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A Three Point Quadrature Rule for Functions of Bounded Variation and Applications

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Abstract. A three point quadrature rule approximating the Riemann integral for a function of bounded variation f by a linear combination with real coefficients of the values $f(a), f(x)$ and $f(b)$ with $x \in [a,b]$ whose sum equalizes $b-a$ is given. Applications for special means inequalities and in establishing *a priory* error bounds for the approximation of selfadjoint operators in Hilbert spaces by spectral families are provided as well.

1. Introduction

In 1999, see [7, Proposition 2] or [11, p. 11], S.S. Dragomir has obtained the following bound for the three point approximation of the Riemann integral $\int_a^b f(t)\,dt$

$$\left|\int_a^b f(t)\,dt-(\alpha-a)f(a)-(\beta-\alpha)f(x)-(b-\beta)f(b)\right| \tag{1.1}$$
$$\leq\left\{\frac{1}{4}(b-a)+\frac{1}{2}\left[\left|x-\frac{a+b}{2}\right|+\left|\alpha-\frac{a+x}{2}\right|+\left|\beta-\frac{x+b}{2}\right|\right]\right.$$
$$\left.+\frac{1}{2}\left|\left|\alpha-\frac{a+x}{2}\right|-\left|\beta-\frac{x+b}{2}\right|\right|\right\}\bigvee_a^b(f)$$

where $a\leq\alpha\leq x\leq\beta\leq b$ and $f:[a,b]\to\mathbb{R}$ is a function of bounded variation on $[a,b]$ while $\bigvee_a^b(f)$ denotes the total variation of f on $[a,b]$.

For $\alpha=a$ and $\beta=b$, we get from (1.1) the following Ostrowski's type inequality firstly obtained in 1999 in [7]

$$\left|\int_a^b f(t)\,dt-(b-a)f(x)\right|\leq\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(f) \tag{1.2}$$

for all $x\in[a,b]$. The constant $\frac{1}{2}$ cannot be replaced by a smaller quantity.

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The best inequality one can get from (1.2) is the following midpoint inequality

$$\left|\int_a^b f(t)\,dt-(b-a)\,f\left(\frac{a+b}{2}\right)\right|\le\frac{1}{2}(b-a)\bigvee_a^b(f). \tag{1.3}$$

Here the constant $\frac{1}{2}$ is also best.

For some recent Ostrowski's type inequalities, see [12], [13], [14], [15], [16], [17], [18] and the references therein.

If $\alpha=x=\beta$, then we get from (1.1) the following generalized trapezoidal inequality also obtained in 1999 [7, Proposition 1]

$$\begin{aligned}&\left|\int_a^b f(t)\,dt-(x-a)\,f(a)-(b-x)\,f(b)\right|\\&\le\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(f)\end{aligned} \tag{1.4}$$

for all $x\in[a,b]$. The constant $\frac{1}{2}$ cannot be replaced by a smaller quantity.

The best inequality one can get from (1.2) is the following trapezoidal inequality

$$\left|\int_a^b f(t)\,dt-\frac{f(a)+f(b)}{2}(b-a)\right|\le\frac{1}{2}(b-a)\bigvee_a^b(f). \tag{1.5}$$

Here the constant $\frac{1}{2}$ is also best.

For recent results on trapezoidal inequality, see [1], [3], [4], [10] and the references therein.

Now, if we take $x=\frac{a+b}{2}$ in (1.1), then we get the inequality

$$\begin{aligned}&\left|\int_a^b f(t)\,dt-(\alpha-a)\,f(a)-(\beta-\alpha)\,f\left(\frac{a+b}{2}\right)-(b-\beta)\,f(b)\right|\\&\le\left\{\frac{1}{4}(b-a)+\frac{1}{2}\left[\left|\alpha-\frac{3a+b}{4}\right|+\left|\beta-\frac{a+3b}{4}\right|\right]\right.\\&\left.+\frac{1}{2}\left|\left|\alpha-\frac{3a+b}{4}\right|-\left|\beta-\frac{a+3b}{4}\right|\right|\right\}\bigvee_a^b(f)\end{aligned} \tag{1.6}$$

for $a\le\alpha\le\frac{a+b}{2}\le\beta\le b$. The best inequality one can obtain from (1.6), as pointed out by Cerone and Dragomir in [11, p. 202], is obtained for $\alpha=\frac{3a+b}{4}$ and $\beta=\frac{a+3b}{4}$ and has the form

$$\left|\int_a^b f(t)\,dt-\frac{b-a}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]\right|\le\frac{1}{4}(b-a)\bigvee_a^b(f). \tag{1.7}$$

The constant $\frac{1}{4}$ is best possible in (1.7).

For other three point quadrature rules with positive coefficients see [2], [5], [6], [9] and the references therein.

We observe that the three point quadrature formula $(\alpha-a)\,f(a)+(\beta-\alpha)\,f(x)+(b-\beta)\,f(b)$ approximating the Riemann integral $\int_a^b f(t)\,dt$ has nonnegative coefficients since $a\le\alpha\le x\le\beta\le b$. The sum of these coefficients is $(b-a)$.

It is therefore natural to put the more general question of approximating $\int_a^b f(t)\,dt$ by a linear combination with real coefficients of the values $f(a),f(x)$

and $f(b)$ with $x \in [a,b]$ whose sum equalize the same $(b-a)$. Some results that address this question for functions of bounded variation are presented below.

Out of these results, some are applied for special means inequalities and in establishing *a priory* error bounds for the approximation of selfadjoint operators in Hilbert spaces by spectral families.

2. The Results

The first result is:

THEOREM 1. *Let $f : [a,b] \to \mathbb{C}$ be a function of bounded variation on $[a,b]$ and α, β, γ be real numbers with $\alpha + \beta + \gamma \neq 0$. Then for any $x \in [a,b]$ we have the inequalities*

$$\left| \frac{\alpha f(a) + \beta f(x) + \gamma f(b)}{\alpha + \beta + \gamma} - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \tag{2.1}$$
$$\leq \max\left\{ \left| \frac{\alpha}{\alpha+\beta+\gamma} \right|, \frac{1}{b-a} \left| x - \frac{(\beta+\gamma)a + \alpha b}{\alpha+\beta+\gamma} \right| \right\} \bigvee_a^x (f)$$
$$+ \max\left\{ \frac{1}{b-a} \left| x - \frac{\gamma a + (\alpha+\beta) b}{\alpha+\beta+\gamma} \right|, \left| \frac{\gamma}{\alpha+\beta+\gamma} \right| \right\} \bigvee_x^b (f)$$
$$:= B_{\alpha,\beta,\gamma}(a,b,x)$$

where

$$B_{\alpha,\beta,\gamma}(a,b,x) \leq \bigvee_a^b (f) \max\left\{ \left| \frac{\alpha}{\alpha+\beta+\gamma} \right|, \left| \frac{\gamma}{\alpha+\beta+\gamma} \right|, \frac{1}{b-a} \left| x - \frac{(\beta+\gamma)a + \alpha b}{\alpha+\beta+\gamma} \right|, \frac{1}{b-a} \left| x - \frac{\gamma a + (\alpha+\beta) b}{\alpha+\beta+\gamma} \right| \right\} \tag{2.2}$$

and

$$B_{\alpha,\beta,\gamma}(a,b,x) \leq \left[\frac{1}{2} \bigvee_a^b (f) + \left| \bigvee_a^x (f) - \bigvee_x^b (f) \right| \right] \tag{2.3}$$
$$\times \left\{ \max\left\{ \left| \frac{\alpha}{\alpha+\beta+\gamma} \right|, \frac{1}{b-a} \left| x - \frac{(\beta+\gamma)a + \alpha b}{\alpha+\beta+\gamma} \right| \right\} + \max\left\{ \frac{1}{b-a} \left| x - \frac{\gamma a + (\alpha+\beta) b}{\alpha+\beta+\gamma} \right|, \left| \frac{\gamma}{\alpha+\beta+\gamma} \right| \right\} \right\}$$

and $\bigvee_a^b (f)$ denotes the total variation of f on $[a,b]$.

As a particular case of interest we have:

Corollary 1. *With the assumptions of Theorem 1 we have the inequalities*

$$\left|\frac{\alpha f(a)+\beta f\left(\frac{a+b}{2}\right)+\gamma f(b)}{\alpha+\beta+\gamma}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{2.4}$$
$$\leq \frac{1}{2|\alpha+\beta+\gamma|}\max\{2|\alpha|,|\beta+\gamma-\alpha|\}\bigvee_a^{\frac{a+b}{2}}(f)$$
$$+\frac{1}{2|\alpha+\beta+\gamma|}\max\{|\alpha+\beta-\gamma|,2|\gamma|\}\bigvee_{\frac{a+b}{2}}^{b}(f)$$
$$:= C_{\alpha,\beta,\gamma}(a,b)$$

where

$$C_{\alpha,\beta,\gamma}(a,b)\leq\frac{1}{2|\alpha+\beta+\gamma|}\times\max\{2|\alpha|,2|\gamma|,|\beta+\gamma-\alpha|,|\alpha+\beta-\gamma|\}\bigvee_a^b(f), \tag{2.5}$$

and

$$C_{\alpha,\beta,\gamma}(a,b)\leq\frac{1}{2|\alpha+\beta+\gamma|}\left[\frac{1}{2}\bigvee_a^b(f)+\left|\bigvee_a^{\frac{a+b}{2}}(f)-\bigvee_{\frac{a+b}{2}}^{b}(f)\right|\right]\times[\max\{2|\alpha|,|\beta+\gamma-\alpha|\}+\max\{|\alpha+\beta-\gamma|,2|\gamma|\}]. \tag{2.6}$$

Remark 1. *We observe that, if $\alpha=\gamma=0$ in the inequality in (2.1), then we get the Ostrowski type inequality*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\left(\frac{x-a}{b-a}\right)\bigvee_a^x(f)+\left(\frac{b-x}{b-a}\right)\bigvee_x^b(f) \tag{2.7}$$

for any $x\in[a,b]$.

Since

$$(x-a)\bigvee_a^x(f)+(b-x)\bigvee_x^b(f)\leq\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(f)$$

and

$$(x-a)\bigvee_a^x(f)+(b-x)\bigvee_x^b(f)\leq\left[\frac{1}{2}\bigvee_a^b(f)+\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|\right](b-a)$$

then we get from (2.7) the known result (1.2) and

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\left[\frac{1}{2}\bigvee_a^b(f)+\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|\right] \tag{2.8}$$

for all $x\in[a,b]$.

If $v \in [a,b]$ is a median point in the sense of bounded variation for the function f on $[a,b]$, namely $\bigvee_a^v (f) = \bigvee_v^b (f)$, then we also have

$$\left| f(v) - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \le \frac{1}{2} \bigvee_a^b (f) . \tag{2.9}$$

Remark 2. *We notice that if $\beta = 2\delta$ and $\gamma = \alpha$ in (2.4), then*

$$\left| \frac{\alpha}{\alpha+\delta} \frac{f(a)+f(b)}{2} + \frac{\delta}{\alpha+\delta} f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \le \frac{1}{2} \frac{\max\{|\alpha|,|\delta|\}}{|\alpha+\delta|} \bigvee_a^b (f) \tag{2.10}$$

where $\alpha, \delta \in \mathbb{R}$ with $\alpha + \delta \neq 0$.

In particular, for $\alpha = \delta = 1$ we get from (2.10) the known inequality (1.7).

If we take in (2.10) $\alpha = 2$ and $\delta = -1$, then we get

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{1}{b-a} \int_a^b f(t)\,dt \right] \right| \le \frac{1}{2} \bigvee_a^b (f) \tag{2.11}$$

The inequality (2.11) is sharp.

For the choice $\alpha = 2, \delta = -4$ we get from (2.10) that

$$\left| \frac{1}{2}\left[\frac{f(a)+f(b)}{2} + \frac{1}{b-a} \int_a^b f(t)\,dt \right] - f\left(\frac{a+b}{2}\right) \right| \le \frac{1}{2} \bigvee_a^b (f) . \tag{2.12}$$

The inequality (2.12) is sharp.

3. Proofs

We use the Montgomery type identity established in [**8**] to write for the functions of bounded variation $f : [a,b] \to \mathbb{C}$ that

$$f(x) = \frac{1}{b-a} \int_a^b f(t)\,dt + \frac{1}{b-a} \int_a^b K(t,x)\,df(t) \tag{3.1}$$

for any $x \in [a,b]$, where the second integral is taken in the Riemann-Stieltjes sense and the kernel K is defined as $K : [a,b] \times [a,b] \to \mathbb{R}$ with

$$K(t,x) = \begin{cases} t-a \text{ for } a \le t \le x, \\ t-b \text{ for } x < t \le b. \end{cases}$$

Writing the representation for a and b we have

$$f(a) = \frac{1}{b-a} \int_a^b f(t)\,dt + \frac{1}{b-a} \int_a^b (t-b)\,df(t) \tag{3.2}$$

and

$$f(b) = \frac{1}{b-a} \int_a^b f(t)\,dt + \frac{1}{b-a} \int_a^b (t-a)\,df(t) . \tag{3.3}$$

Now, if we multiply (3.2) with α, (3.1) with β, (3.3) with γ, add the obtained equalities and divide the sum with $\alpha+\beta+\gamma\neq 0$ we deduce the more general three point representation

$$\frac{\alpha f(a)+\beta f(x)+\gamma f(b)}{\alpha+\beta+\gamma}=\frac{1}{b-a}\int_a^b f(t)\,dt+\frac{1}{b-a}\int_a^b K_{\alpha,\beta,\gamma}(t,x)\,df(t) \tag{3.4}$$

for any $x\in[a,b]$, where the kernel $K_{\alpha,\beta,\gamma}:[a,b]\times[a,b]\to\mathbb{R}$ is given by

$$K_{\alpha,\beta,\gamma}(t,x)=\begin{cases} t-\frac{(\beta+\gamma)a+\alpha b}{\alpha+\beta+\gamma} & \text{for } a\leq t\leq x,\\ \\ t-\frac{\gamma a+(\alpha+\beta)b}{\alpha+\beta+\gamma} & \text{for } x<t\leq b.\end{cases}$$

It is well known that if $p:[c,d]\to\mathbb{C}$ is a continuous function and $v:[c,d]\to\mathbb{C}$ is of bounded variation, then the Riemann-Stieltjes integral $\int_c^d p(t)\,dv(t)$ exists and the following inequality holds

$$\left|\int_c^d p(t)\,dv(t)\right|\leq\max_{t\in[c,d]}|p(t)|\bigvee_c^d(v) \tag{3.5}$$

where $\bigvee_c^d(v)$ denotes the total variation of v on $[c,d]$.

Utilizing (3.4) and (3.7), we have successively

$$\begin{aligned}&\left|\frac{\alpha f(a)+\beta f(x)+\gamma f(b)}{\alpha+\beta+\gamma}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\ &\leq\frac{1}{b-a}\left|\int_a^x\left(t-\frac{(\beta+\gamma)a+\alpha b}{\alpha+\beta+\gamma}\right)df(t)\right| \\ &+\frac{1}{b-a}\left|\int_x^b\left(t-\frac{\gamma a+(\alpha+\beta)b}{\alpha+\beta+\gamma}\right)df(t)\right| \\ &\leq\frac{1}{b-a}\max_{t\in[a,x]}\left|t-\frac{(\beta+\gamma)a+\alpha b}{\alpha+\beta+\gamma}\right|\bigvee_a^x(f) \\ &+\frac{1}{b-a}\max_{t\in[x,b]}\left|t-\frac{\gamma a+(\alpha+\beta)b}{\alpha+\beta+\gamma}\right|\bigvee_x^b(f)\end{aligned} \tag{3.6}$$

for any $x\in[a,b]$, which is an inequality of interest in itself.

Further, we observe that for any c a real number we have the equality

$$\max_{t\in[a,b]}|t-c|=\max\{|c-a|,|b-c|\}. \tag{3.7}$$

Indeed, if $c<a$ then $\max_{t\in[a,b]}|t-c|=b-c$ and $\max\{|c-a|,|b-c|\}=b-c$, if $c\in[a,b]$ then $\max_{t\in[a,b]}|t-c|=\max\{c-a,b-c\}=\frac{1}{2}(b-a)+\left|c-\frac{a+b}{2}\right|$ and if $c>b$ then $\max_{t\in[a,b]}|t-c|=c-a$ and $\max\{|c-a|,|b-c|\}=c-a$.

Now, on making use of (3.7) we have

$$\begin{aligned} &\max_{t\in[a,x]}\left|t-\frac{(\beta+\gamma)a+\alpha b}{\alpha+\beta+\gamma}\right| \\ &=\max\left\{\left|a-\frac{(\beta+\gamma)a+\alpha b}{\alpha+\beta+\gamma}\right|,\left|x-\frac{(\beta+\gamma)a+\alpha b}{\alpha+\beta+\gamma}\right|\right\} \\ &=\max\left\{\left|\frac{\alpha}{\alpha+\beta+\gamma}\right|(b-a),\left|x-\frac{(\beta+\gamma)a+\alpha b}{\alpha+\beta+\gamma}\right|\right\} \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} &\max_{t\in[x,b]}\left|t-\frac{\gamma a+(\alpha+\beta)b}{\alpha+\beta+\gamma}\right| \\ &=\max\left\{\left|x-\frac{\gamma a+(\alpha+\beta)b}{\alpha+\beta+\gamma}\right|,\left|b-\frac{\gamma a+(\alpha+\beta)b}{\alpha+\beta+\gamma}\right|\right\} \\ &=\max\left\{\left|x-\frac{\gamma a+(\alpha+\beta)b}{\alpha+\beta+\gamma}\right|,\left|\frac{\gamma}{\alpha+\beta+\gamma}\right|(b-a)\right\}. \end{aligned} \tag{3.9}$$

Therefore (3.6), (3.8) and (3.9) produce the inequality in (2.1).

The inequalities (2.2) and (2.3) follow by the elementary fact that

$$my+nz\le(m+n)\max\{y,z\}$$

where m,n,y,z are nonnegative real numbers.

Now, in order to prove the sharpness of the inequality (2.11), assume that there exists a $C>0$ such that

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{1}{b-a}\int_a^b f(t)\,dt\right]\right|\le C\bigvee_a^b(f) \tag{3.10}$$

holds for any function of bounded variation and on any interval $[a,b]$.

If we choose $f:[a,b]\to\mathbb{R}$,

$$f(t)=\begin{cases} 1 \text{ if } t=a,\\ 0 \text{ if } t\in(a,b)\\ 1 \text{ if } t=b\end{cases}$$

which is of bounded variation on $[a,b]$ then we get from (3.10) that $1\le 2C$, which proves the sharpness of the inequality.

Similarly, in order to prove the sharpness the inequality (2.12), if we assume that there exists a constant $D>0$ such that

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+\frac{1}{b-a}\int_a^b f(t)\,dt\right]\right|\le D\bigvee_a^b(f). \tag{3.11}$$

If we consider the function $f:[a,b]\to\mathbb{R}$ given by

$$f(t)=\begin{cases} -1 \text{ if } t\in[a,\frac{a+b}{2}),\\ \\ 1 \text{ if } t\in\left[\frac{a+b}{2},b\right]\end{cases}$$

then f is of bounded variation on $[a,b]$, and by (3.11) we get $1\le 2D$ which implies that $D\ge\frac{1}{2}$.

4. A Compounding Rule

We consider the following partition of the interval $[a,b]$,

$$\Delta_n : a = x_0 < x_1 < ... < x_{n-1} < x_n = b.$$

Define $h_k := x_{k+1} - x_k$, $0 \le k \le n-1$ and $\nu(\Delta_n) = \max\{h_k : 0 \le k \le n-1\}$ the norm of the partition Δ_n.

In order to exemplify how we can use the above results in order to produce compounding quadrature rules to approximate the integral $\int_a^b f(t)\,dt$, we consider for $\alpha, \delta \in \mathbb{R}$ with $\alpha + \delta \neq 0$, the two parameters family of *three point quadrature rules*

$$\begin{aligned} T_n(f, \Delta_n, \alpha, \delta) &:= \frac{\alpha}{\alpha+\delta} \sum_{k=0}^{n-1} \left[\frac{f(x_k) + f(x_{k+1})}{2} \right] h_k \\ &\quad + \frac{\delta}{\alpha+\delta} \sum_{k=0}^{n-1} f\left(\frac{x_k + x_{k+1}}{2} \right) h_k. \end{aligned} \tag{4.1}$$

We notice that the family of quadrature rules (4.1) contain the trapezoid rule ($\delta = 0$), the midpoint rule ($\alpha = 0$), the Simpson rule ($\alpha = 1, \delta = 2$) and the arithmetic mean of the trapezoid and midpoint rules ($\alpha = 1, \beta = 1$).

The following proposition provides *a priory* error bounds in approximating the integral $\int_a^b f(t)\,dt$ of the bounded variation f by the compounding quadrature rule $T_n(f, \Delta_n, \alpha, \delta)$.

Proposition 1. *Let $f : [a,b] \to \mathbb{C}$ be a function of bounded variation on $[a,b]$ and α, β be real numbers with $\alpha + \beta \neq 0$. Then*

$$\int_a^b f(t)\,dt = T_n(f, \Delta_n, \alpha, \delta) + R_n(f, \Delta_n, \alpha, \delta) \tag{4.2}$$

and the remainder $R_n(f, \Delta_n, \alpha, \delta)$ satisfies the bounds

$$\begin{aligned} |R_n(f, \Delta_n, \alpha, \delta)| &\le \frac{1}{2} \frac{\max\{|\alpha|, |\delta|\}}{|\alpha+\delta|} \sum_{k=0}^{n-1} h_k \bigvee_{x_k}^{x_{k+1}} (f) \\ &\le \frac{1}{2} \frac{\max\{|\alpha|, |\delta|\}}{|\alpha+\delta|} \nu(\Delta_n) \bigvee_a^b (f). \end{aligned} \tag{4.3}$$

PROOF. Utilizing the generalized triangle inequality and (2.10) we have successively that

$$
\begin{aligned}
&\left|R_n\left(f,\Delta_n,\alpha,\delta\right)\right| \\
&=\left|\sum_{k=0}^{n-1}\int_{x_k}^{x_{k+1}} f\left(t\right)dt-\frac{\alpha}{\alpha+\delta}\sum_{k=0}^{n-1}\left[\frac{f\left(x_k\right)+f\left(x_{k+1}\right)}{2}\right]h_k\right. \\
&\left.-\frac{\delta}{\alpha+\delta}\sum_{k=0}^{n-1}f\left(\frac{x_k+x_{k+1}}{2}\right)h_k\right| \\
&\leq\sum_{k=0}^{n-1}\left|\int_{x_k}^{x_{k+1}} f\left(t\right)dt-\frac{\alpha}{\alpha+\delta}\left[\frac{f\left(x_k\right)+f\left(x_{k+1}\right)}{2}\right]h_k\right. \\
&\left.-\frac{\delta}{\alpha+\delta}\sum_{k=0}^{n-1}f\left(\frac{x_k+x_{k+1}}{2}\right)h_k\right| \\
&\leq\frac{1}{2}\frac{\max\left\{\left|\alpha\right|,\left|\delta\right|\right\}}{\left|\alpha+\delta\right|}\sum_{k=0}^{n-1}h_k\bigvee_{x_k}^{x_{k+1}}\left(f\right)\leq\frac{1}{2}\frac{\max\left\{\left|\alpha\right|,\left|\delta\right|\right\}}{\left|\alpha+\delta\right|}\nu\left(\Delta_n\right)\bigvee_a^b\left(f\right),
\end{aligned}
$$

and the proof is complete. □

5. Applications for Special Means

It is well-known that, if $f:[a,b]\to\mathbb{R}$ is a convex function, then the celebrated Hermite-Hadamard inequality state that

$$
f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f\left(t\right)dt\leq\frac{f\left(a\right)+f\left(b\right)}{2}. \tag{5.1}
$$

Utilizing this fact and the inequalities (2.11) and (2.12) we can state the following result:

PROPOSITION 2. *Assume that $f:[a,b]\to\mathbb{R}$ is a convex function on $[a,b]$. Then*

$$
0\leq\frac{f\left(a\right)+f\left(b\right)}{2}-\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{1}{b-a}\int_a^b f\left(t\right)dt\right]\leq\frac{1}{2}\bigvee_a^b\left(f\right) \tag{5.2}
$$

and

$$
0\leq\frac{1}{2}\left[\frac{f\left(a\right)+f\left(b\right)}{2}+\frac{1}{b-a}\int_a^b f\left(t\right)dt\right]-f\left(\frac{a+b}{2}\right)\leq\frac{1}{2}\bigvee_a^b\left(f\right). \tag{5.3}
$$

The case for concave functions g is similar by applying these inequalities for $f=-g$.

Let us recall the following means:

a) The *arithmetic mean*

$$A\left(a,b\right):=\frac{a+b}{2},\ \ a,b>0,$$

b) The *geometric mean*

$$G\left(a,b\right):=\sqrt{ab};\ \ a,b\geq 0,$$

c) The *harmonic mean*

$$H(a,b) := \frac{2}{\frac{1}{a}+\frac{1}{b}}; \quad a,b>0,$$

d) The *identric mean*

$$I(a,b) := \begin{cases} \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} & \text{if} \quad b \neq a \\ a & \text{if} \quad b=a \end{cases}; \ a,b>0$$

e) The *logarithmic mean*

$$L(a,b) := \begin{cases} \frac{b-a}{\ln b - \ln a} & \text{if} \quad b \neq a \\ a & \text{if} \quad b=a \end{cases}; \quad a,b>0$$

f) The $p-$*logarithmic mean*

$$L_p(a,b) := \begin{cases} \left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}} & \text{if} \quad b \neq a, \ \ p \in \mathbb{R}\backslash\{-1,0\} \\ a & \text{if} \quad b=a \end{cases}; \quad a,b>0.$$

It is well known that, if $L_{-1} := L$ and $L_0 := I$, then the function $\mathbb{R} \ni p \to L_p$ is monotonically strictly increasing. In particular, we have

$$H(a,b) \leq G(a,b) \leq L(a,b) \leq I(a,b) \leq A(a,b).$$

Now, if we consider the power function $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ given by $f(t) = t^p$ then we observe that for $p \in (-\infty,0) \cup [1,\infty)$ the function is convex while for $p \in (0,1)$ the function is concave.

Now, if we apply the inequality (5.2) for the convex function $f(t) = t^p$ we can state that

$$0 \leq A(a^p,b^p) - \frac{1}{2}\left[A^p(a,b) + L_p^p(a,b)\right] \leq \frac{1}{2} \times \begin{cases} b^p - a^p & \text{if } p \geq 1 \\ a^p - b^p & \text{if } p \in (-\infty,0)\setminus\{-1\}. \end{cases} \tag{5.4}$$

In the case of concave functions, the same inequality (5.2) produces the inequality

$$0 \leq \frac{1}{2}\left[A^p(a,b) + L_p^p(a,b)\right] - A(a^p,b^p) \leq \frac{1}{2}(b^p - a^p) \ \text{ if } p \in (0,1). \tag{5.5}$$

Now, if we consider the convex function $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ given by $f(t) = \frac{1}{t}$, then by (5.2) we also have

$$0 \leq H^{-1}(a,b) - \frac{1}{2}\left[A^{-1}(a,b) + L^{-1}(a,b)\right] \leq \frac{1}{2} \cdot \frac{b-a}{ab}. \tag{5.6}$$

Moreover the inequality (5.2) applied for the concave function $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ given by $f(t) = \ln t$ produces the result

$$0 \leq \frac{1}{2}\left[\ln A(a,b) + \ln I(a,b)\right] - \ln G(a,b) \leq \frac{1}{2}\ln\left(\frac{b}{a}\right)$$

which is equivalent with

$$1 \leq \frac{\sqrt{A(a,b) I(a,b)}}{G(a,b)} \leq \sqrt{\frac{b}{a}}. \tag{5.7}$$

Similar results can be obtained if one uses the inequality (5.3), however the details are left to the interested reader.

6. Applications for Selfadjoint Operators in Hilbert Spaces

Let U be a selfadjoint operator on the complex Hilbert space $(H, \langle ., . \rangle)$ with the spectrum $Sp(U)$ included in the interval $[m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its *spectral family*. It is well known that we have the following *spectral representation in terms of the Riemann-Stieltjes integral*:

$$U = \int_{m-0}^{M} \lambda dE_\lambda, \tag{6.1}$$

which in terms of vectors can be written as

$$\langle Ux, y \rangle = \int_{m-0}^{M} \lambda d \langle E_\lambda x, y \rangle, \tag{6.2}$$

for any $x, y \in H$. The function $g_{x,y}(\lambda) := \langle E_\lambda x, y \rangle$ is of *bounded variation* on the interval $[m, M]$ and

$$g_{x,y}(m-0) = 0 \text{ and } g_{x,y}(M) = \langle x, y \rangle$$

for any $x, y \in H$. It is also well known that $g_x(\lambda) := \langle E_\lambda x, x \rangle$ is *monotonic nondecreasing* and *right continuous* on $[m, M]$.

We can state and prove now the following result concerning the numerical approximation of a selfadjoint operator on the complex Hilbert space $(H, \langle ., . \rangle)$.

Theorem 2. *Let A be a selfadjoint operator on the complex Hilbert space $(H, \langle ., . \rangle)$ with the spectrum $Sp(A)$ included in the interval $[m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. We consider the following partition of the interval $[m, M]$*

$$\Delta_n : m = \lambda_0 < \lambda_1 < ... < \lambda_{n-1} < \lambda_n = M.$$

Define $h_k := \lambda_{k+1} - \lambda_k$, $0 \leq k \leq n-1$ and $\nu(\Delta_n) = \max\{h_k : 0 \leq k \leq n-1\}$ the norm of the partition Δ_n. Then for each $\alpha, \delta \in \mathbb{R}$ with $\alpha + \delta \neq 0$ and $x, y \in H$ we have

$$\langle Ax, y \rangle = M \langle x, y \rangle + T_n(\alpha, \delta, \Delta_n, x, y) + R_n(\alpha, \delta, \Delta_n, x, y) \tag{6.3}$$

where

$$\begin{aligned} T_n(\alpha, \delta, \Delta_n, x, y) := & -\frac{\alpha}{\alpha+\delta} \sum_{k=0}^{n-1} \left\langle \left(\frac{E_{\lambda_k} + E_{\lambda_{k+1}}}{2} \right) x, y \right\rangle h_k \\ & - \frac{\delta}{\alpha+\delta} \sum_{k=0}^{n-1} \left\langle E_{\frac{\lambda_k + \lambda_{k+1}}{2}} x, y \right\rangle h_k \end{aligned} \tag{6.4}$$

and the reminder $R_n(\alpha,\delta,\Delta_n,x,y)$ *satisfies the bound*

$$|R_n(\alpha,\delta,\Delta_n,x,y)| \le \frac{1}{2}\frac{\max\{|\alpha|,|\delta|\}}{|\alpha+\delta|}\nu(\Delta_n)\bigvee_m^M(\langle E_{(\cdot)}x,y\rangle) \le \frac{1}{2}\frac{\max\{|\alpha|,|\delta|\}}{|\alpha+\delta|}\nu(\Delta_n)\|x\|\|y\|. \tag{6.5}$$

Proof. On making use of the representation (6.2) and the integration by parts for the Riemann-Stieltjes integral, we have

$$\langle Ax,y\rangle = \int_{m-0}^{M}\lambda d\langle E_\lambda x,y\rangle = \lambda\langle E_\lambda x,y\rangle\Big|_{m-0}^{M} - \int_m^M \langle E_\lambda x,y\rangle d\lambda = M\langle x,y\rangle - \int_m^M \langle E_\lambda x,y\rangle d\lambda \tag{6.6}$$

where the last integral is a Riemann integral, and, similarly

$$\langle Ax,y\rangle = m\langle x,y\rangle + \int_m^M \langle (I-E_\lambda)x,y\rangle d\lambda \tag{6.7}$$

for any $x,y\in H$, where I denotes the identity operator on H.

For $\varphi\in[0,1]$, if we multiply (6.7) with φ, (6.6) with $1-\varphi$ and add the equalities, we get

$$\langle Ax,y\rangle = [\varphi m + (1-\varphi)M]\langle x,y\rangle + \int_m^M \langle (\varphi I - E_\lambda)x,y\rangle d\lambda \tag{6.8}$$

for any $x,y\in H$, which is an inequality of interest in itself as well.

Consider the function $f:[m,M]\to\mathbb{C}$ given by $f(\lambda)=\langle(\varphi I-E_\lambda)x,y\rangle$. If we apply Proposition 1 for this function, we get the representation

$$\begin{aligned}\int_m^M \langle (\varphi I - E_\lambda)x,y\rangle d\lambda &= \frac{\alpha}{\alpha+\delta}\sum_{k=0}^{n-1}\left[\frac{\langle(\varphi I - E_{\lambda_k})x,y\rangle + \langle(\varphi I - E_{\lambda_{k+1}})x,y\rangle}{2}\right]h_k \\ &+ \frac{\delta}{\alpha+\delta}\sum_{k=0}^{n-1}\left\langle\left(\varphi I - E_{\frac{\lambda_k+\lambda_{k+1}}{2}}\right)x,y\right\rangle h_k + R_n(\alpha,\delta,\Delta_n,x,y)\end{aligned} \tag{6.9}$$

for any $x,y\in H$.

Since

$$\begin{aligned}&\sum_{k=0}^{n-1}\left[\frac{\langle(\varphi I - E_{\lambda_k})x,y\rangle + \langle(\varphi I - E_{\lambda_{k+1}})x,y\rangle}{2}\right]h_k \\ &= \varphi(M-m)\langle x,y\rangle - \sum_{k=0}^{n-1}\left\langle\left(\frac{E_{\lambda_k}+E_{\lambda_{k+1}}}{2}\right)x,y\right\rangle h_k\end{aligned}$$

and

$$\sum_{k=0}^{n-1}\left\langle\left(\varphi I - E_{\frac{\lambda_k+\lambda_{k+1}}{2}}\right)x,y\right\rangle h_k = \varphi(M-m)\langle x,y\rangle - \sum_{k=0}^{n-1}\left\langle E_{\frac{\lambda_k+\lambda_{k+1}}{2}}x,y\right\rangle h_k$$

then we get from (6.9)

$$\int_m^M \langle(\varphi I - E_\lambda) x, y\rangle d\lambda = \varphi(M-m)\langle x,y\rangle + T_n(\alpha,\delta,\Delta_n,x,y) + R_n(\alpha,\delta,\Delta_n,x,y)$$

where $T_n(\alpha,\delta,\Delta_n,x,y)$ is defined by (6.4).

Now, on utilizing the equality (6.8) we deduce the representation (6.3).

From Proposition 1 we have the following bound for the remainder $R_n(\alpha,\delta,\Delta_n,x,y)$

$$|R_n(\alpha,\delta,\Delta_n,x,y)| \le \frac{1}{2}\frac{\max\{|\alpha|,|\delta|\}}{|\alpha+\delta|}\nu(\Delta_n)\bigvee_m^M\left(\left\langle\left(\varphi I - E_{(\cdot)}\right)x,y\right\rangle\right). \tag{6.10}$$

Utilizing the properties of total variation, we observe that

$$\bigvee_m^M\left(\left\langle\left(\varphi I - E_{(\cdot)}\right)x,y\right\rangle\right) = \bigvee_m^M\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)$$

for any $\varphi \in [0,1]$ and for any $x,y \in H$, and the first inequality in (6.5) is proved.

If P is a nonnegative operator on H, i.e., $\langle Px,x\rangle \ge 0$ for any $x \in H$, then the following inequality is a generalization of the Schwarz inequality in the Hilbert space H

$$|\langle Px,y\rangle|^2 \le \langle Px,x\rangle\langle Py,y\rangle,$$

for any $x,y \in H$.

To prove the last part of (6.5), observe that if $d : m = t_0 < t_1 < ... < t_{n-1} < t_n = M$ is an arbitrary partition of the interval $[m,M]$, then we have by the Schwarz inequality for nonnegative operators that

$$\begin{aligned}
&\bigvee_m^M\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\\
&= \sup_d\left\{\sum_{i=0}^{n-1}\left|\left\langle\left(E_{t_{i+1}} - E_{t_i}\right)x,y\right\rangle\right|\right\}\\
&\le \sup_d\left\{\sum_{i=0}^{n-1}\left[\left\langle\left(E_{t_{i+1}} - E_{t_i}\right)x,x\right\rangle^{1/2}\left\langle\left(E_{t_{i+1}} - E_{t_i}\right)y,y\right\rangle^{1/2}\right]\right\} := I.
\end{aligned}$$

By the Cauchy-Bunyakovsky-Schwarz inequality for sequences of real numbers we also have that

$$\begin{aligned}
I &\le \sup_d\left\{\left[\sum_{i=0}^{n-1}\left\langle\left(E_{t_{i+1}} - E_{t_i}\right)x,x\right\rangle\right]^{1/2}\left[\sum_{i=0}^{n-1}\left\langle\left(E_{t_{i+1}} - E_{t_i}\right)y,y\right\rangle\right]^{1/2}\right\}\\
&\le \sup_d\left[\sum_{i=0}^{n-1}\left\langle\left(E_{t_{i+1}} - E_{t_i}\right)x,x\right\rangle\right]^{1/2}\sup_d\left[\sum_{i=0}^{n-1}\left\langle\left(E_{t_{i+1}} - E_{t_i}\right)y,y\right\rangle\right]^{1/2}\\
&= \left[\bigvee_m^M\left(\left\langle E_{(\cdot)}x,x\right\rangle\right)\right]^{1/2}\left[\bigvee_m^M\left(\left\langle E_{(\cdot)}y,y\right\rangle\right)\right]^{1/2} = \|x\|\,\|y\|
\end{aligned}$$

for any $x,y \in H$. □

Remark 3. *We also remark that if $\alpha = \delta = 1$ then we get*

$$\langle Ax,y\rangle = M\langle x,y\rangle + T_n(\Delta_n,x,y) + R_n(\Delta_n,x,y) \tag{6.11}$$

where

$$T_n(\Delta_n, x, y) := -\frac{1}{2}\left[\sum_{k=0}^{n-1}\left\langle\left(\frac{E_{\lambda_k}+E_{\lambda_{k+1}}}{2}\right)x, y\right\rangle h_k + \sum_{k=0}^{n-1}\left\langle E_{\frac{\lambda_k+\lambda_{k+1}}{2}}x, y\right\rangle h_k\right] \tag{6.12}$$

and the reminder $R_n(\Delta_n, x, y)$ *satisfies the bound*

$$|R_n(\Delta_n, x, y)| \le \frac{1}{4}\nu(\Delta_n)\bigvee_m^M\left(\left\langle E_{(\cdot)}x, y\right\rangle\right) \le \frac{1}{4}\nu(\Delta_n)\|x\|\|y\|. \tag{6.13}$$

The identity (6.8) established before can be utilized to provide an upper bound for the error in approximating $\langle Ax, y\rangle$ by $[\varphi m + (1-\varphi)M]\langle x, y\rangle$ where $x, y \in H$, as follows:

Proposition 3. *Let A be a selfadjoint operator on the complex Hilbert space $(H, \langle ., .\rangle)$ with the spectrum $Sp(A)$ included in the interval $[m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $\varphi \in [0, 1]$, then*

$$\begin{aligned}&|\langle Ax, y\rangle - [\varphi m + (1-\varphi)M]\langle x, y\rangle| \\ &\le \int_m^M |\langle(\varphi I - E_\lambda)x, y\rangle| d\lambda \le \|y\|\int_m^M \|(\varphi I - E_\lambda)x\| d\lambda \\ &\le (M-m)\left[\left|\varphi - \frac{1}{2}\right| + \frac{1}{2}\right]\|x\|\|y\|\end{aligned} \tag{6.14}$$

for any $x, y \in H$.

Proof. On making use of the representation (6.8), we have by Schwarz's inequality in Hilbert space $(H, \langle ., .\rangle)$ that

$$\begin{aligned}|\langle Ax, y\rangle - [\varphi m + (1-\varphi)M]\langle x, y\rangle| &\le \int_m^M |\langle(\varphi I - E_\lambda)x, y\rangle| d\lambda \\ &\le \|y\|\int_m^M \|(\varphi I - E_\lambda)x\| d\lambda\end{aligned} \tag{6.15}$$

for any $x, y \in H$.

Now, we observe that, by the triangle inequality we have

$$\begin{aligned}\|\varphi x - E_\lambda x\| &= \left\|\left(\varphi - \frac{1}{2}\right)x + \left(\frac{1}{2}x - E_\lambda x\right)\right\| \\ &\le \left|\varphi - \frac{1}{2}\right|\|x\| + \left\|\frac{1}{2}x - E_\lambda x\right\|\end{aligned}$$

for any $x \in H$ and $\lambda \in [m, M]$.

Also, since E_λ are projectors, then we have

$$\left\|E_\lambda x - \frac{1}{2}x\right\| = \left[\langle E_\lambda x, E_\lambda x\rangle - \langle E_\lambda x, x\rangle + \frac{1}{4}\|x\|^2\right]^{1/2} = \frac{1}{2}\|x\|$$

for any $x \in H$ and $\lambda \in [m, M]$.

Consequently

$$\|\varphi x - E_\lambda x\| \le \left[\left|\varphi - \frac{1}{2}\right| + \frac{1}{2}\right]\|x\|$$

for any $x \in H$ and $\lambda \in [m, M]$, which implies that

$$\int_m^M \|(\varphi I - E_\lambda)\, x\|\, d\lambda \leq (M - m) \left[\left|\varphi - \frac{1}{2}\right| + \frac{1}{2}\right] \|x\|$$

for any $x \in H$, and by (6.15) we deduce the desired result (6.14). □

REMARK 4. *The case $\varphi = \frac{1}{2}$ produces the simple inequality*

$$\left|\langle Ax, y\rangle - \frac{m+M}{2}\langle x, y\rangle\right| \leq \int_m^M \left|\left\langle \left(\frac{1}{2}I - E_\lambda\right) x, y\right\rangle\right| d\lambda \leq \frac{1}{2}(M - m)\|x\|\,\|y\| \tag{6.16}$$

for any $x, y \in H$.

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