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Power Series Inequalities Via Buzano's Result and Applications

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ABSTRACT. In this paper, some new inequalities for power series via Buzano's result are obtained. Applications for some fundamental complex functions are also provided.

1. Introduction

If we consider an analytic function $f(z)$ defined by the power series $\sum_{n=0}^{\infty} a_n z^n$ with complex coefficients a_n and apply the well-know Cauchy-Bunyakovsky-Schwarz (CBS)-inequality [**9**],

$$\left|\sum_{j=1}^{n} x_j y_j\right|^2 \leq \sum_{j=1}^{n} |x_j|^2 \sum_{j=1}^{n} |y_j|^2 \tag{1.1}$$

holding for the complex numbers $x_j, y_j, j \in \{1, 2, ..., n\}$, then we can deduce that

$$|f(z)|^2 = \left|\sum_{n=0}^{\infty} a_n z^n\right|^2 \leq \sum_{n=0}^{\infty} |a_n|^2 \sum_{n=0}^{\infty} |z|^{2n} = \frac{1}{1-|z|^2} \cdot \sum_{n=0}^{\infty} |a_n|^2, \tag{1.2}$$

for any $z \in D(0, R) \cap D(0, 1)$, where R is the radius of convergence of f.

If we restrict ourselves more and assume that the coefficients in the representation $f(z) = \sum_{n=0}^{\infty} a_n z^n$ are nonnegative, and the assumption incorporates various examples of complex functions of interest indicated in the sequel, on utilising the weighted version of the (CBS)-inequality, namely

$$\left|\sum_{j=1}^{n} w_j x_j y_j\right|^2 \leq \sum_{j=1}^{n} w_j |x_j|^2 \sum_{j=1}^{n} w_j |y_j|^2 \tag{1.3}$$

where $w_j \geq 0$, while $x_j, y_j \in \mathbb{C}$, $j \in \{1, 2, ..., n\}$, we can state that

$$|f(zw)|^2 = \left|\sum_{n=0}^{\infty} a_n z^n w^n\right|^2 \leq \sum_{n=0}^{\infty} a_n |z|^{2n} \sum_{n=0}^{\infty} a_n |w|^{2n} = f\left(|z|^2\right) f\left(|w|^2\right), \tag{1.4}$$

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for any $z, w \in \mathbb{C}$ with $zw, |z|^2, |w|^2 \in D(0, R)$. In particular, if $w = a \in \mathbb{R}$ then we get from (1.4)

$$|f(az)|^2 \leq f(a^2) f\left(|z|^2\right), \tag{1.5}$$

for any $a \in \mathbb{R}$, $z \in \mathbb{C}$ with $az, a^2, |w|^2 \in D(0, R)$.

A refinement of the celebrated (CBS)-inequality (1.1) has been obtained by de Bruijn in [**5**], namely

$$\left|\sum_{k=1}^{n} b_k z_k\right|^2 \leq \frac{1}{2} \sum_{k=1}^{n} b_k^2 \left[\sum_{k=1}^{n} |z_k|^2 + \left|\sum_{k=1}^{n} z_k^2\right|\right], \tag{1.6}$$

where $b_k \in \mathbb{R}, z_k \in \mathbb{C}, k \in \{1, 2, ..., n\}$ and the equality holds in (1.6) if and only if $b_k = \operatorname{Re}(\lambda z_k)$, where λ is a complex number such that the $\lambda^2 \sum_{k=1}^{n} z_k$ is a nonnegative real number.

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an analytic function defined by a power series with nonnegative coefficients and convergent on the open disk $D(0, R) \subset \mathbb{C}$, if a is a real number and z a complex number such that $az, a^2, z^2, |z|^2 \in D(0, R)$, Cerone and Dragomir [**4**] proved that

$$|f(az)|^2 \leq \frac{1}{2} f(a^2) \left[f\left(|z|^2\right) + \left|f(z^2)\right|\right], \tag{1.7}$$

which is a refinement of the inequality (1.5). They also applied this inequality for various examples of complex functions including the exponential, trigonometric, hyperbolic, hypergeometric and polylogarithmic function.

For other inequalities concerning power series, see [**2**], [**6**], [**7**], [**8**] and the references therein.

Motivated by the above result (1.7), we investigate in this paper some natural extensions of de Bruijn inequality by utilising a result in inner product spaces obtained by Buzano in [**3**], and apply them for functions defined by power series. Particular examples that are related to some fundamental complex functions are also presented.

2. Some Inequalities Via Buzano's Result

In [**4**], S.S. Dragomir has observed that, on utilizing Buzano's inequality [**3**] in the complex inner product space $(H; \langle\cdot,\cdot\rangle)$,

$$|\langle a, x\rangle \langle x, b\rangle| \leq \frac{1}{2} [\|a\| \|b\| + |\langle a, b\rangle|] \|x\|^2, \tag{2.1}$$

for $a, b, x \in H$, which for $a = b$ reduces to the Schwarz inequality,

$$|\langle a, x\rangle|^2 \leq \|a\|^2 \|x\|^2 ; \; a, x \in H, \tag{2.2}$$

with equality if and only if there exists a scalar $\lambda \in \mathbb{C}$ such that $x = \lambda a$, one can obtain the discrete inequality

$$\begin{aligned} (2.3) \qquad & \left| \sum_{j=1}^{n} p_j c_j \overline{x_j} \sum_{j=1}^{n} p_j x_j \overline{b_j} \right| \\ & \leq \frac{1}{2} \left[\left(\sum_{j=1}^{n} p_j |c_j|^2 \sum_{j=1}^{n} p_j |b_j|^2 \right)^{1/2} + \left| \sum_{j=1}^{n} p_j c_j \overline{b_j} \right| \right] \sum_{j=1}^{n} p_j |x_j|^2 , \end{aligned}$$

where $p_j \geq 0$, $x_j, b_j, c_j \in \mathbb{C}$, $j \in \{1, \dots, n\}$.

If we take in (2.3) $b_j = \overline{c_j}$, for $j \in \{1, 2, ..., n\}$, then we obtain

$$(2.4) \qquad \left| \sum_{j=1}^{n} p_j c_j \overline{x_j} \sum_{j=1}^{n} p_j c_j x_j \right| \leq \frac{1}{2} \left[\sum_{j=1}^{n} p_j |c_j|^2 + \left| \sum_{j=1}^{n} p_j c_j^2 \right| \right] \sum_{j=1}^{n} p_j |x_j|^2 ,$$

for any $p_j \geq 0, x_j, c_j \in \mathbb{C}, j \in \{1, 2, ..., n\}$.

As pointed out in [**4**], if $x_j, j \in \{1, 2, ..., n\}$ are real numbers, then (2.3) generates the de Bruijn refinement of the celebrated weighted (CBS)-inequality

$$(2.5) \qquad \left| \sum_{j=1}^{n} p_j x_j z_j \right|^2 \leq \frac{1}{2} \sum_{j=1}^{n} p_j x_j^2 \left[\sum_{j=1}^{n} p_j |z_j|^2 + \left| \sum_{j=1}^{n} p_j z_j^2 \right| \right] ,$$

where $p_j \geq 0, x_j \in \mathbb{R}, z_j \in \mathbb{C}, j \in \{1, 2, ..., n\}$.

The following result holds:

THEOREM 1. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with nonnegative coefficients a_n and convergent in the open disk $D(0, R)$. If $x, \alpha, \beta \in \mathbb{C}$ so that $\alpha\overline{x}$, $\overline{\beta}x$, $|\alpha|^2$, β^2, $a\overline{\beta}$, $|x|^2 \in D(0, R)$, then*

$$(2.6) \qquad \left| f(\alpha\overline{x}) f(\overline{\beta}x) \right| \leq \frac{1}{2} \left[\left[f\left(|\alpha|^2\right) f\left(|\beta|^2\right) \right]^{1/2} + \left| f(\alpha\overline{\beta}) \right| \right] f\left(|x|^2\right) .$$

PROOF. On utilizing the inequality (2.3), for the choices $p_n = a_n$, $c_n = \alpha^n$, $x_n = x^n$, $b_n = \beta^n$, $n \geq 0$ we have

$$\begin{aligned} (2.7) \qquad & \left| \sum_{n=0}^{m} a_n \alpha^n (\overline{x})^n \sum_{n=0}^{m} a_n \left(\overline{\beta}\right)^n x^n \right| \\ & \leq \frac{1}{2} \left[\left(\sum_{n=0}^{m} a_n |\alpha|^{2n} \sum_{n=0}^{m} a_n |\beta|^{2n} \right)^{1/2} + \left| \sum_{n=0}^{m} a_n \alpha^n (\overline{\beta})^n \right| \right] \sum_{n=0}^{m} a_n |x|^{2n} , \end{aligned}$$

for any $m \geq 0$.

Since $\alpha\overline{x}, \overline{\beta}x, |\alpha|^2, |\beta|^2, \alpha\overline{\beta}, |x|^2$ belong to the convergence disk $D(0, R)$, hence the series in (2.7) are convergent and letting $m \to \infty$, we deduce the desired inequality (2.6). □

A particular case of interest is as follows:

COROLLARY 1. *Let f be as in Theorem 1 and $z, x \in \mathbb{C}$ with $z\overline{x}$, zx, $|z|^2$, z^2, $|x|^2 \in D(0, R)$. Then*

$$(2.8) \qquad |f(z\overline{x}) f(zx)| \leq \frac{1}{2} \left[f\left(|z|^2\right) + \left| f\left(z^2\right) \right| \right] f\left(|x|^2\right) .$$

This follows from (2.6) by choosing $\alpha = z$, $\beta = \overline{z}$.

Remark 1. *In particular, if $x = a \in \mathbb{R}$, then from (2.8) we deduce the inequality (1.7) above,* [**4**].

The above result (2.6) has some natural applications for particular complex functions of interest as follows:

1. If we apply the inequality (2.6) for $f(z) = \frac{1}{1-z}$, $z \in D(0,1)$, then we get

$$\begin{aligned} &\left| \frac{1}{1-\alpha\overline{x}} \cdot \frac{1}{1-\overline{\beta}x} \right| \\ &\leq \frac{1}{2} \left[\left(\frac{1}{1-|\alpha|^2} \cdot \frac{1}{1-|\beta|^2} \right)^{1/2} + \left| \frac{1}{1-\alpha\overline{\beta}} \right| \right] \frac{1}{1-|x|^2}, \end{aligned} \tag{2.9}$$

for any $x, \alpha, \beta \in D(0,1)$. This is equivalent with

$$\begin{aligned} &2\left(1-|x|^2\right)\left|1-\alpha\overline{\beta}\right| \sqrt{\left(1-|\alpha|^2\right)(1-|\beta|)^2} \\ &\leq |1-\alpha\overline{x}|\left|1-\overline{\beta}x\right| \left[\left|1-\alpha\overline{\beta}\right| + \sqrt{\left(1-|\alpha|^2\right)(1-|\beta|)^2} \right], \end{aligned} \tag{2.10}$$

for $x, \alpha, \beta \in D(0,1)$. In particular, if $\beta = \overline{\alpha}$, then we get from (2.10) that

$$2(1-|x|)\left(1-|\alpha|^2\right)\left|1-\alpha^2\right| \leq |1-\alpha\overline{x}|\,|1-\alpha x| \left[\left|1-\alpha^2\right| + 1 - |\alpha|^2 \right], \tag{2.11}$$

for any $x, \alpha \in D(0,1)$.

2. If we apply (2.6) for $f(z) = \exp(z)$, $z \in \mathbb{C}$, then we get the inequality

$$\begin{aligned} &\left|\exp\left(\alpha\overline{x} + \overline{\beta}x\right)\right| \\ &\leq \frac{1}{2} \left[\left(\exp\left(|\alpha|^2 + |\beta|^2\right)\right)^{1/2} + \left|\exp\left(\alpha\overline{\beta}\right)\right| \right] \exp\left(|x|^2\right), \end{aligned} \tag{2.12}$$

for any $\alpha, \beta, x \in \mathbb{C}$. In particular, if $\alpha = \overline{\beta}$, then we get from (2.12) that

$$|\exp(2\alpha \operatorname{Re}(x))| \leq \frac{1}{2} \left[\left(\exp(2|\alpha|^2)\right)^{1/2} + \left|\exp\left(\alpha^2\right)\right| \right] \exp\left(|x|^2\right), \tag{2.13}$$

for any $\alpha, x \in \mathbb{C}$.

3. If we apply (2.6) for the Koebe function $f(z) = \frac{z}{(1-z)^2}$, $z \in D(0,1)$, then we get

$$\begin{aligned} &\left| \frac{\alpha\overline{\beta}\,|x|^2}{(1-\alpha\overline{x})^2\left(1-\overline{\beta}x\right)^2} \right| \\ &\leq \frac{1}{2} \left(\frac{|\alpha\beta|}{\left(1-|\alpha|^2\right)\left(1-|\beta|^2\right)} + \left| \frac{\alpha\overline{\beta}}{\left(1-\alpha\overline{\beta}\right)^2} \right| \right) \frac{|x|^2}{\left(1-|x|^2\right)^2}, \end{aligned} \tag{2.14}$$

for any $x, \alpha, \beta \in D(0,1)$. If we simplify (2.14), then we get

$$\frac{1-|x|^2}{\left|(1-\alpha\overline{x})\left(1-\overline{\beta}x\right)\right|} \leq \left(\frac{1}{2\left(1-|\alpha|^2\right)\left(1-|\beta|^2\right)} + \frac{1}{2\left|1-\alpha\overline{\beta}\right|^2} \right)^{1/2}, \tag{2.15}$$

for any $\alpha, \beta, x \in D(0,1)$. In particular if $\beta = \overline{\alpha}$, then we get from (2.15) that

$$\frac{1-|x|^2}{|(1-\alpha\overline{x})(1-\alpha x)|} \le \left(\frac{1}{2\left(1-|\alpha|^2\right)^2} + \frac{1}{2\left|1-\alpha^2\right|^2} \right)^{1/2}, \tag{2.16}$$

for any $\alpha, x \in D(0,1)$.

4. If we apply the same inequality (2.6) for the function

$$f(z) = \cosh(z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}, \ z \in \mathbb{C},$$

then we obtain

$$\begin{aligned} &\left|\cosh\left(\alpha\overline{x} + \overline{\beta}x\right) + \cosh\left(\alpha\overline{x} - \overline{\beta}x\right)\right| \\ &\le \left(\left[\frac{1}{2}\left(\cosh\left(|\alpha|^2 + |\beta|^2\right) + \cosh\left(|\alpha|^2 - |\beta|^2\right) \right) \right]^{1/2} + \left|\cosh\left(\alpha\overline{\beta}\right)\right| \right) \\ &\times \cosh\left(|x|^2\right), \end{aligned} \tag{2.17}$$

for any $x, \alpha, \beta \in \mathbb{C}$. In particular, for $\beta = \overline{\alpha}$, we get that from (2.17)

$$\begin{aligned} &\left|\cosh\left(2\alpha \operatorname{Re}(x)\right) + \cosh\left(2i\alpha \operatorname{Im}(x)\right)\right| \\ &\le \left[\cosh\left(|\alpha|^2\right) + \left|\cosh\left(\alpha^2\right)\right|\right] \cosh\left(|x|^2\right), \end{aligned} \tag{2.18}$$

that holds for any $\alpha, x \in \mathbb{C}$.

In the following, we construct another power series that have coefficients as the absolute values of coefficients of the original series, namely

$$f_A(z) = \sum_{n=0}^{\infty} |a_n| z^n.$$

This new power series also have the same radius of convergence as the original series. The following result gives an inequality connecting the function f with its transform f_A.

Theorem 2. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a function defined by a power series with real coefficients and convergent on the open disk $D(0,R) \subset \mathbb{C}$, $R > 0$. If α, β, x are complex numbers such that $\alpha\overline{x}, \overline{\beta}x, \alpha\overline{\beta}, |\alpha|^2, |\beta|^2, \left|x^2\right| \in D(0,R)$, then*

$$\left|f(\alpha\overline{x}) f\left(\overline{\beta}x\right)\right| \le \frac{1}{2}\left(\left[f_A\left(|\alpha|^2\right) f_A\left(|\beta|^2\right) \right]^{1/2} + \left|f_A\left(\alpha\overline{\beta}\right)\right| \right) f_A\left(|x|^2\right). \tag{2.19}$$

Proof. Firstly, observe that for each $n \ge 0$, $a_n = |a_n|\operatorname{sgn}(a_n)$ where the $\operatorname{sgn}(x)$ is the sgn function defined to be 1 if $x > 0$, -1 if $x < 0$ and 0 if $x = 0$. By choosing

$p_n = |a_n|$, $c_n = \alpha^n$, $b_n = \beta^n$ and $x_n = sgn(a_n) x^n$, $n \geq 0$ in (2.3) we have

$$\begin{aligned}
(2.20)\qquad & \left| \sum_{n=0}^{m} a_n (\alpha\overline{x})^n \sum_{n=0}^{m} a_n (\overline{\beta}x)^n \right| \\
&= \left| \sum_{n=0}^{m} |a_n| \, sgn(a_n) \, \alpha^n (\overline{x})^n \sum_{n=0}^{m} |a_n| \, sgn(a_n) \, x^n (\overline{\beta})^n \right|, \\
&\leq \frac{1}{2} \left(\left[\sum_{n=0}^{m} |a_n| \left(|\alpha|^2 \right)^n \sum_{n=0}^{m} |a_n| \left(|\beta|^2 \right)^n \right]^{1/2} + \left| \sum_{n=0}^{m} |a_n| \left(\alpha\overline{\beta} \right)^n \right| \right) \\
&\times \sum_{n=0}^{m} |a_n| \left(|x|^2 \right)^n,
\end{aligned}$$

for any $\alpha, \beta, x \in \mathbb{C}$ with $\alpha\overline{x}$, $\overline{\beta}x$, $\alpha\overline{\beta}$, $|\alpha|^2, |\beta|^2, |x|^2 \in D(0,R)$. Taking the limit as $m \to \infty$ in (2.20) and noticing that all the involved series are convergent, then we deduce the desired inequality (2.19). $\square$

In what follows we provide some applications of the inequality (2.19) for particular functions of interest.

1. If we take the function

$$f(z) = \frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n, \ z \in D(0,1),$$

then

$$f_A(z) = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}, \ z \in D(0,1).$$

Applying Theorem 2, we get the following inequality

$$\begin{aligned}
(2.21)\qquad & 2\left|1-\alpha\overline{\beta}\right| \left(1-|x|^2\right) \left[\left(1-|\alpha|^2\right)\left(1-|\beta|^2\right) \right]^{1/2} \\
&\leq |1+\alpha\overline{x}| \left|1+\overline{\beta}x\right| \left(\left|1-\alpha\overline{\beta}\right| + \left[\left(1-|\alpha|^2\right)\left(1-|\beta|^2\right) \right]^{1/2} \right),
\end{aligned}$$

for any $\alpha, \beta, x \in D(0,1)$. In particular, if $\alpha = \overline{\beta}$, then from (2.21) we obtain

$$(2.22)\quad 2\left|1-\alpha^2\right| \left(1-|\alpha|^2\right)\left(1-|x|^2\right) \leq |1+\alpha x| \, |1+\alpha\overline{x}| \left(1-|\alpha|^2 + \left|1-\alpha^2\right|\right),$$

for any $\alpha, x \in D(0,1)$.

2. For the function

$$f(z) = e^{-z} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} z^n, \ z \in \mathbb{C},$$

we have the transform

$$f_A(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z, \ z \in \mathbb{C}.$$

Utilising the inequality (2.19) we obtain

$$\begin{aligned}(2.23)\qquad & \frac{1}{\left|\exp\left(\alpha\overline{x}+\overline{\beta}x\right)\right|}\\ &\leq \frac{1}{2}\left(\exp\left[\frac{1}{2}\left(|\alpha|^2+|\beta|^2\right)+|x|^2\right]+\exp\left(|x|^2\right)\left|\exp\left(\alpha\overline{\beta}\right)\right|\right),\end{aligned}$$

for any $\alpha, \beta, x \in \mathbb{C}$. In particular, if $\alpha = \overline{\beta}$ in (2.23), then we get

$$\frac{1}{\left|\exp\left(2\alpha\operatorname{Re}(x)\right)\right|} \leq \frac{1}{2}\left[\exp\left(|\alpha|^2+|\beta|^2\right)+\exp\left(|x|^2\right)\left|\exp\left(\alpha^2\right)\right|\right], \tag{2.24}$$

for any $\alpha, x \in \mathbb{C}$.

3. If in (2.19) we choose the function

$$f(z) = \cos z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n},\ \ z \in \mathbb{C},$$

then

$$f_A(z) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} z^{2n} = \cosh(z) = \frac{1}{2}\left(e^z + e^{-z}\right).$$

Applying the inequality (2.19) will produce the result

$$\begin{aligned}(2.25)\qquad & \left|\cos\left(\alpha\overline{x}\right)\cos\left(\overline{\beta}x\right)\right|\\ &\leq \frac{1}{2}\left(\left[\cosh\left(|\alpha|^2\right)\cosh\left(|\beta|^2\right)\right]^{1/2}+\left|\cosh\left(\alpha\overline{\beta}\right)\right|\right)\cosh\left(|x|^2\right),\end{aligned}$$

for any $\alpha, \beta, x \in \mathbb{C}$. In particular, if we choose $\alpha = \overline{\beta}$ in (2.25), then we obtain the inequality

$$\left|\cos\left(\alpha\overline{x}\right)\cos\left(\alpha x\right)\right| \leq \frac{1}{2}\left[\cosh\left(|\alpha|^2\right)+\left|\cosh\left(\alpha^2\right)\right|\right]\cosh\left(|x|^2\right), \tag{2.26}$$

for any $\alpha, x \in \mathbb{C}$.

From a different perspective, we can state the following result which establishes a connection between two and three power series, one having positive coefficients.

Theorem 3. *Let $g(z) = \sum_{n=0}^{\infty} g_n z^n$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be two power series with $g_n \in \mathbb{C}$ and $a_n > 0$ for $n \geq 0$. If f and g are convergent on $D(0, R_1)$ and $D(0, R_2)$ respectively, and the numerical series $\sum_{n=0}^{\infty} \frac{|g_n|^2}{a_n}$ is convergent, then we have the inequality:*

$$\left|g(z)\,g(\overline{z})\right| \leq \frac{1}{2}\sum_{n=0}^{\infty}\frac{|g_n|^2}{a_n}\left[f\left(|z|^2\right)+\left|f\left(z^2\right)\right|\right] \tag{2.27}$$

for any $z \in \mathbb{C}$ with $z, z^2, |z|^2 \in D(0, R_1) \cap D(0, R_2)$.

Proof. On utilising the inequality (2.4) for the choices $p_n = a_n$, $c_n = z^n$, $x_n = \frac{g_n}{a_n}$, $n \geq 0$ we have

$$\left|\sum_{n=0}^{m} g_n z^n \sum_{n=0}^{m} \overline{g_n} z^n\right| \leq \frac{1}{2}\left[\sum_{n=0}^{m} a_n \left(|z|^2\right)^n + \left|\sum_{n=0}^{m} a_n \left(z^2\right)^n\right|\right]\sum_{n=0}^{m}\frac{|g_n|^2}{a_n}, \tag{2.28}$$

for any $m \geq 0$.

Observe that $\sum_{n=0}^{m}\overline{g_n}z^n=\overline{\sum_{n=0}^{m}g_n\left(\overline{z}\right)^n}$ and then $\left|\sum_{n=0}^{m}\overline{g_n}z^n\right|=\left|\sum_{n=0}^{m}g_n\left(\overline{z}\right)^n\right|$. Replacing this in (2.28) we get

$$\left|\sum_{n=0}^{m}g_nz^n\sum_{n=0n}^{m}g_n\left(\overline{z}\right)^n\right|\leq\frac{1}{2}\sum_{n=0}^{m}\frac{\left|g_n\right|^2}{a_n}\left[\sum_{n=0}^{m}a_n\left(\left|z\right|^2\right)^n+\left|\sum_{n=0}^{m}a_n\left(z^2\right)^n\right|\right]. \tag{2.29}$$

Since $z,z^2,\left|z\right|^2\in D\left(0,R_1\right)\cap D\left(0,R_2\right)$, hence the series in (2.29) are convergent and letting $m\to\infty$, we deduce the desired inequality (2.27). □

Remark 2. *If the coefficients g_n, $n\geq0$ are real, then we recapture the inequality (27) from the paper* [**4**], *namely*

$$\left|g\left(z\right)\right|^2\leq\frac{1}{2}\sum_{n=0}^{\infty}\frac{g_n^2}{a_n}\left[f\left(\left|z\right|^2\right)+\left|f\left(z^2\right)\right|\right],$$

for any $z\in D\left(0,R_2\right)$ with $z^2,\left|z\right|^2\in D\left(0,R_1\right)$.

Corollary 2. *Let $g\left(z\right)=\sum_{n=0}^{\infty}g_nz^n$ be a power series with complex coefficients and convergent on the open disk $D\left(0,R\right)$. If the numerical series $\sum_{n=0}^{\infty}\left|g_n\right|^2$ is convergent, then*

$$|g(z)g(\overline{z})|\leq\frac{1}{2}\sum_{n=0}^{\infty}\left|g_n\right|^2\left[\frac{1-\left|z\right|^2+\left|1-z^2\right|}{\left(1-\left|z\right|^2\right)\left|1-z^2\right|}\right], \tag{2.30}$$

for any $z\in D\left(0,1\right)\cap D\left(0,R\right)$.

This follows from (2.27) for $f\left(z\right)=\frac{1}{1-z}$, $z\in D\left(0,1\right)$.

Remark 3. *If we consider the series expansion*

$$\frac{1}{iz}\ln\left(\frac{1}{1-iz}\right)=\sum_{n=0}^{\infty}\frac{i^n}{n+1}z^n;\ z\in D\left(0,1\right)\setminus\left\{0\right\},$$

then, on utilising the inequality (2.30) for the choice $g_n=\frac{i^n}{n+1}$ and taking into account that

$$\sum_{n=0}^{\infty}\frac{1}{\left(n+1\right)^2}=\sum_{n=1}^{\infty}\frac{1}{n^2}=\zeta\left(2\right)=\frac{\pi^2}{6}, \tag{2.31}$$

we can state the inequality

$$\left|\ln\left(\frac{1}{1-iz}\right)\ln\left(\frac{1}{1-i\overline{z}}\right)\right|\leq\frac{\pi^2}{12}\left(\frac{\left|z\right|^2}{1-\left|z\right|^2}\right)\left(\frac{1-\left|z\right|^2+\left|1-z^2\right|}{\left|1-z^2\right|}\right), \tag{2.32}$$

for any $z\in D\left(0,1\right)$.

Corollary 3. *Let $g\left(z\right)=\sum_{n=0}^{\infty}g_nz^n$ be a power series with complex coefficients and convergent on the open disk $D\left(0,R\right)$. If the numerical series $\sum_{n=0}^{\infty}n!\left|g_n\right|^2$ is convergent, then*

$$|g(z)g(\overline{z})|\leq\frac{1}{2}\sum_{n=0}^{\infty}n!\left|g_n\right|^2\left[\exp\left(\left|z\right|^2\right)+\left|\exp\left(z^2\right)\right|\right], \tag{2.33}$$

for any $z\in D\left(0,R\right)$.

This follows from Theorem (3) by choosing $f(z) = \exp(z)$.

Some applications of the inequality (2.33) are as follows.

1. If we apply the inequality (2.33) for the function $\sin(iz) = \sum_{n=0}^{\infty} \frac{i}{(2n+1)!} z^{2n+1}$, then we obtain the inequality

$$|\sin(iz)\sin(i\overline{z})| \le \frac{1}{2} \sum_{n=0}^{\infty} \frac{n!}{[(2n+1)!]^2} \left[\exp\left(|z|^2\right) + \left|\exp\left(z^2\right)\right| \right],$$

for any $z \in \mathbb{C}$.

2. If we apply the inequality (2.33) for the function $\sinh(iz) = \sum_{n=0}^{\infty} \frac{(-1)^n i}{(2n+1)!} z^{2n+1}$, then we obtain the inequality

$$|\sin(z)|^2 \le \frac{1}{2} \sum_{n=0}^{\infty} \frac{n!}{[(2n+1)!]^2} \left[\exp\left(|z|^2\right) + \left|\exp\left(z^2\right)\right| \right], \tag{2.34}$$

for any $z \in \mathbb{C}$.

Indeed, observing that

$$|\sinh(iz)\sinh(i\overline{z})| = |i\sin(z) \cdot i\sin(\overline{z})| = |\sin(z)\sin(\overline{z})| = |\sin z|^2,$$

and by (2.33) we have

$$|\sinh(iz)\sinh(i\overline{z})| \le \frac{1}{2} \sum_{n=0}^{\infty} \frac{n!}{[(2n+1)!]^2} \left[\exp\left(|z|^2\right) + \left|\exp\left(z^2\right)\right| \right],$$

then we deduce desired inequality (2.34).

Theorem 4. *Let $g(z) = \sum_{n=0}^{\infty} g_n z^n$, $h(z) = \sum_{n=0}^{\infty} h_n z^n$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be three power series with $g_n, h_n \in \mathbb{C}$ and $a_n > 0$ for $n \ge 0$. If f, g and h are convergent on $D(0,R_1)$, $D(0,R_2)$ and $D(0,R_3)$ respectively, and the numerical series $\sum_{n=0}^{\infty} \frac{|g_n|^2}{a_n}$, $\sum_{n=0}^{\infty} \frac{|h_n|^2}{a_n}$ and $\sum_{n=0}^{\infty} \frac{g_n \overline{h_n}}{a_n}$ are convergent, then we have the inequality:*

$$|g(z)h(z)| \le \frac{1}{2} \left(\left[\sum_{n=0}^{\infty} \frac{|g_n|^2}{a_n} \sum_{n=0}^{\infty} \frac{|h_n|^2}{a_n} \right]^{1/2} + \left| \sum_{n=0}^{\infty} \frac{g_n \overline{h_n}}{a_n} \right| \right) f\left(|z|^2\right), \tag{2.35}$$

for any $z \in \mathbb{C}$ with $z, |z|^2 \in D(0,R_1) \cap D(0,R_2) \cap D(0,R_3)$.

Proof. On utilising the Buzano inequality (2.3) for the choices $p_n = a_n$, $c_n = \frac{g_n}{a_n}$, $b_n = \frac{h_n}{a_n}$, $x_n = z^n$, $n \ge 0$, we can state that

$$\begin{aligned}
|g(z)h(z)| &= \left| \sum_{n=0}^{\infty} a_n \left(\frac{g_n}{a_n}\right) z^n \sum_{n=0}^{\infty} a_n \left(\frac{h_n}{a_n}\right) z^n \right|, \\
&\le \frac{1}{2} \left(\left[\sum_{n=0}^{\infty} a_n \left|\frac{g_n}{a_n}\right|^2 \sum_{n=0}^{\infty} a_n \left|\frac{h_n}{a_n}\right|^2 \right]^{1/2} + \left| \sum_{n=0}^{\infty} a_n \left(\frac{g_n}{a_n}\right) \overline{\left(\frac{h_n}{a_n}\right)} \right| \right) \\
&\times \sum_{n=0}^{\infty} a_n \left(|z|^2\right)^n, \\
&= \frac{1}{2} \left(\left[\sum_{n=0}^{\infty} \frac{|g_n|^2}{a_n} \sum_{n=0}^{\infty} \frac{|h_n|^2}{a_n} \right]^{1/2} + \left| \sum_{n=0}^{\infty} \frac{g_n \overline{h_n}}{a_n} \right| \right) f\left(|z|^2\right),
\end{aligned}$$

for any $z \in \mathbb{C}$ with $z, |z|^2 \in D(0, R_1) \cap D(0, R_2) \cap D(0, R_3)$. □

Remark 4. *In particular, if $g_n = h_n$, then from (2.35) we obtain*

$$|g(z)|^2 \leq f\left(|z|^2\right) \sum_{n=0}^{\infty} \frac{|g_n|^2}{a_n}, \tag{2.36}$$

for any $z, |z|^2 \in D(0, R_1) \cap D(0, R_2)$.

Remark 5. *Also if $h_n = \overline{g_n}$, then we get the following inequality*

$$|g(z)g(\overline{z})| \leq \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{|g_n|^2}{a_n} + \left|\sum_{n=0}^{\infty} g_n^2\right|\right) f\left(|z|^2\right), \tag{2.37}$$

for any $z, |z|^2 \in D(0, R_1) \cap D(0, R_2)$.

Corollary 4. *Let $g(z)$ and $h(z)$ be power series as in Theorem 4. If the numerical series $\sum_{n=0}^{\infty} |g_n|^2$, $\sum_{n=0}^{\infty} |h_n|^2$ and $\sum_{n=0}^{\infty} \left|g_n \overline{h_n}\right|$ are convergent, then*

$$|g(z)h(z)| \leq \frac{1}{2\left(1 - |z|^2\right)}\left(\left[\sum_{n=0}^{\infty} |g_n|^2 \sum_{n=0}^{\infty} |h_n|^2\right]^{1/2} + \left|\sum_{n=0}^{\infty} g_n \overline{h_n}\right|\right), \tag{2.38}$$

for any $z \in D(0, 1) \cap D(0, R_2) \cap D(0, R_3)$.

Remark 6. *If we consider the series*

$$\frac{1}{iz} \ln\left(\frac{1}{1 - iz}\right) = \sum_{n=0}^{\infty} \frac{i^n}{(n+1)} z^n, \ z \in D(0, 1) \setminus \{0\}$$

and

$$\ln\left(\frac{1}{1 + iz}\right) = \sum_{n=1}^{\infty} \frac{(-i)^n}{n} z^n, \ z \in D(0, 1),$$

then on utilising the inequality (2.38) for the choices $g_0 = h_0 = 0$, $g_n = \frac{i^n}{n+1}$, $h_n = \frac{(-i)^n}{n}$, $n \geq 1$ and taking into account that

$$\sum_{n=0}^{\infty} g_n \overline{h_n} = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1 \tag{2.39}$$

and the equality (2.31), we obtain the following inequality

$$\left|\ln\left(\frac{1}{1 - iz}\right) \ln\left(\frac{1}{1 + iz}\right)\right| \leq \frac{\pi^2 + 6}{12}\left(\frac{|z|}{1 - |z|^2}\right), \tag{2.40}$$

for any $z \in D(0, 1)$.

3. Some Inequalities for the Polylogarithm

Before we start our results for the polylogarithm that can be obtained on utilising the Buzano inequality, we recall some concepts that will be used in the sequel.

The polylogarithm $Li_n(z)$, also known as the *de Jonquières function* is the function defined by

$$Li_n(z) := \sum_{k=1}^{\infty} \frac{z^k}{k^n}, \tag{3.1}$$

defined in the complex plane over the unit disk $D(0,1)$.

The special case, $z=1$ reduces to $Li_s(z)=\zeta(s)$, where ζ is the Riemann zeta function. The polylogarithm of nonnegative order arises in the sums of the form

$$Li_{-n}(r)=\sum_{k=1}^{\infty}k^n r^k=\frac{1}{(1-r)^{n+1}}\sum_{i=0}^{n}E_{n,i}r^{n-i},$$

where $E_{n,i}$ is an Eulerian number, namely, we recall that

$$E_{n,k}:=\sum_{j=0}^{k+1}(-1)^j\left(\begin{array}{c}n+1\\ j\end{array}\right)(k-j+1)^n.$$

Polylogarithms also arise in sum of generalized harmonic numbers $H_{n,r}$ as

$$\sum_{n=1}^{\infty}H_{n,r}z^n=\frac{Li_r(z)}{1-z},$$

for $z\in D(0,1)$, where we recall that

$$H_{n,r}:=\sum_{k=1}^{\infty}\frac{1}{k^r}\text{ and }H_{n,1}:=H_n=\sum_{k=1}^{\infty}\frac{1}{k}.$$

Special forms of low-order polylogarithms include

$$\begin{aligned}Li_{-2}(z)&=\frac{z(z+1)}{(1-z)^3},\ Li_{-1}(z)=\frac{z}{(1-z)^2},\\ Li_0(z)&=\frac{z}{(1-z)},\ Li_1(z)=-\ln(1-z),\ z\in D(0,1).\end{aligned}$$

At argument $z=-1$, the general polylogarithms becomes $Li_n(-1)=-\eta(x)$, where $\eta(x)$ is the Dirichlet eta function.

It is clear that Li_n being a power series with nonnegative coefficients and convergent on the open unit disk $D(0,1)$, so that all the above results hold true. Therefore we have, for instance the inequality:

$$\begin{aligned}&\left|Li_n(\alpha\overline{x})\,Li_n(\overline{\beta}x)\right|\\ &\leq\frac{1}{2}\left(\left[Li_n\left(|\alpha|^2\right)Li_n\left(|\beta|^2\right)\right]^{1/2}+\left|Li_n(\alpha\overline{\beta})\right|\right)Li_n\left(|x|^2\right),\end{aligned}\tag{3.2}$$

for any $\alpha,\beta,x\in\mathbb{C}$ with $\alpha\overline{x},\overline{\beta}x,|\alpha|^2,|\beta|^2,\alpha\overline{\beta},|x|^2\in D(0,1)$ and n is a negative or positive integer.

In the following, we present some results that connect different order polylogarithms:

Theorem 5. *Let $\alpha,\beta,x\in\mathbb{C}$, $\alpha\overline{x},\overline{\beta}x,|\alpha|^2,|\beta|^2,\alpha\overline{\beta},|x|^2\in D(0,1)$ and p,q,r integers such that the following series exist. Then*

$$\begin{aligned}&\left|Li_{r+p+q}(\alpha\overline{x})\,Li_{r+p+q}(\overline{\beta}x)\right|\\ &\leq\frac{1}{2}\left(\left[Li_{r+2q}\left(|\alpha|^2\right)Li_{r+2q}\left(|\beta|^2\right)\right]^{1/2}+\left|Li_{r+2q}(\alpha\overline{\beta})\right|\right)Li_{r+2p}\left(|x|^2\right).\end{aligned}\tag{3.3}$$

Proof. Utilising the Buzano inequality (2.3) for $p_k = \frac{1}{k^r}$, $c_k = \frac{\alpha^k}{k^q}$, $b_k = \frac{\beta^k}{k^q}$ and $x_k = \frac{x^k}{k^p}$, we have

$$\begin{aligned}
&\left|Li_{r+p+q}\left(\alpha\overline{x}\right)Li_{r+p+q}\left(\overline{\beta}x\right)\right| \\
&= \left|\sum_{k=1}^{\infty}\frac{1}{k^r}\frac{\alpha^k}{k^q}\frac{\left(\overline{x}\right)^k}{k^p}\sum_{k=1}^{\infty}\frac{1}{k^r}\frac{\left(\overline{\beta}\right)^k}{k^q}\frac{\left(x\right)^k}{k^p}\right|, \\
&\leq \frac{1}{2}\left(\left[\sum_{k=1}^{\infty}\frac{1}{k^r}\frac{\left(|\alpha|^2\right)^k}{k^{2q}}\sum_{k=1}^{\infty}\frac{1}{k^r}\frac{\left(|\beta|^2\right)^k}{k^{2q}}\right]^{1/2} + \left|\sum_{k=1}^{\infty}\frac{1}{k^r}\frac{\left(\alpha\overline{\beta}\right)^k}{k^{2q}}\right|\right) \\
&\times \sum_{k=1}^{\infty}\frac{1}{k^r}\frac{\left(|x|^2\right)^k}{k^{2p}}, \\
&= \frac{1}{2}\left(\left[Li_{r+2q}\left(|\alpha|^2\right)Li_{r+2q}\left(|\beta|^2\right)\right]^{1/2} + \left|Li_{r+2q}\left(\alpha\overline{\beta}\right)\right|\right) \\
&\times Li_{r+2p}\left(|x|^2\right),
\end{aligned}$$

and the inequality is proved. $\square$

On making use of the above result (3.3), we can get some simpler inequalities as follows:

1. If $\alpha = z$, $\beta = \overline{z}$, then from (3.3) we can state that

$$\begin{aligned}
&\left|Li_{r+p+q}\left(z\overline{x}\right)Li_{r+p+q}\left(zx\right)\right| \\
&\leq \frac{1}{2}\left[Li_{r+2q}\left(|z|^2\right) + \left|Li_{r+2q}\left(z^2\right)\right|\right]Li_{r+2p}\left(|x|^2\right).
\end{aligned} \tag{3.4}$$

2. Moreover, if $x = a \in R$, then from (3.4) we deduce the inequality (33) in paper [**4**], namely

$$\left|Li_{r+p+q}\left(az\right)\right|^2 \leq \frac{1}{2}Li_{r+2p}\left(a^2\right)\left[Li_{r+2q}\left(|z|^2\right) + \left|Li_{r+2q}\left(z^2\right)\right|\right].$$

From a different perspective, we can state the following inequality which incorporates the zeta function, ζ:

Corollary 5. *Let $\alpha, \beta \in D(0,1)$ and p, q, r integers such that $r + 2p > 1$. Then*

$$\begin{aligned}
&\left|Li_{r+p+q}\left(-\alpha i\right)Li_{r+p+q}\left(\overline{\beta}i\right)\right| \\
&\leq \frac{1}{2}\zeta\left(r+2p\right)\left(\left[Li_{r+2q}\left(|\alpha|^2\right)Li_{r+2q}\left(|\beta|^2\right)\right]^{1/2} + \left|Li_{r+2q}\left(\alpha\overline{\beta}\right)\right|\right).
\end{aligned} \tag{3.5}$$

The proof follows by Theorem 5 for $x = i$.

Remark 7. *On utilising (3.5) and taking into account that some particular values of ζ are known, such as $\zeta(2) = \frac{\pi^2}{6}$, $\zeta(4) = \frac{\pi^4}{90}$, then we can state the following results:*

$$\left|Li_{q+1}\left(-\alpha i\right)Li_{q+1}\left(\overline{\beta}i\right)\right| \leq \frac{\pi^2}{12}\left(\left[Li_{2q}\left(|\alpha|^2\right)Li_{2q}\left(|\beta|^2\right)\right]^{1/2} + \left|Li_{2q}\left(\alpha\overline{\beta}\right)\right|\right),$$

$$\left|Li_{q+2}\left(-\alpha i\right)Li_{q+2}\left(\overline{\beta}i\right)\right| \leq \frac{\pi^2}{180}\left(\left[Li_{2q}\left(\left|\alpha\right|^2\right)Li_{2q}\left(\left|\beta\right|^2\right)\right]^{1/2}+\left|Li_{2q}\left(\alpha\overline{\beta}\right)\right|\right)$$

and

$$\begin{aligned}&\left|Li_{q+3}\left(-\alpha i\right)Li_{q+3}\left(\overline{\beta}i\right)\right|\\ &\leq \frac{\pi^4}{180}\left(\left[Li_{2(q+1)}\left(\left|\alpha\right|^2\right)Li_{2(q+1)}\left(\left|\beta\right|^2\right)\right]^{1/2}+\left|Li_{2(q+1)}\left(\alpha\overline{\beta}\right)\right|\right),\end{aligned}$$

for any $\alpha, \beta \in D(0,1)$ and q an integer.

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