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ON SOME HADAMARD-TYPE INEQUALITIES FOR PRODUCT OF TWO $s-$CONVEX FUNCTIONS ON THE CO-ORDINATES

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Abstract. In this paper Hadamard-type inequalities for product of $s-$convex in the second sense on the co-ordinates on the rectangle from the plane are established.

1. INTRODUCTION

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function and $a, b \in I$ with $a < b$. Then the following double inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

is known as Hadamard's inequality for convex mappings. For particular choice of the function f in (1.1) yields some classical inequalities of means. Both inequalities in (1.1) hold in reversed direction if f is concave. In [12], Orlicz introduced two definitions of $s-$convexity of real valued functions:

Definition 1. *Let $s \in (0, 1]$ be fixed real number. A function $f : (0, \infty] \to \mathbb{R}$ is said to be $s-$convex (in the second sense), or that f belongs to the class K_s^2, if*

$$f(\alpha x + (1-\alpha)y) \leq \alpha^s f(x) + (1-\alpha)^s f(y)$$

holds for all $x, y \in (0, \infty]$ and $\alpha \in [0, 1]$.

Definition 2. *Let $s \in (0, 1]$ be fixed real number. A function $f : (0, \infty] \to \mathbb{R}$ is said to be $s-$convex (in the first sense), or that f belongs to the class K_s^1, if*

$$f(\alpha x + \beta y) \leq \alpha^s f(x) + \beta^s f(y)$$

holds for all $x, y \in (0, \infty]$ with $\alpha^s + \beta^s = 1$, $\alpha, \beta \geq 0$.

It is clear that $s-$convexity mean just the convexity when $s = 1$. In [7], Dragomir and Fitzpatrick proved the following variant of Hadamard's inequality which hold for $s-$convex functions in the second sense:

Theorem 1. *Suppose that $f : [0, \infty) \to [0, \infty)$ is an $s-$convex function in the second sense, where $s \in (0, 1)$ and let $a, b \in [0, \infty)$, $a < b$. If $f \in L_1([a, b])$, then the following inequalities hold:*

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{s+1} \tag{1.2}$$

The constant $k = \frac{1}{s+1}$ is the best possible in the second inequality in (1.2).

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Again in [7], Dragomir and Fitzpatrick also proved the following Hadamard-type inequality for $s-$convex functions in the first sense:

Theorem 2. *Suppose that $f:[0,\infty)\to[0,\infty)$ is an $s-$convex function in the first sense, where $s\in(0,1)$ and let $a,b\in[0,\infty)$. If $f\in L_1([a,b])$ then the following inequalities hold:*

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{f(a)+sf(b)}{s+1} \tag{1.3}$$

The above inequalities are sharp.

A modification for convex functions which is also known as co-ordinated convex functions was introduced as following by Dragomir in [6]. Let us consider a bidimensional interval $\Delta=:[a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b$ and $c<d$. A mapping $f:\Delta\to\mathbb{R}$ is said to be convex on Δ if the following inequality:

$$f(\alpha x+(1-\alpha)z,\alpha y+(1-\alpha)w)\leq\alpha f(x,y)+(1-\alpha)f(z,w)$$

holds, for all $(x,y),(z,w)\in\Delta$ and $\alpha\in[0,1]$.

A function $f:\Delta\to\mathbb{R}$ is said to be convex on the co-ordinates on Δ if the partial mappings $f_y:[a,b]\to\mathbb{R}$, $f_y(u)=f(u,y)$ and $f_x:[c,d]\to\mathbb{R}$, $f_x(v)=f(x,v)$ are convex where defined for all $x\in[a,b]$, $y\in[c,d]$. In the same paper, Dragomir established the following Hadamard-type inequalities for convex functions on the co-ordinates on a rectangle from the plane $\mathbb{R}^2$.

Theorem 3. *Suppose $f:\Delta=[a,b]\times[c,d]\subseteq[0,\infty)\to\mathbb{R}$ is convex function on the co-ordinates on Δ. Then one has the inequalities:*

$$\begin{aligned} f\left(\frac{a+b}{2},\frac{c+d}{2}\right) &\leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx \\ &\leq \frac{f(a,c)+f(b,c)+f(a,d)+f(b,d)}{4} \end{aligned} \tag{1.4}$$

The concept of $s-$convex functions on the co-ordinates in both sense was introduced by Alomari and Darus in [1] and [4].

Definition 3. *Consider the bidimensional interval $\Delta=:[a,b]\times[c,d]$ in $[0,\infty)^2$ with $a<b$ and $c<d$. The mapping $f:\Delta\to\mathbb{R}$ is $s-$convex in the first sense (in the second sense) on Δ if*

$$f(\alpha x+\beta z,\alpha y+\beta w)\leq\alpha^s f(x,y)+\beta^s f(z,w)$$

, holds for all $(x,y),(z,w)\in\Delta$, $\alpha,\beta\geq0$ with $\alpha^s+\beta^s=1$ $(\alpha+\beta=1)$ and for some fixed $s\in(0,1]$. We write $f\in K_s^i$ $(i=1,2)$ which means that f is $s-$convex in the first sense when $i=1$, (in the second sense when $i=2$).

A function $f:\Delta=:[a,b]\times[c,d]\subseteq[0,\infty)^2\to\mathbb{R}$ is called $s-$convex in first sense (in the second sense) on the co-ordinates on Δ if the partial mappings $f_y:[a,b]\to\mathbb{R}$, $f_y(u)=f(u,y)$ and $f_x:[c,d]\to\mathbb{R}$, $f_x(v)=f(x,v)$, are $s-$convex in the first sense (in the second sense) for all $y\in[c,d]$, $x\in[a,b]$ and $s\in(0,1]$, i.e, the partial mappings f_y and f_x are $s-$convex in the first sense (second sense) with some fixed $s\in(0,1]$.

In [1], Alomari and Darus proved the following inequalities for $s-$convex functions (in the second sense) on the co-ordinates on a rectangle from the plane $\mathbb{R}^2$.

Theorem 4. *Suppose $f : \Delta = [a, b] \times [c, d] \subseteq [0, \infty) \to \mathbb{R}$ is $s-$convex function (in the second sense) on the co-ordinates on Δ. Then one has the inequalities:*

$$4^{s-1} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dy dx \leq \frac{f(a, c) + f(b, c) + f(a, d) + f(b, d)}{(s+1)^2} \tag{1.5}$$

Also in [4] (see also [5]), Alomari and Darus established the following inequalities for $s-$convex functions (in the first sense) on the co-ordinates on a rectangle from the plane $\mathbb{R}^2$.

Theorem 5. *Suppose $f : \Delta = [a, b] \times [c, d] \subseteq [0, \infty) \to \mathbb{R}$ is $s-$convex function (in the first sense) on the co-ordinates on Δ . Then one has the inequalities:*

$$f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dy dx \leq \frac{f(a, c) + s f(b, c) + s f(a, d) + s^2 f(b, d)}{(s+1)^2} \tag{1.6}$$

The above inequalities are sharp.

For refinements, counterparts, generalizations and new Hadamard's type inequalities see the papers [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 14, 15]. In [3] (see also [5]), Alomari and Darus introduced new classes of $s-$convex functions on the co-ordinates as following:

Definition 4. *Consider the bidimensional interval $\Delta =: [a, b] \times [c, d]$ in $[0, \infty)^2$ with $a < b$ and $c < d$. The mapping $f : \Delta \to \mathbb{R}$ is $s-$convex in the first sense on Δ if there exist $s_1, s_2 \in (0, 1]$ such that $s = \frac{s_1+s_2}{2}$,*

$$f(\alpha x + \beta z, \alpha y + \beta w) \leq \alpha^{s_1} f(x, y) + \beta^{s_2} f(z, w)$$

holds for all (x, y), $(z, w) \in \Delta$ with $\alpha, \beta \geq 0$ with $\alpha^{s_1} + \beta^{s_2} = 1$ and for some fixed $s_1, s_2 \in (0, 1]$. We denote this class of functions by $MWO^1_{s_1,s_2}$.

Definition 5. *Consider the bidimensional interval $\Delta =: [a, b] \times [c, d]$ in $[0, \infty)^2$ with $a < b$ and $c < d$. The mapping $f : \Delta \to \mathbb{R}$ is $s-$convex in the second sense on Δ if there exist $s_1, s_2 \in (0, 1]$ such that $s = \frac{s_1+s_2}{2}$,*

$$f(\alpha x + \beta z, \alpha y + \beta w) \leq \alpha^{s_1} f(x, y) + \beta^{s_2} f(z, w)$$

holds for all $(x, y), (z, w) \in \Delta$ with $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ and for all fixed $s_1, s_2 \in (0, 1]$. We denote this class of functions by $MWO^2_{s_1,s_2}$.

In [13], Pachpatte established some inequalities for product of convex functions as followings:

Theorem 6. *Let $f, g : [a, b] \subseteq \mathbb{R} \to [0, \infty)$ be convex functions on $[a, b]$, $a < b$. Then*

$$\frac{1}{b-a} \int_a^b f(x) g(x) dx \leq \frac{1}{3} M(a, b) + \frac{1}{6} N(a, b) \tag{1.7}$$

and

$$2 f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) g(x) dx + \frac{1}{6} M(a, b) + \frac{1}{3} N(a, b) \tag{1.8}$$

where $M(a,b) = f(a)g(a) + f(b)g(b)$ *and* $N(a,b) = f(a)g(b) + f(b)g(a)$.

Similar results for $s-$convex functions is due to Kirmaci *et al.* [11] as followings:

Theorem 7. *Let* $f, g : [a,b] \subseteq \mathbb{R} \to \mathbb{R}$ $a, b \in [0,\infty)$, $a < b$, *be functions such that* g *and* fg *are in* $L_1([a,b])$. *If* f *is convex and non-negative on* $[a,b]$ *and if* g *is* $s-$*convex on* $[a,b]$ *for some* $s \in (0,1)$, *then*

$$\begin{aligned} & 2^s f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)g(x)dx \\ & \leq \frac{1}{(s+1)(s+2)}M(a,b) + \frac{1}{s+2}N(a,b) \end{aligned} \tag{1.9}$$

where $M(a,b) = f(a)g(a) + f(b)g(b)$ *and* $N(a,b) = f(a)g(b) + f(b)g(a)$.

Theorem 8. *Let* $f, g : [a,b] \subseteq \mathbb{R} \to \mathbb{R}$ $a, b \in [0,\infty)$, $a < b$, *be functions such that* g *and* fg *are in* $L_1([a,b])$. *If* f *is convex and non-negative on* $[a,b]$ *and if* g *is* $s-$*convex on* $[a,b]$ *for some* $s \in (0,1)$, *then*

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \leq \frac{1}{s+2}M(a,b) + \frac{1}{(s+1)(s+2)}N(a,b) \tag{1.10}$$

where $M(a,b) = f(a)g(a) + f(b)g(b)$ *and* $N(a,b) = f(a)g(b) + f(b)g(a)$.

Theorem 9. *Let* $f, g : [a,b] \subseteq \mathbb{R} \to \mathbb{R}$ $a, b \in [0,\infty)$, $a < b$, *be functions such that* f, g *and* fg *are in* $L_1([a,b])$. *If* f *is* s_1-*convex and* g *is* s_2-*convex on* $[a,b]$ *for some fixed* $s_1, s_2 \in (0,1)$, *then*

$$\begin{aligned} \frac{1}{b-a}\int_a^b f(x)g(x)dx &\leq \frac{1}{s_1+s_2+1}M(a,b) + B(s_1+1, s_2+1)N(a,b) \\ &= \frac{1}{s_1+s_2+1}\left[M(a,b) + s_1 s_2 \frac{\Gamma(s_1)\Gamma(s_2)}{\Gamma(s_1+s_2+1)}N(a,b)\right] \end{aligned} \tag{1.11}$$

where $M(a,b) = f(a)g(a) + f(b)g(b)$ *and* $N(a,b) = f(a)g(b) + f(b)g(a)$.

In the last theorem Beta function of Euler type, defined by

$$B(x,y) = \int_0^t t^{x-1}(1-t)^{y-1}dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

has been used.

The main purpose of the present paper is to establish new Hadamard-type inequalities similar to the above inequalities, but now for product of $s-$convex functions (in the second sense) on the co-ordinates on the rectangle from the plane $\mathbb{R}^2$.

2. MAIN RESULTS

We will start with the following theorem;

Theorem 10. *Let* $f, g : \Delta = [a,b] \times [c,d] \subseteq [0,\infty)^2 \to \mathbb{R}$, $a < b$, $c < d$, *be functions such that* g *and* fg *are in* $L^2(\Delta)$. *If* f *is non-negative and convex on the* *co-ordinates on Δ and if g is $s-$convex in the second sense on the co-ordinates on Δ, for all $s_1, s_2 \in (0,1)$, such that $s = \frac{s_1+s_2}{2}$, then one has the inequality;*

(2.1)

$$\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)dydx \leq \frac{1}{2}\left(p^2+r^2\right)L(a,b,c,d)+\frac{1}{2}\left(pq+rt\right)M(a,b,c,d)+\frac{1}{2}\left(q^2+t^2\right)N(a,b,c,d)$$

where

$$\begin{aligned}
L(a,b,c,d) &= f(a,c)g(a,c)+f(b,c)g(b,c)+f(a,d)g(a,d)+f(b,d)g(b,d)\\
M(a,b,c,d) &= f(a,c)g(a,d)+f(a,d)g(a,c)+f(b,c)g(b,d)+f(b,d)g(b,c)\\
&\quad + f(b,c)g(a,c)+f(b,d)g(a,d)+f(a,c)g(b,c)+f(a,d)g(b,d)\\
N(a,b,c,d) &= f(b,c)g(a,d)+f(b,d)g(a,c)+f(a,c)g(b,d)+f(a,d)g(b,c)
\end{aligned}$$

$$p=\frac{1}{s_2+2},\quad q=\frac{1}{(s_2+1)(s_2+2)},\quad r=\frac{1}{s_1+2},\quad t=\frac{1}{(s_1+1)(s_1+2)}$$

Proof. Since f is convex and g is $s-$convex in the second sense on the co-ordinates on Δ. Therefore the partial mappings

$$f_y:[a,b]\to[0,\infty),\ f_y(x)=f(x,y)$$

and

$$f_x:[c,d]\to[0,\infty),\ f_x(y)=f(x,y)$$

are convex and non-negative on $[a,b]$ and $[c,d]$, respectively. The partial mappings

$$g_y:[a,b]\to[0,\infty),\ g_y(x)=g(x,y)$$

and

$$g_x:[c,d]\to[0,\infty),\ g_x(y)=g(x,y)$$

are s_1-, s_2-convex on $[a,b]$ and $[c,d]$, respectively, for all $x\in[a,b]$, $y\in[c,d]$, for all s_1, $s_2\in(0,1]$, such that $s=\frac{s_1+s_2}{2}$. Now by applying $f_x(y)g_x(y)$ to (1.10) on $[c,d]$, we get

$$\frac{1}{d-c}\int_c^d f_x(y)g_x(y)dy \leq p\left[f_x(c)g_x(c)+f_x(d)g_x(d)\right] + q\left[f_x(c)g_x(d)+f_x(d)g_x(c)\right].$$

That is

$$\frac{1}{d-c}\int_c^d f(x,y)g(x,y)dy \leq p\left[f(x,c)g(x,c)+f(x,d)g(x,d)\right] + q\left[f(x,c)g(x,d)+f(x,d)g(x,c)\right].$$

Integrating over $[a,b]$ with respect to x and dividing both sides by $b-a$, we have

$$\begin{aligned}
&\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)dydx \\
&\leq p\left[\frac{1}{b-a}\int_a^b f(x,c)g(x,c)dx+\frac{1}{b-a}\int_a^b f(x,d)g(x,d)dx\right] \\
&+q\left[\frac{1}{b-a}\int_a^b f(x,c)g(x,d)dx+\frac{1}{b-a}\int_a^b f(x,d)g(x,c)dx\right].
\end{aligned} \tag{2.2}$$

Now by applying (1.10) to each integral on right hand side of (2.2) again, we get

$$\begin{aligned}
\frac{1}{b-a}\int_a^b f(x,c)g(x,c)dx &\leq p\left[f(a,c)g(a,c)+f(b,c)g(b,c)\right] \\
&+q\left[f(a,c)g(b,c)+f(b,c)g(a,c)\right].
\end{aligned}$$

$$\begin{aligned}
\frac{1}{b-a}\int_a^b f(x,d)g(x,d)dx &\leq p\left[f(a,d)g(a,d)+f(b,d)g(b,d)\right] \\
&+q\left[f(a,d)g(b,d)+f(b,d)g(a,d)\right].
\end{aligned}$$

$$\begin{aligned}
\frac{1}{b-a}\int_a^b f(x,c)g(x,d)dx &\leq p\left[f(a,c)g(a,d)+f(b,c)g(b,d)\right] \\
&+q\left[f(a,c)g(b,d)+f(b,c)g(a,d)\right].
\end{aligned}$$

$$\begin{aligned}
\frac{1}{b-a}\int_a^b f(x,d)g(x,c)dx &\leq p\left[f(a,d)g(a,c)+f(b,d)g(b,c)\right] \\
&+q\left[f(a,d)g(b,c)+f(b,d)g(a,c)\right].
\end{aligned}$$

On substitution of these inequalities in (2.2), we obtain

$$\begin{aligned}
&\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)dydx \\
&\leq p^2L(a,b,c,d)+pqM(a,b,c,d)+q^2N(a,b,c,d).
\end{aligned}$$

Similarly, if we apply $f_y(x)g_y(x)$ to (1.10) on $[a,b]$, we get the following result:

$$\begin{aligned}
&\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)dydx \\
&\leq r^2L(a,b,c,d)+rtM(a,b,c,d)+t^2N(a,b,c,d)
\end{aligned}$$

Adding these inequalities and dividing by 2 we get (2.1). □

Theorem 11. *Let $f,g:\Delta=[a,b]\times[c,d]\subseteq\mathbb{R}^2\to\mathbb{R}$, $a<b$, $c<d$, be functions such that g and fg are in $L^2(\Delta)$. If f is non-negative and convex on the co-ordinates on Δ and if g is $s-$convex on the co-ordinates on Δ, for all $s_1,s_2\in(0,1)$, such that*

$s = \frac{s_1+s_2}{2}$, *then one has the inequality;*

$$2^{s_1+s_2} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) g\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)g(x,y)dydx + \frac{1}{2}(q^2+t^2+2pt+2qr)L(a,b,c,d) + \frac{1}{2}(pq+rt+2qt+2rp)M(a,b,c,d) + \frac{1}{2}(p^2+r^2+2pt+2rq)N(a,b,c,d) \tag{2.3}$$

where $L(a,b,c,d)$, $M(a,b,c,d)$, $N(a,b,c,d)$, p, q, r *and* t *as in Theorem 10.*

Proof. Applying $2^{s_1} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) g\left(\frac{a+b}{2}, \frac{c+d}{2}\right)$ to (1.9) and multiplying both sides by 2^{s_2}, we get

$$\begin{aligned} &2^{s_1+s_2} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) g\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\ &\leq \frac{2^{s_2}}{b-a} \int_a^b f\left(x, \frac{c+d}{2}\right) g\left(x, \frac{c+d}{2}\right) dx \\ &+ t\left[2^{s_2} f\left(a, \frac{c+d}{2}\right) g\left(a, \frac{c+d}{2}\right) + 2^{s_2} f\left(b, \frac{c+d}{2}\right) g\left(b, \frac{c+d}{2}\right)\right] \\ &+ r\left[2^{s_2} f\left(a, \frac{c+d}{2}\right) g\left(b, \frac{c+d}{2}\right) + 2^{s_2} f\left(b, \frac{c+d}{2}\right) g\left(a, \frac{c+d}{2}\right)\right]. \end{aligned} \tag{2.4}$$

Similarly, by applying $2^{s_2} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) g\left(\frac{a+b}{2}, \frac{c+d}{2}\right)$ to (1.9) and multiplying both sides by 2^{s_1}, we get

$$\begin{aligned} &2^{s_1+s_2} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) g\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\ &\leq \frac{2^{s_1}}{d-c} \int_c^d f\left(\frac{a+b}{2}, y\right) g\left(\frac{a+b}{2}, y\right) dy \\ &+ q\left[2^{s_1} f\left(\frac{a+b}{2}, c\right) g\left(\frac{a+b}{2}, c\right) + 2^{s_1} f\left(\frac{a+b}{2}, d\right) g\left(\frac{a+b}{2}, d\right)\right] \\ &+ p\left[2^{s_1} f\left(\frac{a+b}{2}, c\right) g\left(\frac{a+b}{2}, d\right) + 2^{s_1} f\left(\frac{a+b}{2}, d\right) g\left(\frac{a+b}{2}, c\right)\right]. \end{aligned} \tag{2.5}$$

Adding (2.4) and (2.5), we have

(2.6)

$$\begin{aligned}
&2^{s_1+s_2+1}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)g\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\
&\leq \frac{2^{s_2}}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)g\left(x,\frac{c+d}{2}\right)dx+\frac{2^{s_1}}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)g\left(\frac{a+b}{2},y\right)dy\\
&+t\left[2^{s_2}f\left(a,\frac{c+d}{2}\right)g\left(a,\frac{c+d}{2}\right)+2^{s_2}f\left(b,\frac{c+d}{2}\right)g\left(b,\frac{c+d}{2}\right)\right]\\
&+r\left[2^{s_2}f\left(a,\frac{c+d}{2}\right)g\left(b,\frac{c+d}{2}\right)+2^{s_2}f\left(b,\frac{c+d}{2}\right)g\left(a,\frac{c+d}{2}\right)\right]\\
&+q\left[2^{s_1}f\left(\frac{a+b}{2},c\right)g\left(\frac{a+b}{2},c\right)+2^{s_1}f\left(\frac{a+b}{2},d\right)g\left(\frac{a+b}{2},d\right)\right]\\
&+p\left[2^{s_1}f\left(\frac{a+b}{2},c\right)g\left(\frac{a+b}{2},d\right)+2^{s_1}f\left(\frac{a+b}{2},d\right)g\left(\frac{a+b}{2},c\right)\right].
\end{aligned}$$

Applying (1.9) to each term within the brackets, we get

$$\begin{aligned}
2^{s_2}f\left(a,\frac{c+d}{2}\right)g\left(a,\frac{c+d}{2}\right) &\leq \frac{1}{d-c}\int_c^d f(a,y)\,g(a,y)\,dy\\
&+t\left[f(a,c)\,g(a,c)+f(a,d)\,g(a,d)\right]\\
&+r\left[f(a,c)\,g(a,d)+f(a,d)\,g(a,c)\right].
\end{aligned}$$

$$\begin{aligned}
2^{s_2}f\left(b,\frac{c+d}{2}\right)g\left(b,\frac{c+d}{2}\right) &\leq \frac{1}{d-c}\int_c^d f(b,y)\,g(b,y)\,dy\\
&+t\left[f(b,c)\,g(b,c)+f(b,d)\,g(b,d)\right]\\
&+r\left[f(b,c)\,g(b,d)+f(b,d)\,g(b,c)\right].
\end{aligned}$$

$$\begin{aligned}
2^{s_2}f\left(a,\frac{c+d}{2}\right)g\left(b,\frac{c+d}{2}\right) &\leq \frac{1}{d-c}\int_c^d f(a,y)\,g(b,y)\,dy\\
&+t\left[f(a,c)\,g(b,c)+f(a,d)\,g(b,d)\right]\\
&+r\left[f(a,c)\,g(b,d)+f(a,d)\,g(b,c)\right].
\end{aligned}$$

$$\begin{aligned}
2^{s_2}f\left(b,\frac{c+d}{2}\right)g\left(a,\frac{c+d}{2}\right) &\leq \frac{1}{d-c}\int_c^d f(b,y)\,g(a,y)\,dy\\
&+t\left[f(b,c)\,g(a,c)+f(b,d)\,g(a,d)\right]\\
&+r\left[f(b,c)\,g(a,d)+f(b,d)\,g(a,c)\right].
\end{aligned}$$

$$\begin{aligned}
2^{s_1}f\left(\frac{a+b}{2},c\right)g\left(\frac{a+b}{2},c\right) &\leq \frac{1}{b-a}\int_a^b f(x,c)g(x,c)dx\\
&+q\left[f(a,c)\,g(a,c)+f(b,c)\,g(b,c)\right]\\
&+p\left[f(a,c)\,g(b,c)+f(b,c)\,g(a,c)\right].
\end{aligned}$$

$$2^{s_1} f\left(\frac{a+b}{2}, d\right) g\left(\frac{a+b}{2}, d\right) \leq \frac{1}{b-a}\int_a^b f(x,d)g(x,d)dx + q\left[f(a,d)\,g(a,d) + f(b,d)\,g(b,d)\right] + p\left[f(a,d)\,g(b,d) + f(b,d)\,g(a,d)\right].$$

$$2^{s_1} f\left(\frac{a+b}{2}, c\right) g\left(\frac{a+b}{2}, d\right) \leq \frac{1}{b-a}\int_a^b f(x,c)g(x,d)dx + q\left[f(a,c)\,g(a,d) + f(b,c)\,g(b,d)\right] + p\left[f(a,c)\,g(b,d) + f(b,c)\,g(a,d)\right].$$

$$2^{s_1} f\left(\frac{a+b}{2}, d\right) g\left(\frac{a+b}{2}, c\right) \leq \frac{1}{b-a}\int_a^b f(x,d)g(x,c)dx + q\left[f(a,d)\,g(a,c) + f(b,d)\,g(b,c)\right] + p\left[f(a,d)\,g(b,c) + f(b,d)\,g(a,c)\right].$$

Substituting these inequalities in (2.6) and simplifying, we obtain

$$\begin{aligned}
&2^{s_1+s_2+1} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) g\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
&\leq \frac{2^{s_2}}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right) g\left(x, \frac{c+d}{2}\right) dx \\
&+ \frac{2^{s_1}}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right) g\left(\frac{a+b}{2}, y\right) dy \\
&+ t\frac{1}{d-c}\int_c^d f(a,y)\,g(a,y)\,dy + t\frac{1}{d-c}\int_c^d f(b,y)\,g(b,y)\,dy \\
&+ r\frac{1}{d-c}\int_c^d f(a,y)\,g(b,y)\,dy + r\frac{1}{d-c}\int_c^d f(b,y)\,g(a,y)\,dy \\
&+ q\frac{1}{b-a}\int_a^b f(x,c)g(x,c)dx + q\frac{1}{b-a}\int_a^b f(x,d)g(x,d)dx \\
&+ p\frac{1}{b-a}\int_a^b f(x,c)g(x,d)dx + p\frac{1}{b-a}\int_a^b f(x,d)g(x,c)dx \\
&+ (q^2+t^2)L(a,b,c,d) + (pq+rt)M(a,b,c,d) + (p^2+r^2)N(a,b,c,d).
\end{aligned} \tag{2.7}$$

Now by applying $2^{s_1} f\left(\frac{a+b}{2}, y\right) g\left(\frac{a+b}{2}, y\right)$ to (1.9), integrating over $[c,d]$ and dividing both sides by $d-c$, we get

$$\begin{aligned}
&\frac{2^{s_1}}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right) g\left(\frac{a+b}{2}, y\right) dy \\
&\leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,g(x,y)\,dxdy \\
&+ t\frac{1}{d-c}\int_c^d f(a,y)\,g(a,y)\,dy + t\frac{1}{d-c}\int_c^d f(b,y)\,g(b,y)\,dy \\
&+ r\frac{1}{d-c}\int_c^d f(a,y)\,g(b,y)\,dy + r\frac{1}{d-c}\int_c^d f(b,y)\,g(a,y)\,dy.
\end{aligned} \tag{2.8}$$

Again by applying $2^{s_2} f\left(x, \frac{c+d}{2}\right) g\left(x, \frac{c+d}{2}\right)$ to (1.9), integrating over $[a,b]$ and dividing both sides by $b-a$, we get

$$
\begin{aligned}
&\frac{2^{s_2}}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right) g\left(x,\frac{c+d}{2}\right) dx \\
&\leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,g(x,y)\,dydx \\
&+ q\frac{1}{b-a}\int_a^b f(x,c)g(x,c)dx + q\frac{1}{b-a}\int_a^b f(x,d)g(x,d)dx \\
&+ p\frac{1}{b-a}\int_a^b f(x,c)g(x,d)dx + p\frac{1}{b-a}\int_a^b f(x,d)g(x,c)dx.
\end{aligned}
\tag{2.9}
$$

Adding (2.8) and (2.9), we have

$$
\begin{aligned}
&\frac{2^{s_2}}{b-a}\int_c^d f\left(x,\frac{c+d}{2}\right) g\left(x,\frac{c+d}{2}\right) dx + \frac{2^{s_1}}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right) g\left(\frac{a+b}{2},y\right) dy \\
&\leq \frac{2}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,g(x,y)\,dydx \\
&+ t\frac{1}{d-c}\int_c^d f(a,y)\,g(a,y)\,dy + t\frac{1}{d-c}\int_c^d f(b,y)\,g(b,y)\,dy \\
&+ r\frac{1}{d-c}\int_c^d f(a,y)\,g(b,y)\,dy + r\frac{1}{d-c}\int_c^d f(b,y)\,g(a,y)\,dy \\
&+ q\frac{1}{b-a}\int_a^b f(x,c)g(x,c)dx + q\frac{1}{b-a}\int_a^b f(x,d)g(x,d)dx \\
&+ p\frac{1}{b-a}\int_a^b f(x,c)g(x,d)dx + p\frac{1}{b-a}\int_a^b f(x,d)g(x,c)dx.
\end{aligned}
\tag{2.10}
$$

Therefore from (2.7) and (2.10), we obtain

$$
\begin{aligned}
&2^{s_1+s_2+1} f\left(\frac{a+b}{2},\frac{c+d}{2}\right) g\left(\frac{a+b}{2},\frac{c+d}{2}\right) \\
&\leq \frac{2}{(b-a)(d-c)}\int_a^b f(x,y)\,g(x,y)\,dxdy \\
&+ 2t\frac{1}{d-c}\int_c^d f(a,y)\,g(a,y)\,dy + 2t\frac{1}{d-c}\int_c^d f(b,y)\,g(b,y)\,dy \\
&+ 2r\frac{1}{d-c}\int_c^d f(a,y)\,g(b,y)\,dy + 2r\frac{1}{d-c}\int_c^d f(b,y)\,g(a,y)\,dy \\
&+ 2q\frac{1}{b-a}\int_a^b f(x,c)g(x,c)dx + 2q\frac{1}{b-a}\int_a^b f(x,d)g(x,d)dx \\
&+ 2p\frac{1}{b-a}\int_a^b f(x,c)g(x,d)dx + 2p\frac{1}{b-a}\int_a^b f(x,d)g(x,c)dx \\
&+ (q^2+t^2)L(a,b,c,d) + (pq+rt)M(a,b,c,d) + (p^2+r^2)N(a,b,c,d).
\end{aligned}
\tag{2.11}
$$

By applying (1.10) to each of the integral in (2.11) and simplifying, we get

$$\begin{aligned}
&2^{s_1+s_2+1} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) g\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
&\leq \frac{2}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)\, g(x,y)\, dydx \\
&+ (q^2+t^2+2pt+2qr)L(a,b,c,d) + (pq+rt+2qt+2rp)M(a,b,c,d) \\
&+ (p^2+r^2+2pt+2rq)N(a,b,c,d)
\end{aligned}$$

Dividing both sides by 2, we get required result. □

Theorem 12. *Let $f, g : \Delta = [a,b] \times [c,d] \subseteq \mathbb{R}^2 \to \mathbb{R}$, $a < b$, $c < d$, be functions such that f, g and fg are in $L_2(\Delta)$. If f is s_1-convex on the co-ordinates on Δ and g is s_2-convex on the co-ordinates on Δ, for some fixed $s_{11}, s_{12}, s_{21}, s_{22} \in (0,1)$, such that $s_1 = \frac{s_{11}+s_{12}}{2}$, $s_2 = \frac{s_{21}+s_{22}}{2}$, then one has the following inequality;*

(2.12)

$$\begin{aligned}
&\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)g(x,y)dydx \\
&\leq \frac{1}{2}\left(p_1^2+r_1^2\right) L(a,b,c,d) + \frac{1}{2}\left(p_1q_1+r_1t_1\right) M(a,b,c,d) + \frac{1}{2}\left(q_1^2+t_1^2\right) N(a,b,c,d)
\end{aligned}$$

where $L(a,b,c,d)$, $M(a,b,c,d)$ and $N(a,b,c,d)$ as defined in Theorem 10 and

$$p_1 = \frac{1}{s_{12}+s_{22}+1}, \quad q_1 = B(s_{12}+1, s_{22}+1), \quad r_1 = \frac{1}{s_{11}+s_{21}+1}, \quad t_1 = B(s_{11}+1, s_{21}+1).$$

Proof. By a similar way to Theorem 10 with $p_1 = \frac{1}{s_{12}+s_{22}+1}$, $q_1 = B(s_{12}+1, s_{22}+1)$, $r_1 = \frac{1}{s_{11}+s_{21}+1}$, $t_1 = B(s_{11}+1, s_{21}+1)$ and thus (2.12) is established. □

Theorem 13. *Let $f, g : \Delta = [a,b] \times [c,d] \subseteq \mathbb{R}^2 \to \mathbb{R}$, $a < b$, $c < d$, be functions such that f, g and fg are in $L_2(\Delta)$. If f is s_1-convex on the co-ordinates on Δ and g is s_2-convex on the co-ordinates on Δ, for some fixed $s_{11}, s_{12}, s_{21}, s_{22} \in (0,1)$, such that $s_1 = \frac{s_{11}+s_{12}}{2}$, $s_2 = \frac{s_{21}+s_{22}}{2}$, then one has the following inequality;*

$$\begin{aligned}
&2^{s_{11}+s_{21}+s_{12}+s_{22}-2} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) g\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
&\leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)g(x,y)dydx \\
&+ \frac{1}{2}(q_1^2+t_1^2+2p_1t_1+2q_1r_1)L(a,b,c,d) \\
&+ \frac{1}{2}(p_1q_1+r_1t_1+2q_1t_1+2r_1p_1)M(a,b,c,d) \\
&+ \frac{1}{2}(p_1^2+r_1^2+2p_1t_1+2r_1q_1)N(a,b,c,d)
\end{aligned} \tag{2.13}$$

where $L(a,b,c,d)$, $M(a,b,c,d)$ and $N(a,b,c,d)$ as defined in Theorem 10 and $p_1 = \frac{1}{s_{12}+s_{22}+1}$, $q_1 = B(s_{12}+1, s_{22}+1)$, $r_1 = \frac{1}{s_{11}+s_{21}+1}$, $t_1 = B(s_{11}+1, s_{21}+1)$.

Proof. By a similar way to Theorem 10 with $p_1 = \frac{1}{s_{12}+s_{22}+1}$, $q_1 = B(s_{12}+1, s_{22}+1)$, $r_1 = \frac{1}{s_{11}+s_{21}+1}$, $t_1 = B(s_{11}+1, s_{21}+1)$, the proof is completed. □

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