$ and $\alpha \in [0,1]$. If the inequality reversed then f is said to be concave on Δ. A function $f : \Delta \to \mathbb{R}$ is said to be convex on the co-ordinates on Δ if the partial mappings $f_y : [a,b] \to \mathbb{R}$, $f_y(u) = f(u,y)$ and $f_x : [c,d] \to \mathbb{R}$, $f_x(v) = f(x,v)$ are convex where defined for all $x \in [a,b]$, $y \in [c,d]$.

A formal definition for co-ordinated convex functions may be stated as follow [see [24]]:

Definition 4. *A function $f : \Delta \to \mathbb{R}$ is said to be convex on the co-ordinates on Δ if the following inequality:*

$$\begin{aligned} &f(tx + (1-t)y, su + (1-s)w) \\ &\leq tsf(x,u) + t(1-s)f(x,w) + s(1-t)f(y,u) + (1-t)(1-s)f(y,w) \end{aligned}$$

holds for all $t, s \in [0,1]$ and $(x,u),(x,w),(y,u),(y,w) \in \Delta$.

Clearly, every convex mapping $f : \Delta \to \mathbb{R}$ is convex on the co-ordinates. Furthermore, there exists co-ordinated convex function which is not convex, (see [10]). In [8], Dragomir established the following inequalities of Hadamard's type for convex functions on the co-ordinates on a rectangle from the plane $\mathbb{R}^2$.

Theorem 3. *Suppose $f : \Delta = [a,b] \times [c,d] \subseteq [0,\infty) \to \mathbb{R}$ is convex function on the co-ordinates on Δ. Then one has the inequalities:*

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \le \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx \le \frac{f(a,c)+f(b,c)+f(a,d)+f(b,d)}{4} \tag{1.4}$$

In [1], Alomari and Darus proved the following inequalities of Hadamard-type as above for $s-$convex functions in the second sense on the co-ordinates on a rectangle from the plane $\mathbb{R}^2$.

Theorem 4. *Suppose $f : \Delta = [a,b] \times [c,d] \subseteq [0,\infty) \to \mathbb{R}$ is $s-$convex function (in the second sense) on the co-ordinates on Δ. Then one has the inequalities:*

$$4^{s-1} f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \le \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx \le \frac{f(a,c)+f(b,c)+f(a,d)+f(b,d)}{(s+1)^2} \tag{1.5}$$

Also in [4] (see also [5]), Alomari and Darus established the following inequalities of Hadamard-type similar to (1.5) for $s-$convex functions in the first sense on the co-ordinates on a rectangle from the plane $\mathbb{R}^2$.

Theorem 5. *Suppose $f : \Delta = [a,b] \times [c,d] \subseteq [0,\infty) \to \mathbb{R}$ is $s-$convex function on the co-ordinates on Δ in the first sense. Then one has the inequalities:*

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \le \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx \le \frac{f(a,c)+sf(b,c)+sf(a,d)+s^2f(b,d)}{(s+1)^2} \tag{1.6}$$

The above inequalities are sharp.

For refinements, counterparts, generalizations and new Hadamard-type inequalities see the papers [1, 2, 3, 4, 5, 8, 9, 10, 11, 13, 22, 23, 24, 25].

In [18], Pachpatte established two Hadamard-type inequalities for product of convex functions. An analogous results for $s-$convex functions is due to Kırmacı *et al.* [14].

Theorem 6. *Let $f, g : [a,b] \subseteq \mathbb{R} \to [0,\infty)$ be convex functions on $[a,b]$, $a < b$. Then*

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \le \frac{1}{3}M(a,b) + \frac{1}{6}N(a,b) \tag{1.7}$$

and

$$2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)g(x)dx + \frac{1}{6}M(a,b) + \frac{1}{3}N(a,b) \tag{1.8}$$

where $M(a,b) = f(a)g(a) + f(b)g(b)$ and $N(a,b) = f(a)g(b) + f(b)g(a)$.

Theorem 7. *Let $f, g : [a,b] \subseteq \mathbb{R} \to \mathbb{R}$ $a, b \in [0,\infty)$, $a < b$, be functions such that g and fg are in $L_1([a,b])$. If f is convex and non-negative on $[a,b]$ and if g is s-convex on $[a,b]$ for some $s \in (0,1)$. Then*

$$\begin{aligned} & 2^s f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \\ & \leq \frac{1}{b-a}\int_a^b f(x)g(x)dx + \frac{1}{(s+1)(s+2)}M(a,b) + \frac{1}{s+2}N(a,b) \end{aligned} \tag{1.9}$$

and

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \leq \frac{1}{s+2}M(a,b) + \frac{1}{(s+1)(s+2)}N(a,b) \tag{1.10}$$

where $M(a,b) = f(a)g(a) + f(b)g(b)$ and $N(a,b) = f(a)g(b) + f(b)g(a)$.

The class of $h-$convex functions was introduced by S. Varosaneć in [20] (see [20] for further properties of $h-$convex functions).

Definition 5. *Let $h : J \subseteq \mathbb{R} \to \mathbb{R}$, where $(0,1) \subseteq J$, be a positive function. A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be $h-$convex or that f is said to belong to the class $SX(h,I)$, if f is non-negative and for all $x, y \in I$ and $\alpha \in (0,1)$, we have*

$$f(\alpha x + (1-\alpha)y) \leq h(\alpha)f(x) + h(1-\alpha)f(y)$$

if the inequality is reversed then f is said to be $h-$concave and we say that f belongs to the class $SV(h,I)$.

Remark 1. *Obviously, if $h(\alpha) = \alpha$, then all the non-negative convex functions belong to the class $SX(h,I)$ and all non-negative concave functions belong to the class $SV(h,I)$. Also note that if $h(\alpha) = \frac{1}{\alpha}$, then $SX(h,I) = Q(I)$; if $h(\alpha) = 1$, then $SX(h,I) \supseteq P(I)$; and if $h(\alpha) = \alpha^s$, where $s \in (0,1)$, then $SX(h,I) \supseteq K_s^2$.*

In [19], Sarıkaya *et al.* established the following inequalities of Hadamard's type for product of $h-$convex functions.

Theorem 8. *Let $f \in SX(h_1,I)$, $g \in SX(h_2,I)$, $a, b \in I$, $a < b$, be functions such that $fg \in L_1([a,b])$ and $h_1h_2 \in L_1([0,1])$, then*

$$\begin{aligned} & \frac{1}{b-a}\int_a^b f(x)g(x)dx \\ & \leq M(a,b)\int_0^1 h_1(t)h_2(t)dt + N(a,b)\int_0^1 h_1(t)h_2(1-t)dt \end{aligned} \tag{1.11}$$

where $M(a,b) = f(a)g(a) + f(b)g(b)$ and $N(a,b) = f(a)g(b) + f(b)g(a)$.

Theorem 9. *Let $f \in SX(h_1,I)$, $g \in SX(h_2,I)$, $a, b \in I$, $a < b$, be functions such that $fg \in L_1([a,b])$ and $h_1h_2 \in L_1([0,1])$, then*

$$\begin{aligned} & \frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})} f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)g(x)dx \\ & \leq M(a,b)\int_0^1 h_1(t)h_2(1-t)dt + N(a,b)\int_0^1 h_1(t)h_2(t)dt \end{aligned} \tag{1.12}$$

where $M(a,b) = f(a)g(a) + f(b)g(b)$ and $N(a,b) = f(a)g(b) + f(b)g(a)$.

In [21], Sarıkaya *et al.* established the following inequality of Hadamard's type which involving $h-$convex functions;

Theorem 10. *Let $f \in SX(h,I)$, $a,b \in I$ with $a<b$, $f \in L_1([a,b])$ and $g:[a,b] \to \mathbb{R}$ is non-negative, integrable and symmetric about $\frac{a+b}{2}$. Then*

(1.13)
$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \le \frac{f(a)+f(b)}{2}\int_a^b \left(h\left(\frac{b-x}{b-a}\right)+h\left(\frac{x-a}{b-a}\right)\right)g(x)dx.$$

In [15], authors proved the following results for product of two convex functions on the co-ordinates on rectagle from the plane $\mathbb{R}^2$.

Theorem 11. *Let $f,g:\Delta=[a,b]\times[c,d]\subseteq\mathbb{R}^2\to[0,\infty)$ be convex functions on the co-ordinates on Δ with $a<b, c<d$. Then*

(1.14)
$$\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)dydx \le \frac{1}{9}L(a,b,c,d)+\frac{1}{18}M(a,b,c,d)+\frac{1}{36}N(a,b,c,d)$$

where

$$\begin{aligned}
L(a,b,c,d) &= f(a,c)g(a,c)+f(b,c)g(b,c)+f(a,d)g(a,d)+f(b,d)g(b,d)\\
M(a,b,c,d) &= f(a,c)g(a,d)+f(a,d)g(a,c)+f(b,c)g(b,d)+f(b,d)g(b,c)\\
&\quad + f(b,c)g(a,c)+f(b,d)g(a,d)+f(a,c)g(b,c)+f(a,d)g(b,d)\\
N(a,b,c,d) &= f(b,c)g(a,d)+f(b,d)g(a,c)+f(a,c)g(b,d)+f(a,d)g(b,c)
\end{aligned}$$

Theorem 12. *Let $f,g:\Delta=[a,b]\times[c,d]\subseteq\mathbb{R}^2\to[0,\infty)$ be convex functions on the co-ordinates on Δ with $a<b, c<d$. Then*

(1.15)
$$\begin{aligned}
4f\left(\frac{a+b}{2},\frac{c+d}{2}\right)g\left(\frac{a+b}{2},\frac{c+d}{2}\right) &\le \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx\\
&\quad+\frac{5}{36}L(a,b,c,d)+\frac{7}{36}M(a,b,c,d)+\frac{2}{9}N(a,b,c,d)
\end{aligned}$$

where $L(a,b,c,d), M(a,b,c,d)$, and $N(a,b,c,d)$ as in Theorem 10.

Similar to definition of co-ordinated convex functions Latif and Alomari gave the notion of $h-$convexity of a function f on a rectangle from the plane $\mathbb{R}^2$ and $h-$convexity on the co-ordinates on a rectangle from the plane $\mathbb{R}^2$ in [24], as follows:

Definition 6. *(See [24]) Let us consider a bidimensional interval $\Delta =: [a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b$ and $c<d$. Let $h:J\subseteq\mathbb{R}\to\mathbb{R}$, where $(0,1)\subseteq J$, be a positive function. A mapping $f:\Delta =: [a,b]\times[c,d]\to\mathbb{R}$ is said to be $h-$convex on Δ, if f is non-negative and if the following inequality:*

$$f(\alpha x+(1-\alpha)z,\alpha y+(1-\alpha)w)\le h(\alpha)f(x,y)+h(1-\alpha)f(z,w)$$

holds, for all $(x,y),(z,w)\in\Delta$ and $\alpha\in(0,1)$. Let us denote this class of functions by $SX(h,\Delta)$. The function f is said to be $h-$concave if the inequality reversed. We denote this class of functions by $SV(h,\Delta)$.

A function $f:\Delta\to\mathbb{R}$ is said to be $h-$convex on the co-ordinates on Δ if the partial mappings $f_y:[a,b]\to\mathbb{R}$, $f_y(u)=f(u,y)$ and $f_x:[c,d]\to\mathbb{R}$, $f_x(v)=f(x,v)$ are $h-$convex where defined for all $x\in[a,b]$, $y\in[c,d]$. A formal definition of $h-$convex functions may also be stated as follows:

Definition 7. *(See [24]) A function $f:\Delta\to\mathbb{R}$ is said to be $h-$convex on the co-ordinates on Δ, if the following inequality:*

$$f(tx+(1-t)y, su+(1-s)w) \le h(t)h(s)f(x,u)+h(t)h(1-s)f(x,w) + h(s)h(1-t)f(y,u)+h(1-t)h(1-s)f(y,w)$$

holds for all $t,s\in[0,1]$ and $(x,u),(x,w),(y,u),(y,w)\in\Delta$.

Lemma 1. *(See [24]) Every $h-$convex mapping $f:\Delta\to\mathbb{R}$ is $h-$convex on the co-ordinates, but the converse is not generally true.*

The converse of this Lemma is not true in general. To prove this fact we consider the same function as it was taken in [10], with $h(\alpha)=\alpha$.

The main purpose of the present paper is to establish new Hadamard-type inequalities like those given above in the Theorem 10-11, but now for product of two $h-$convex functions on the co-ordinates on rectangle from the plane $\mathbb{R}^2$.

2. MAIN RESULTS

In this section we establish some Hadamard's type inequalities for product of two $h-$convex functions on the co-ordinates on rectangle from the plane. In the sequel of the paper h_1 and h_2 are positive functions defined on J, where $(0,1)\subseteq J\subseteq\mathbb{R}$ and f and g are non-negative functions defined on $\Delta=[a,b]\times[c,d]\subseteq\mathbb{R}^2$.

Theorem 13. *Let $f,g:\Delta=[a,b]\times[c,d]\subseteq\mathbb{R}^2\to\mathbb{R}$, where $a<b$ and $c<d$, be functions such that $fg\in L^2(\Delta)$, $h_1h_2\in L_1[0,1])$. If f is h_1-convex on the co-ordinates on Δ and if g is h_2-convex on the co-ordinates on Δ, then*

$$\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)dydx \le p^2L(a,b,c,d)+pqM(a,b,c,d)+q^2N(a,b,c,d) \tag{2.1}$$

where $L(a,b,c,d)$, $M(a,b,c,d)$, $N(a,b,c,d)$ as in Theorem 10 and $p=\int_0^1 h_1(t)h_2(t)dt$ and $q=\int_0^1 h_1(t)h_2(1-t)dt$.

Proof. Since $f,g:\Delta=[a,b]\times[c,d]\subseteq\mathbb{R}^2\to\mathbb{R}$ be functions such that $fg\in L^2(\Delta)$ and f is h_1-convex on the co-ordinates on Δ and g is h_2-convex on the co-ordinates on Δ, therefore the partial mappings

$$f_y:[a,b]\to\mathbb{R},\ f_y(x)=f(x,y)$$
$$g_y:[a,b]\to\mathbb{R}, g_y(x)=g(x,y)$$

and

$$f_x:[c,d]\to\mathbb{R},\ f_x(y)=f(x,y)$$
$$g_x:[c,d]\to\mathbb{R}, g_x(y)=g(x,y)$$

are h_1-,h_2-convex on $[a,b]$ and $[c,d]$, respectively, for all $x\in[a,b]$, $y\in[c,d]$. Now by applying (1.11) to $f_x(y)g_x(y)$ on $[c,d]$ we get

$$\frac{1}{d-c}\int_c^d f_x(y)g_x(y)dy\le p\left[f_x(c)g_x(c)+f_x(d)g_x(d)\right]+q\left[f_x(c)g_x(d)+f_x(d)g_x(c)\right].$$

That is

$$\frac{1}{d-c}\int_c^d f(x,y)g(x,y)dy\le p\left[f(x,c)g(x,c)+f(x,d)g(x,d)\right]+q\left[f(x,c)g(x,d)+f(x,d)g(x,c)\right].$$

Integrating over $[a,b]$ and dividing both sides by $b-a$, we have

$$\begin{aligned}(2.2)\qquad &\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)dydx\\ &\le p\left[\frac{1}{b-a}\int_a^b f(x,c)g(x,c)dx+\frac{1}{b-a}\int_a^b f(x,d)g(x,d)dx\right]\\ &+q\left[\frac{1}{b-a}\int_a^b f(x,c)g(x,d)dx+\frac{1}{b-a}\int_a^b f(x,d)g(x,c)dx\right].\end{aligned}$$

Now by applying (1.11) to each integral on R.H.S of (2.2) again, we get

$$\frac{1}{b-a}\int_a^b f(x,c)g(x,c)dx\le p\left[f(a,c)g(a,c)+f(b,c)g(b,c)\right]+q\left[f(a,c)g(b,c)+f(b,c)g(a,c)\right]$$

$$\frac{1}{b-a}\int_a^b f(x,d)g(x,d)dx\le p\left[f(a,d)g(a,d)+f(b,d)g(b,d)\right]+q\left[f(a,d)g(b,d)+f(b,d)g(a,d)\right]$$

$$\frac{1}{b-a}\int_a^b f(x,c)g(x,d)dx\le p\left[f(a,c)g(a,d)+f(b,c)g(b,d)\right]+q\left[f(a,c)g(b,d)+f(b,c)g(a,d)\right]$$

$$\frac{1}{b-a}\int_a^b f(x,d)g(x,c)dx\le p\left[f(a,d)g(a,c)+f(b,d)g(b,c)\right]+q\left[f(a,d)g(b,c)+f(b,d)g(a,c)\right].$$

On substitution of these inequalities in (2.2) yields

$$\begin{aligned}&\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)dydx\\ &\le p^2\left[f(a,c)g(a,c)+f(b,c)g(b,c)\right]+pq\left[f(a,c)g(b,c)+f(b,c)g(a,c)\right]\\ &+p^2\left[f(a,d)g(a,d)+f(b,d)g(b,d)\right]+pq\left[f(a,d)g(b,d)+f(b,d)g(a,d)\right]\\ &+pq\left[f(a,c)g(a,d)+f(b,c)g(b,d)\right]+q^2\left[f(a,c)g(b,d)+f(b,c)g(a,d)\right]\\ &+pq\left[f(a,d)g(a,c)+f(b,d)g(b,c)\right]+q^2\left[f(a,d)g(b,c)+f(b,d)g(a,c)\right]\\ &=p^2L(a,b,c,d)+pqM(a,b,c,d)+q^2N(a,b,c,d).\end{aligned}$$

This completes the proof. □

Remark 2. *If we take $h_1(t)=h_2(t)=t$, then inequality (2.1) reduces to the inequality (1.14).*

Theorem 14. *Let $f,g:\Delta=[a,b]\times[c,d]\subseteq\mathbb{R}^2\to\mathbb{R}$, where $a<b$ and $c<d$, be functions such that $fg\in L^2(\Delta)$, $h_1h_2\in L_1[0,1])$. If f is h_1-convex on the co-ordinates on Δ and if g is h_2-convex on the co-ordinates on Δ, then*

$$\begin{aligned}(2.3)\qquad &\frac{1}{4h_1^2(\frac{1}{2})h_2^2(\frac{1}{2})}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)g\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\ &\le\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx\\ &+(q^2+2pq)L(a,b,c,d)+(p^2+pq+q^2)M(a,b,c,d)\\ &+(p^2+2pq)N(a,b,c,d)\end{aligned}$$

where $L(a,b,c,d), M(a,b,c,d)$, and $N(a,b,c,d)$ as in Theorem 10 and $p=\int_0^1 h_1(t)h_2(t)dt$ and $q=\int_0^1 h_1(t)h_2(1-t)dt$.

Proof. Now applying (1.12) to $\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)g\left(\frac{a+b}{2},\frac{c+d}{2}\right)$, we get

$$\begin{aligned}(2.4)\qquad &\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)g\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\ &\leq \frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)g\left(x,\frac{c+d}{2}\right)dx\\ &+q\left[f\left(a,\frac{c+d}{2}\right)g\left(a,\frac{c+d}{2}\right)+f\left(b,\frac{c+d}{2}\right)g\left(b,\frac{c+d}{2}\right)\right]\\ &+p\left[f\left(a,\frac{c+d}{2}\right)g\left(b,\frac{c+d}{2}\right)+f\left(b,\frac{c+d}{2}\right)g\left(a,\frac{c+d}{2}\right)\right]\end{aligned}$$

and

$$\begin{aligned}(2.5)\qquad &\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)g\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\ &\leq \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)g\left(\frac{a+b}{2},y\right)dy\\ &+q\left[f\left(\frac{a+b}{2},c\right)g\left(\frac{a+b}{2},c\right)+f\left(\frac{a+b}{2},d\right)g\left(\frac{a+b}{2},d\right)\right]\\ &+p\left[f\left(\frac{a+b}{2},c\right)g\left(\frac{a+b}{2},d\right)+f\left(\frac{a+b}{2},d\right)g\left(\frac{a+b}{2},c\right)\right].\end{aligned}$$

Adding (2.4) and (2.5) and multiplying both sides by $\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}$, we get

(2.6)

$$\begin{aligned}&\frac{1}{2\left[h_1(\frac{1}{2})h_2(\frac{1}{2})\right]^2}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)g\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\ &\leq \frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)g\left(x,\frac{c+d}{2}\right)dx\\ &+\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)g\left(\frac{a+b}{2},y\right)dy\\ &+q\left[\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}f\left(a,\frac{c+d}{2}\right)g\left(a,\frac{c+d}{2}\right)+\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}f\left(b,\frac{c+d}{2}\right)g\left(b,\frac{c+d}{2}\right)\right]\\ &+p\left[\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}f\left(a,\frac{c+d}{2}\right)g\left(b,\frac{c+d}{2}\right)+\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}f\left(b,\frac{c+d}{2}\right)g\left(a,\frac{c+d}{2}\right)\right]\\ &+q\left[\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}f\left(\frac{a+b}{2},c\right)g\left(\frac{a+b}{2},c\right)+\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}f\left(\frac{a+b}{2},d\right)g\left(\frac{a+b}{2},d\right)\right]\\ &+p\left[\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}f\left(\frac{a+b}{2},c\right)g\left(\frac{a+b}{2},d\right)+\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}f\left(\frac{a+b}{2},d\right)g\left(\frac{a+b}{2},c\right)\right].\end{aligned}$$

Applying (1.12) to each term within the brackets, we have

$$\begin{aligned}\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}f\left(a,\frac{c+d}{2}\right)g\left(a,\frac{c+d}{2}\right) &\leq \frac{1}{d-c}\int_c^d f\left(a,y\right)g\left(a,y\right)dy\\ &+q\left[f\left(a,c\right)g\left(a,c\right)+f\left(a,d\right)g\left(a,d\right)\right]\\ &+p\left[f\left(a,c\right)g\left(a,d\right)+f\left(a,d\right)g\left(a,c\right)\right]\end{aligned}$$

$$\begin{aligned}\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}f\left(b,\frac{c+d}{2}\right)g\left(b,\frac{c+d}{2}\right) &\leq \frac{1}{d-c}\int_c^d f\left(b,y\right)g\left(b,y\right)dy\\ &+q\left[f\left(b,c\right)g\left(b,c\right)+f\left(b,d\right)g\left(b,d\right)\right]\\ &+p\left[f\left(b,c\right)g\left(b,d\right)+f\left(b,d\right)g\left(b,c\right)\right]\end{aligned}$$

$$\begin{aligned}\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}f\left(a,\frac{c+d}{2}\right)g\left(b,\frac{c+d}{2}\right) &\leq \frac{1}{d-c}\int_c^d f\left(a,y\right)g\left(b,y\right)dy\\ &+q\left[f\left(a,c\right)g\left(b,c\right)+f\left(a,d\right)g\left(b,d\right)\right]\\ &+p\left[f\left(a,c\right)g\left(b,d\right)+f\left(a,d\right)g\left(b,c\right)\right]\end{aligned}$$

$$\begin{aligned}\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}f\left(b,\frac{c+d}{2}\right)g\left(a,\frac{c+d}{2}\right) &\leq \frac{1}{d-c}\int_c^d f\left(b,y\right)g\left(a,y\right)dy\\ &+q\left[f\left(b,c\right)g\left(a,c\right)+f\left(b,d\right)g\left(a,d\right)\right]\\ &+p\left[f\left(b,c\right)g\left(a,d\right)+f\left(b,d\right)g\left(a,c\right)\right]\end{aligned}$$

$$\begin{aligned}\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}f\left(\frac{a+b}{2},c\right)g\left(\frac{a+b}{2},c\right) &\leq \frac{1}{b-a}\int_a^b f(x,c)g(x,c)dx\\ &+q\left[f\left(a,c\right)g\left(a,c\right)+f\left(b,c\right)g\left(b,c\right)\right]\\ &+p\left[f\left(a,c\right)g\left(b,c\right)+f\left(b,c\right)g\left(a,c\right)\right]\end{aligned}$$

$$\begin{aligned}\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}f\left(\frac{a+b}{2},d\right)g\left(\frac{a+b}{2},d\right) &\leq \frac{1}{b-a}\int_a^b f(x,d)g(x,d)dx\\ &+q\left[f\left(a,d\right)g\left(a,d\right)+f\left(b,d\right)g\left(b,d\right)\right]\\ &+p\left[f\left(a,d\right)g\left(b,d\right)+f\left(b,d\right)g\left(a,d\right)\right]\end{aligned}$$

$$\begin{aligned}\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}f\left(\frac{a+b}{2},c\right)g\left(\frac{a+b}{2},d\right) &\leq \frac{1}{b-a}\int_a^b f(x,c)g(x,d)dx\\ &+q\left[f\left(a,c\right)g\left(a,d\right)+f\left(b,c\right)g\left(b,d\right)\right]\\ &+p\left[f\left(a,c\right)g\left(b,d\right)+f\left(b,c\right)g\left(a,d\right)\right]\end{aligned}$$

$$\begin{aligned}\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}f\left(\frac{a+b}{2},d\right)g\left(\frac{a+b}{2},c\right) &\leq \frac{1}{b-a}\int_a^b f(x,d)g(x,c)dx\\ &+q\left[f\left(a,d\right)g\left(a,c\right)+f\left(b,d\right)g\left(b,c\right)\right]\\ &+p\left[f\left(a,d\right)g\left(b,c\right)+f\left(b,d\right)g\left(a,c\right)\right].\end{aligned}$$

Substituting these inequalities in (2.6) and simplifying we have;

$$\begin{aligned}
&\frac{1}{2\left[h_1(\frac{1}{2})h_2(\frac{1}{2})\right]^2} f\left(\frac{a+b}{2},\frac{c+d}{2}\right) g\left(\frac{a+b}{2},\frac{c+d}{2}\right) \\
&\leq \frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right) g\left(x,\frac{c+d}{2}\right)dx \\
&+\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right) g\left(\frac{a+b}{2},y\right)dy \\
&+q\frac{1}{d-c}\int_c^d f(a,y)\,g(a,y)\,dy+q\frac{1}{d-c}\int_c^d f(b,y)\,g(b,y)\,dy \\
&+p\frac{1}{d-c}\int_c^d f(a,y)\,g(b,y)\,dy+p\frac{1}{d-c}\int_c^d f(b,y)\,g(a,y)\,dy \\
&+q\frac{1}{b-a}\int_a^b f(x,c)g(x,c)dx+q\frac{1}{b-a}\int_a^b f(x,d)g(x,d)dx \\
&+p\frac{1}{b-a}\int_a^b f(x,c)g(x,d)dx+p\frac{1}{b-a}\int_a^b f(x,d)g(x,c)dx \\
&+2q^2L(a,b,c,d)+2pqM(a,b,c,d)+p^2N(a,b,c,d)
\end{aligned} \tag{2.7}$$

Now by applying (1.12) to $\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}f\left(\frac{a+b}{2},y\right)g\left(\frac{a+b}{2},y\right)$, integrating over $[c,d]$ and dividing both sides by $d-c$, we get

$$\begin{aligned}
&\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right) g\left(\frac{a+b}{2},y\right)dy \\
&\leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,g(x,y)\,dxdy \\
&+q\frac{1}{d-c}\int_c^d f(a,y)\,g(a,y)\,dy+q\frac{1}{d-c}\int_c^d f(b,y)\,g(b,y)\,dy \\
&+p\frac{1}{d-c}\int_c^d f(a,y)\,g(b,y)\,dy+p\frac{1}{d-c}\int_c^d f(b,y)\,g(a,y)\,dy
\end{aligned} \tag{2.8}$$

Now again by applying (1.12) to $\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}f\left(x,\frac{c+d}{2}\right)g\left(x,\frac{c+d}{2}\right)$, integrating over $[a,b]$ and dividing both sides by $b-a$, we get

$$\begin{aligned}
&\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right) g\left(x,\frac{c+d}{2}\right)dx \\
&\leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,g(x,y)\,dydx \\
&+q\frac{1}{b-a}\int_a^b f(x,c)g(x,c)dx+q\frac{1}{b-a}\int_a^b f(x,d)g(x,d)dx \\
&+p\frac{1}{b-a}\int_a^b f(x,c)g(x,d)dx+p\frac{1}{b-a}\int_a^b f(x,d)g(x,c)dx.
\end{aligned} \tag{2.9}$$

Adding (2.8) and (2.9), we have

$$
\begin{aligned}
&\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}\frac{1}{b-a}\int_c^d f\left(x,\frac{c+d}{2}\right)g\left(x,\frac{c+d}{2}\right)dx \\
&+\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)g\left(\frac{a+b}{2},y\right)dy \\
&\leq \frac{2}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,g(x,y)\,dydx \\
&+q\frac{1}{d-c}\int_c^d f(a,y)\,g(a,y)\,dy+q\frac{1}{d-c}\int_c^d f(b,y)\,g(b,y)\,dy \\
&+p\frac{1}{d-c}\int_c^d f(a,y)\,g(b,y)\,dy+p\frac{1}{d-c}\int_c^d f(b,y)\,g(a,y)\,dy \\
&+q\frac{1}{b-a}\int_a^b f(x,c)g(x,c)dx+q\frac{1}{b-a}\int_a^b f(x,d)g(x,d)dx \\
&+p\frac{1}{b-a}\int_a^b f(x,c)g(x,d)dx+p\frac{1}{b-a}\int_a^b f(x,d)g(x,c)dx
\end{aligned}
\tag{2.10}
$$

Therefore from (2.7) and (2.10), we get

$$
\begin{aligned}
&\frac{1}{2\left[h_1(\frac{1}{2})h_2(\frac{1}{2})\right]^2}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)g\left(\frac{a+b}{2},\frac{c+d}{2}\right) \\
&\leq \frac{2}{(b-a)(d-c)}\int_a^b f(x,y)\,g(x,y)\,dxdy \\
&+2q\frac{1}{d-c}\int_c^d f(a,y)\,g(a,y)\,dy+2q\frac{1}{d-c}\int_c^d f(b,y)\,g(b,y)\,dy \\
&+2p\frac{1}{d-c}\int_c^d f(a,y)\,g(b,y)\,dy+2p\frac{1}{d-c}\int_c^d f(b,y)\,g(a,y)\,dy \\
&+2q\frac{1}{b-a}\int_a^b f(x,c)g(x,c)dx+2q\frac{1}{b-a}\int_a^b f(x,d)g(x,d)dx \\
&+2p\frac{1}{b-a}\int_a^b f(x,c)g(x,d)dx+2p\frac{1}{b-a}\int_a^b f(x,d)g(x,c)dx \\
&+2q^2L(a,b,c,d)+2pqM(a,b,c,d)+2p^2N(a,b,c,d)
\end{aligned}
$$

By using (1.11) to each of the above integral and simplifying, we get

$$
\begin{aligned}
&\frac{1}{2\left[h_1(\frac{1}{2})h_2(\frac{1}{2})\right]^2}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)g\left(\frac{a+b}{2},\frac{c+d}{2}\right) \\
&\leq \frac{2}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,g(x,y)\,dydx \\
&+(2q^2+4pq)L(a,b,c,d)+(2p^2+2pq+2q^2)M(a,b,c,d) \\
&+(2p^2+4pq)N(a,b,c,d)
\end{aligned}
$$

Dividing both sides by 2;

$$\begin{aligned}
&\frac{1}{\left[2h_1(\frac{1}{2})h_2(\frac{1}{2})\right]^2} f\left(\frac{a+b}{2},\frac{c+d}{2}\right) g\left(\frac{a+b}{2},\frac{c+d}{2}\right) \\
&\leq \frac{1}{(b-a)(d-c)}\int_a^b f(x,y)\,g(x,y)\,dxdy \\
&+(q^2+2pq)L(a,b,c,d)+(p^2+pq+q^2)M(a,b,c,d) \\
&+(p^2+2pq)N(a,b,c,d)
\end{aligned}$$

This completes the proof of the theorem. □

Remark 3. *If we take $h_1(t) = h_2(t) = t$, then inequality (2.3) reduces to the inequality (1.15).*

Theorem 15. *Suppose that all the assumptions of Theorem 12 are satisfied, if g_x and g_y are symmetric about $\frac{a+b}{2}$ and $\frac{c+d}{2}$, respectively, with $h_1 = h_2 = h$, then one has the inequality;*

$$\begin{aligned}
&\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,g(x,y)\,dydx \\
&\leq \frac{1}{4(b-a)}\int_a^b\int_c^d [f(x,c)+f(x,d)]\left(h\left(\frac{d-y}{d-c}\right)+h\left(\frac{y-c}{d-c}\right)\right)g(x,y)dydx \\
&+\frac{1}{4(d-c)}\int_c^d\int_a^b [f(a,y)+f(b,y)]\left(h\left(\frac{b-x}{b-a}\right)+h\left(\frac{x-a}{b-a}\right)\right)g(x,y)dxdy
\end{aligned}$$

Proof. Since the partial mappings f_x and g_x are $h-$convex, by applying to the inequality (1.13), we can write

$$\begin{aligned}
&\frac{1}{d-c}\int_c^d f_x(y)g_x(y)dy \\
&\leq \frac{f_x(c)+f_x(d)}{2}\int_c^d\left(h\left(\frac{d-y}{d-c}\right)+h\left(\frac{y-c}{d-c}\right)\right)g_x(y)dy.
\end{aligned}$$

That is;

$$\begin{aligned}
&\frac{1}{d-c}\int_c^d f(x,y)g(x,y)dy \\
&\leq \frac{f(x,c)+f(x,d)}{2}\int_c^d\left(h\left(\frac{d-y}{d-c}\right)+h\left(\frac{y-c}{d-c}\right)\right)g(x,y)dy.
\end{aligned}$$

Integrating the result with respect to x on $[a,b]$ and dividing both sides of inequality, we get;

(2.11)

$$\begin{aligned}
&\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,g(x,y)\,dydx \\
&\leq \frac{1}{2(b-a)}\int_a^b\int_c^d [f(x,c)+f(x,d)]\left(h\left(\frac{d-y}{d-c}\right)+h\left(\frac{y-c}{d-c}\right)\right)g(x,y)dydx.
\end{aligned}$$

By a similar argument f_y and g_y are $h-$convex, by applying to the inequality (1.13), we get;

(2.12)

$$\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,g(x,y)\,dydx$$
$$\leq \frac{1}{2(d-c)}\int_c^d\int_a^b [f(a,y)+f(b,y)]\left(h\left(\frac{b-x}{b-a}\right)+h\left(\frac{x-a}{b-a}\right)\right)g(x,y)dxdy.$$

Summing (2.11) and (2.12), we obtain the required result. □

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