Received 23/04/11

GENERALIZATIONS OF INTEGRAL INEQUALITIES FOR FUNCTIONS WHOSE SECOND DERIVATIVES ARE CONVEX AND $m-$CONVEX

M. EMIN ÖZDEMIR[■], ALPER EKINCI[♠,★], AND AHMET OCAK AKDEMIR[♠]

ABSTRACT. In this paper, we establish some new integral inequalities for convex and $m-$convex functions by using a new kernel. The analysis used in the proofs is fairly elementary and based on the classical inequalities. We also give some comparisons and applications to special means for our results.

1. Introduction

The following definition is well known in the literature: a function $f : I \to \mathbb{R}, \emptyset \neq I \subseteq \mathbb{R}$, is said to be convex on I if the inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds for every $x, y \in I$ and $t \in [0,1]$.

Definition 1. *[See [2]] A function $f : [0,b] \to \mathbb{R}$ is said to be $m-$convex, where $m \in [0,1]$, if for every $x, y \in [0,b]$ and $t \in [0,1]$ we have :*

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y).$$

For recent results related to $m-$convex functions we refer the interest of readers to the papers [2, 6, 7, 8, 9, 10].

Many authors have been studied on integral inequalities. One of the well known of these inequalities -Simpson's inequality- is given as following:

Theorem 1. *Let $f : [a,b] \to \mathbb{R}$ be a four times continuously differentiable mapping on (a,b) and $\left\|f^{(4)}\right\|_\infty = \sup_{x\in(a,b)} \left|f^{(4)}(x)\right| < \infty$. Then, the following inequality holds:*

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{1}{2880}\left\|f^{(4)}\right\|_\infty (b-a)^4.$$

In [1], Sarıkaya *et al.* proved following lemma and established some new Simpson type inequalities for convex functions:

Lemma 1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable mapping on I° such that $f'' \in L_1[a,b]$, where $a, b \in I$ with $a < b$, then following equality holds:*

$$\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)dx \tag{1.1}$$

$$= (b-a)^2 \int_0^1 k(t) f''(tb + (1-t)a)\,dt$$

1991 *Mathematics Subject Classification.* 26D15.

Key words and phrases. convex functions, $m-$convex functions, Simpson's Inequality, Power mean inequality.

[★]Corresponding Author.

where

$$k(t)=\begin{cases}\frac{t}{2}\left(\frac{1}{3}-t\right) & , \quad t\in\left[0,\frac{1}{2}\right)\\ (1-t)\left(\frac{t}{2}-\frac{1}{3}\right), & \quad t\in\left[\frac{1}{2},1\right]\end{cases}.$$

Theorem 2. *(See [1]) Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be twice differentiable mapping on I° such that $f''\in L_1[a,b]$, where $a,b\in I$ with $a<b$. if $|f''|$ is a convex on $[a,b]$, then the following inequality holds:*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \tag{1.2}$$
$$\leq \frac{(b-a)^2}{162}\left[\left|f''(a)\right|+\left|f''(b)\right|\right].$$

Theorem 3. *(See [1]) Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be twice differentiable mapping on I° such that $f''\in L_1[a,b]$, where $a,b\in I$ with $a<b$. If $|f''|^q$ is a convex on $[a,b]$ and $q\geq 1$, then the following inequality holds:*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \tag{1.3}$$
$$\leq (b-a)^2\left(\frac{1}{162}\right)^{1-\frac{1}{q}}\left\{\left(\frac{59}{3^5 2^7}\left|f''(b)\right|^q+\frac{133}{3^5 2^7}\left|f''(a)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{133}{3^5 2^7}\left|f''(b)\right|^q+\frac{59}{3^5 2^7}\left|f''(a)\right|^q\right)^{\frac{1}{q}}\right\}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

For recent refinements, counterparts, generalizations and new Simpson's type inequalities, see the papers [3, 4, 5].

The main purpose of this paper is to give some generalizations of integral inequalities for convex and $m-$convex functions by using a more general lemma similar to Lemma 1. Some comparisons and applications to special means related to our results are also given.

2. Main Results

Throughout this paper, we will assume that $I\subset\mathbb{R}$. To prove our main theorems we need the following lemma which involving a new kernel.

Lemma 2. *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be a twice differentiable mapping on I° such that $f''\in L_1[a,b]$, where $a,b\in I$ with $a<b$ and $r\in\mathbb{R}^+$ then the following equality holds:*

$$\frac{1}{r(r+1)}\left[f(a)+f(b)\right]+\frac{2}{r+1}f\left(\frac{a+b}{2}\right)-\frac{2}{r(b-a)}\int_a^b f(x)\,dx \tag{2.1}$$
$$=(b-a)^2\int_0^1 k(t)f''(tb+(1-t)a)\,dt$$

where

$$k(t)=\begin{cases}\frac{t}{r}\left(\frac{1}{r+1}-t\right) & , \quad t\in\left[0,\frac{1}{2}\right)\\ (1-t)\left(\frac{t}{r}-\frac{1}{r+1}\right), & \quad t\in\left[\frac{1}{2},1\right]\end{cases}.$$

Proof. By definition of $k(t)$, we can write

$$\begin{aligned} I &= \int_0^1 k(t) f''(tb+(1-t)a)\,dt \\ &= \int_0^{\frac{1}{2}} \frac{t}{r}\left(\frac{1}{r+1}-t\right) f''(tb+(1-t)a)\,dt + \int_{\frac{1}{2}}^1 (1-t)\left(\frac{t}{r}-\frac{1}{r+1}\right) f''(tb+(1-t)a)\,dt. \end{aligned}$$

Integrating the right hand side of above equality by parts twice, we have

$$\begin{aligned} & \int_0^{\frac{1}{2}} \frac{t}{r}\left(\frac{1}{r+1}-t\right) f''(tb+(1-t)a)\,dt \\ = & -\frac{r-1}{4r(r+1)(b-a)} f'\left(\frac{a+b}{2}\right) \\ & +\frac{1}{(b-a)^2}\left[\frac{1}{r+1} f\left(\frac{a+b}{2}\right)+\frac{1}{r(r+1)} f(a)-\frac{2}{r}\int_0^{\frac{1}{2}} f(tb+(1-t)a)\,dt\right] \end{aligned}$$

and

$$\begin{aligned} & \int_{\frac{1}{2}}^1 (1-t)\left(\frac{t}{r}-\frac{1}{r+1}\right) f''(tb+(1-t)a)\,dt \\ = & \frac{r-1}{4r(r+1)(b-a)} f'\left(\frac{a+b}{2}\right) \\ & +\frac{1}{(b-a)^2}\left[\frac{1}{r+1} f\left(\frac{a+b}{2}\right)+\frac{1}{r(r+1)} f(b)-\frac{2}{r}\int_0^{\frac{1}{2}} f(tb+(1-t)a)\,dt\right]. \end{aligned}$$

By addition and using the change of variable $x = tb+(1-t)a$ for $t \in [0,1]$ and multiplying the both sides by $(b-a)^2$, we obtain (2.1). □

Remark 1. *If we choose* $r = 2$ *in* (2.1), *we get the equality* (1.1).

Theorem 4. *Let* $f : I \subset \mathbb{R} \to \mathbb{R}$ *be a twice differentiable mapping on* I° *such that* $f'' \in L_1[a,b]$, *where* $a, b \in I$ *with* $a < b$ *and* $r \in \mathbb{R}^+$. *If* $|f''|$ *is convex, then one has the following inequality;*

$$\begin{aligned} & \left|\frac{1}{r(r+1)}[f(a)+f(b)]+\frac{2}{r+1} f\left(\frac{a+b}{2}\right)-\frac{2}{r(b-a)}\int_a^b f(x)\,dx\right| \qquad (2.2) \\ \le \; & M(b-a)^2\left[\left|f''(a)\right|+\left|f''(b)\right|\right] \end{aligned}$$

where $M = \frac{r^3-3r+6}{24r(r+1)^3}$.

Proof. From Lemma 2 and by using convexity of $|f''(x)|$, we obtain

$$\left|\frac{1}{r(r+1)}\left[f(a)+f(b)\right]+\frac{2}{r+1}f\left(\frac{a+b}{2}\right)-\frac{2}{r}\int_0^1 f\left(tb+(1-t)a\right)dt\right| \tag{2.3}$$

$$\begin{aligned}
&\leq (b-a)^2\left\{\int_0^{\frac{1}{2}}\left|\frac{t}{r}\left(\frac{1}{r+1}-t\right)\right|\left[t\left|f''(b)\right|+(1-t)\left|f''(a)\right|dt\right]\right.\\
&\quad\left.+\int_{\frac{1}{2}}^1\left|(1-t)\left(\frac{t}{r}-\frac{1}{r+1}\right)\right|\left[t\left|f''(b)\right|+(1-t)\left|f''(a)\right|dt\right]\right\}\\
&= (b-a)^2\left(J_1+J_2\right)
\end{aligned}$$

where

$$\begin{aligned}
J_1 &= \int_0^{\frac{1}{2}}\left|\frac{t}{r}\left(\frac{1}{r+1}-t\right)\right|\left[t\left|f''(b)\right|+(1-t)\left|f''(a)\right|dt\right]\\
J_2 &= \int_{\frac{1}{2}}^1\left|(1-t)\left(\frac{t}{r}-\frac{1}{r+1}\right)\right|\left[t\left|f''(b)\right|+(1-t)\left|f''(a)\right|dt\right].
\end{aligned}$$

By a little hard computation, one can see that

$$\begin{aligned}
J_1 &= \int_0^{\frac{1}{r+1}}\left|\frac{t}{r}\left(\frac{1}{r+1}-t\right)\right|\left[t\left|f''(b)\right|+(1-t)\left|f''(a)\right|dt\right]\\
&\quad+\int_{\frac{1}{r+1}}^{\frac{1}{2}}\left|\frac{t}{r}\left(t-\frac{1}{r+1}\right)\right|\left[t\left|f''(b)\right|+(1-t)\left|f''(a)\right|dt\right]\\
&= M_1\left|f''(b)\right|+M_2\left|f''(a)\right|
\end{aligned}$$

and

$$\begin{aligned}
J_2 &= \int_{\frac{1}{2}}^{\frac{r}{r+1}}\left|(1-t)\left(\frac{1}{r+1}-\frac{t}{r}\right)\right|\left[t\left|f''(b)\right|+(1-t)\left|f''(a)\right|dt\right]\\
&\quad+\int_{\frac{r}{r+1}}^1\left|(1-t)\left(\frac{t}{r}-\frac{1}{r+1}\right)\right|\left[t\left|f''(b)\right|+(1-t)\left|f''(a)\right|dt\right]\\
&= M_1\left|f''(a)\right|+M_2\left|f''(b)\right|
\end{aligned}$$

where

$$M_1=\frac{3r^4+4r^3-6r^2-12r+27}{192r\left(r+1\right)^4}\text{ and }M_2=\frac{5r^4+4r^3-18r^2+36r+21}{192r\left(r+1\right)^4}.$$

By taking into account J_1, J_2, M_1 and M_2 in (2.3), we obtain

$$\begin{aligned}
J_1+J_2 &= \left(M_1+M_2\right)\left[\left|f''(a)\right|+\left|f''(b)\right|\right]\\
&= M\left[\left|f''(a)\right|+\left|f''(b)\right|\right]
\end{aligned}$$

which completes the proof. □

Corollary 1. *If we take* $r=1$ *in* (2.2) *we obtain the following inequality:*

$$\left|\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)-\frac{2}{b-a}\int_a^b f(x)\,dx\right| \tag{2.4}$$
$$\leq \frac{(b-a)^2}{48}\left[\left|f''(a)\right|+\left|f''(b)\right|\right].$$

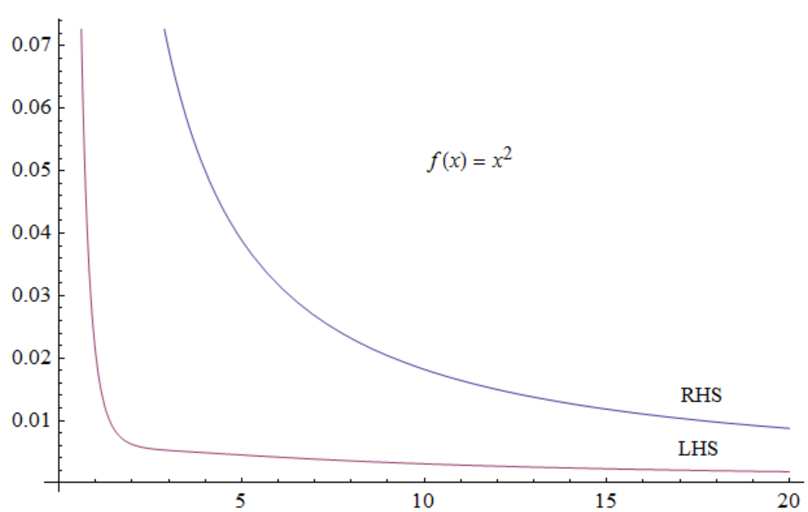
Corollary 2. *Following table shows the results in* (2.2) *for several values of* r*. In the table the left hand side of* (2.2) *is given by LHS and the right one is given by RHS.*

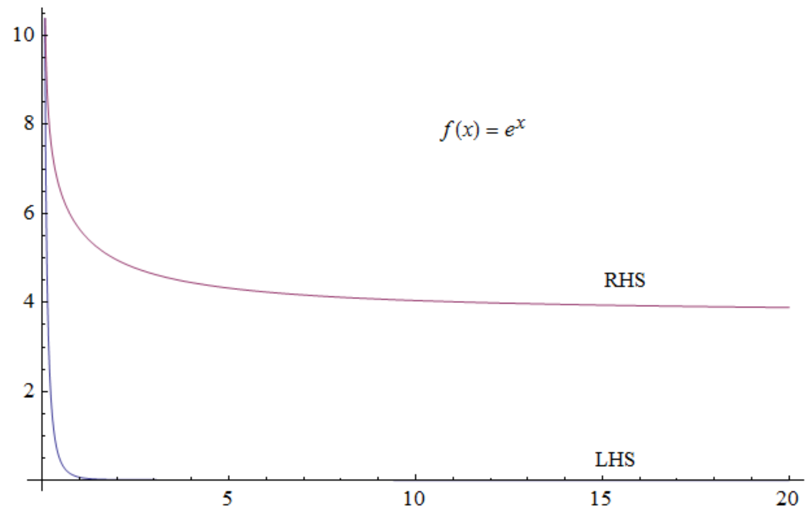
r	*LHS*	*RHS*
1	$\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)-\frac{2}{(b-a)}\int_a^b f(x)\,dx$	$\frac{(b-a)^2}{48}\left[\|f''(a)\|+\|f''(b)\|\right]$
2	$\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{(b-a)}\int_a^b f(x)dx$	$\frac{(b-a)^2}{162}\left[\|f''(a)\|+\|f''(b)\|\right]$
3	$\frac{1}{12}\left[f(a)+6f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{2}{3(b-a)}\int_a^b f(x)dx$	$\frac{(b-a)^2}{192}\left[\|f''(a)\|+\|f''(b)\|\right]$
24	$\frac{1}{600}\left[f(a)+48f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{12(b-a)}\int_a^b f(x)dx$	$\frac{(b-a)^2}{654}\left[\|f''(a)\|+\|f''(b)\|\right]$
30	$\frac{1}{930}\left[f(a)+30f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{15(b-a)}\int_a^b f(x)dx$	$\frac{(b-a)^2}{797}\left[\|f''(a)\|+\|f''(b)\|\right]$

From the table one can see that if we take $r\to\infty$ we obtain smaller upper bounds for the inequality (2.2).

Remark 2. *If we take* $r=2$ *in* (2.2) *we obtain* (1.2).

Example 1. *Under the assumptions on Theorem 4 if we choose* $[a,b]=[0,1]$ *then, we can give following graphics for some special cases of* f *by Mathematica programme . In these graphics* $x-axis$ *shows values of* $r\in\mathbb{R}^+$*:*





Theorem 5. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable mapping on I° such that $f'' \in L_1[a,b]$ and if $|f''|^q$ is convex on $[a,b]$ where $a, b \in I$ with $a < b$, $r \in \mathbb{R}^+$ and $q \geq 1$, then the following inequality holds:*

$$\left| \frac{1}{r(r+1)} [f(a)+f(b)] + \frac{2}{r+1} f\left(\frac{a+b}{2}\right) - \frac{2}{r(b-a)} \int_a^b f(x)\,dx \right|$$
$$\leq (b-a)^2 \left(\frac{r^3-3r+6}{24r(r+1)^3} \right)^{1-\frac{1}{q}} \left\{ \left[\left(M_1 |f''(b)|^q + M_2 |f''(a)|^q \right)^{\frac{1}{q}} \right] + \left[\left(M_2 |f''(b)|^q + M_1 |f''(a)|^q \right)^{\frac{1}{q}} \right] \right\}$$

where $M_1 = \frac{3r^4+4r^3-6r^2-12r+27}{192r(r+1)^4}$ and $M_2 = \frac{5r^4+4r^3-18r^2+36r+21}{192r(r+1)^4}$.

Proof. From Lemma 2 and by using well known power-mean inequality for $q \geq 1$, we have

$$\left| \frac{1}{r(r+1)} [f(a)+f(b)] + \frac{2}{r+1} f\left(\frac{a+b}{2}\right) - \frac{2}{r(b-a)} \int_a^b f(x)\,dx \right|$$
$$\leq (b-a)^2 \int_0^1 |k(t)| \left| f''(tb+(1-t)a) \right| dt$$
$$\leq (b-a)^2 \left(\int_0^{\frac{1}{2}} \left| \frac{t}{r}\left(\frac{1}{r+1}-t\right) \right| \right)^{1-\frac{1}{q}} \left(\int_0^{\frac{1}{2}} \left| \frac{t}{r}\left(\frac{1}{r+1}-t\right) \right| \left| f''(tb+(1-t)a) \right|^q dt \right)^{\frac{1}{q}}$$
$$+ \left(\int_{\frac{1}{2}}^1 \left| (1-t)\left(\frac{t}{r}-\frac{1}{r+1}\right) \right| \right)^{1-\frac{1}{q}} \left(\int_{\frac{1}{2}}^1 \left| (1-t)\left(\frac{t}{r}-\frac{1}{r+1}\right) \right| \left| f''(tb+(1-t)a) \right|^q dt \right)^{\frac{1}{q}}.$$

On the other hand, since $|f''|^q$ is convex on $[a,b]$, we have

$$\begin{aligned} & \int_0^{\frac{1}{2}} \left| \frac{t}{r}\left(\frac{1}{r+1}-t\right)\right| \left|f''\left(tb+(1-t)a\right)\right|^q dt \qquad (2.5) \\ \leq & \int_0^{\frac{1}{2}} \left| \frac{t}{r}\left(\frac{1}{r+1}-t\right)\right| \left[t\left|f''(b)\right|^q+(1-t)\left|f''(a)\right|^q\right] dt \\ = & \int_0^{\frac{1}{r+1}} \left| \frac{t}{r}\left(\frac{1}{r+1}-t\right)\right| \left[t\left|f''(b)\right|^q+(1-t)\left|f''(a)\right|^q\right] dt \\ & +\int_{\frac{1}{r+1}}^{\frac{1}{2}} \left| \frac{t}{r}\left(t-\frac{1}{r+1}\right)\right| \left[t\left|f''(b)\right|^q+(1-t)\left|f''(a)\right|^q\right] dt \\ = & M_1\left|f''(b)\right|^q+M_2\left|f''(a)\right|^q \end{aligned}$$

and

$$\begin{aligned} & \int_{\frac{1}{2}}^{1} \left| (1-t)\left(\frac{t}{r}-\frac{1}{r+1}\right)\right| \left|f''\left(tb+(1-t)a\right)\right|^q dt \qquad (2.6) \\ \leq & \int_{\frac{1}{2}}^{1} \left| (1-t)\left(\frac{t}{r}-\frac{1}{r+1}\right)\right| \left[t\left|f''(b)\right|^q+(1-t)\left|f''(a)\right|^q\right] dt \\ = & \int_{\frac{1}{2}}^{\frac{r}{r+1}} \left| (1-t)\left(\frac{t}{r}-\frac{1}{r+1}\right)\right| \left[t\left|f''(b)\right|^q+(1-t)\left|f''(a)\right|^q\right] dt \\ & +\int_{\frac{r}{r+1}}^{1} \left| (1-t)\left(\frac{1}{r+1}-\frac{t}{r}\right)\right| \left[t\left|f''(b)\right|^q+(1-t)\left|f''(a)\right|^q\right] dt \\ = & M_2\left|f''(b)\right|^q+M_1\left|f''(a)\right|^q \end{aligned}$$

where $M_1=\frac{3r^4+4r^3-6r^2-12r+27}{192r(r+1)^4}$ and $M_2=\frac{5r^4+4r^3-18r^2+36r+21}{192r(r+1)^4}$.

From (2.5) and (2.6) and by using the fact that

$$\int_0^{\frac{1}{2}} \left| \frac{t}{r}\left(\frac{1}{r+1}-t\right)\right| = \int_{\frac{1}{2}}^{1} \left| (1-t)\left(\frac{t}{r}-\frac{1}{r+1}\right)\right| = \frac{r^3-3r+6}{24r(r+1)^3}$$

we deduce

$$\begin{aligned} & \left| \frac{1}{r(r+1)}\left[f(a)+f(b)\right]+\frac{2}{r+1}f\left(\frac{a+b}{2}\right)-\frac{2}{r(b-a)}\int_a^b f(x)\,dx \right| \\ \leq & (b-a)^2\left(\frac{r^3-3r+6}{24r(r+1)^3}\right)^{1-\frac{1}{q}} \left\{\left[\left(M_1\left|f''(b)\right|^q+M_2\left|f''(a)\right|^q\right)^{\frac{1}{q}}\right]\right. \\ & \left.+\left[\left(M_2\left|f''(b)\right|^q+M_1\left|f''(a)\right|^q\right)^{\frac{1}{q}}\right]\right\}. \end{aligned}$$

which completes the proof. □

Corollary 3. *i) Under the assumptions of Theorem 5, if we choose $r = 1$ we get the following inequality:*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)-\frac{2}{r(b-a)}\int_a^b f(x)\,dx\right| \\ \leq\ & (b-a)^2\left(\frac{1}{48}\right)^{1-\frac{1}{q}}\left\{\left[\left(\frac{1}{192}\left|f''(b)\right|^q+\frac{3}{192}\left|f''(a)\right|^q\right)^{\frac{1}{q}}\right]\right. \\ &\left.+\left[\left(\frac{3}{192}\left|f''(b)\right|^q+\frac{1}{192}\left|f''(a)\right|^q\right)^{\frac{1}{q}}\right]\right\}\end{aligned}$$

ii) Let $a_1 = \frac{1}{192}\left|f''(b)\right|^q$, $b_1 = \frac{3}{192}\left|f''(a)\right|^q$, $a_2 = \frac{3}{192}\left|f''(b)\right|^q$, $b_2 = \frac{1}{192}\left|f''(a)\right|^q$.

Here $0 < \frac{1}{q} < 1$, for $q > 1$, using the fact that

$$\sum_{k=1}^{n}(a_k+b_k)^s \leq \sum_{k=1}^{n}(a_k)^s+\sum_{k=1}^{n}(b_k)^s$$

for $0 \leq s \leq 1$, $a_1, ..., a_n \geq 0$, $b_1, ...b_n \geq 0$, we get the following inequality

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)-\frac{2}{r(b-a)}\int_a^b f(x)\,dx\right| \\ \leq\ & (b-a)^2\left(\frac{1}{48}\right)^{1-\frac{1}{q}}\left\{\frac{1}{192^{1/q}}\left|f''(b)\right|+\left(\frac{3}{192}\right)^{1/q}\left|f''(a)\right|\right\}.\end{aligned}$$

Now for $q \to \infty$, we get

$$\left|\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)-\frac{2}{r(b-a)}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)^2}{24}\left(\left|f''(a)\right|+\left|f''(b)\right|\right)$$

Theorem 6. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable mapping on I° such that $f'' \in L_1[a,b]$ and $|f''|$ is $m-$convex function with $m \in (0,1]$, where $a, b \in I$ with $a < b$ and $r \in \mathbb{R}^+$*

$$\begin{aligned}&\left|\frac{1}{r(r+1)}[f(a)+f(b)]+\frac{2}{r+1}f\left(\frac{a+b}{2}\right)-\frac{2}{r(b-a)}\int_a^b f(x)\,dx\right| \qquad (2.7)\\ \leq\ & (b-a)^2 M\left[m\left|f''\left(\frac{a}{m}\right)\right|+\left|f''(b)\right|\right]\end{aligned}$$

where $M = \frac{r^3-3r+6}{24r(r+1)^3}$.

Proof. From Lemma 2 and using the property of absolute value, we can write

$$\begin{aligned}&\left|\frac{1}{r(r+1)}[f(a)+f(b)]+\frac{2}{r+1}f\left(\frac{a+b}{2}\right)-\frac{2}{r(b-a)}\int_a^b f(x)\,dx\right| \\ \leq\ & (b-a)^2\int_0^1\left|k(t)f''(tb+(1-t)a)\right|dt.\end{aligned}$$

Since $|f''|$ is $m-$convex on $[a,b]$, we know that for any $t \in [0,1]$

$$\begin{aligned}\left|f''(tb+(1-t)a)\right| &= \left|f''\left(tb+m(1-t)\frac{a}{m}\right)\right| \\ &\leq t\left|f''(b)\right|+m(1-t)\left|f''\left(\frac{a}{m}\right)\right|\end{aligned}$$

hence it folllows that

$$\begin{aligned}
&(b-a)^2\int_0^1 |k(t)| \left[t\left|f''(b)\right| + m(1-t)\left|f''\left(\frac{a}{m}\right)\right|\right] \\
= \; & (b-a)^2 \left\{ \int_0^{\frac{1}{2}} \left|\frac{t}{r}\left(\frac{1}{r+1}-t\right)\right| \left[t\left|f''(b)\right| + m(1-t)\left|f''\left(\frac{a}{m}\right)\right| dt\right] \right. \\
& \left. + \int_{\frac{1}{2}}^1 \left|(1-t)\left(\frac{t}{r}-\frac{1}{r+1}\right)\right| \left[t\left|f''(b)\right| + m(1-t)\left|f''\left(\frac{a}{m}\right)\right| dt\right] \right\} \\
= \; & J_1 + J_2
\end{aligned}$$

where

$$\begin{aligned}
J_1 &= \int_0^{\frac{1}{2}} \left|\frac{t}{r}\left(\frac{1}{r+1}-t\right)\right| \left[t\left|f''(b)\right| + m(1-t)\left|f''\left(\frac{a}{m}\right)\right| dt\right] \\
J_2 &= \int_{\frac{1}{2}}^1 \left|(1-t)\left(\frac{t}{r}-\frac{1}{r+1}\right)\right| \left[t\left|f''(b)\right| + m(1-t)\left|f''\left(\frac{a}{m}\right)\right| dt\right].
\end{aligned}$$

If we compute J_1 and J_2, we have

$$\begin{aligned}
J_1 &= \int_0^{\frac{1}{r+1}} \left|\frac{t}{r}\left(\frac{1}{r+1}-t\right)\right| \left[t\left|f''(b)\right| + m(1-t)\left|f''\left(\frac{a}{m}\right)\right| dt\right] \\
&\quad + \int_{\frac{1}{r+1}}^{\frac{1}{2}} \left|\frac{t}{r}\left(t-\frac{1}{r+1}\right)\right| \left[t\left|f''(b)\right| + m(1-t)\left|f''\left(\frac{a}{m}\right)\right| dt\right] \\
&= M_1\left|f''(b)\right| + M_2 m\left|f''\left(\frac{a}{m}\right)\right|
\end{aligned}$$

and

$$\begin{aligned}
J_2 &= \int_{\frac{1}{2}}^{\frac{r}{r+1}} \left|(1-t)\left(\frac{1}{r+1}-\frac{t}{r}\right)\right| \left[t\left|f''(b)\right| + m(1-t)\left|f''\left(\frac{a}{m}\right)\right| dt\right] \\
&\quad + \int_{\frac{r}{r+1}}^1 \left|(1-t)\left(\frac{t}{r}-\frac{1}{r+1}\right)\right| \left[t\left|f''(b)\right| + m(1-t)\left|f''\left(\frac{a}{m}\right)\right| dt\right] \\
&= M_1 m\left|f''\left(\frac{a}{m}\right)\right| + M_2\left|f''(b)\right|
\end{aligned}$$

where $M_1 = \frac{3r^4+4r^3-6r^2-12r+27}{192r(r+1)^4}$ and $M_2 = \frac{5r^4+4r^3-18r^2+36r+21}{192r(r+1)^4}$.
By taking into account J_1 and J_2, we obtain the desired result. □

Theorem 7. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable mapping on I° such that $f'' \in L_1[a,b]$ and if $|f''|^q$ is $m-$convex on $[a,b]$ and $m \in (0,1]$ where $a, b \in I$ with $a < b$,*

$r\in\mathbb{R}^{+}$ and $q\geq 1$, then the following inequality holds;

$$\left|\frac{1}{r(r+1)}\left[f(a)+f(b)\right]+\frac{2}{r+1}f\left(\frac{a+b}{2}\right)-\frac{2}{r(b-a)}\int_a^b f(x)\,dx\right| \tag{2.8}$$
$$\leq (b-a)^2\left(\frac{r^3-3r+6}{24r(r+1)^3}\right)^{1-\frac{1}{q}}\left\{\left\{\left(M_1\left|f''(b)\right|^q+M_2m\left|f''(\frac{a}{m})\right|^q\right)^{\frac{1}{q}}\right\}\right.$$
$$\left.+\left(M_2\left|f''(b)\right|^q+M_1m\left|f''(\frac{a}{m})\right|^q\right)^{\frac{1}{q}}\right\}$$

where $M_1=\frac{3r^4+4r^3-6r^2-12r+27}{192r(r+1)^4}$ and $M_2=\frac{5r^4+4r^3-18r^2+36r+21}{192r(r+1)^4}$.

Proof. From Lemma 2 and by using the well known power-mean inequality, we have

$$\left|\frac{1}{r(r+1)}\left[f(a)+f(b)\right]+\frac{2}{r+1}f\left(\frac{a+b}{2}\right)-\frac{2}{r(b-a)}\int_a^b f(x)\,dx\right|$$
$$\leq (b-a)^2\int_0^1 |k(t)|\left|f''(tb+(1-t)a)\right|dt$$
$$\leq (b-a)^2\left(\int_0^{\frac{1}{2}}\left|\frac{t}{r}\left(\frac{1}{r+1}-t\right)\right|\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}\left|\frac{t}{r}\left(\frac{1}{r+1}-t\right)\right|\left|f''(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\left(\int_{\frac{1}{2}}^1\left|(1-t)\left(\frac{t}{r}-\frac{1}{r+1}\right)\right|\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^1\left|(1-t)\left(\frac{t}{r}-\frac{1}{r+1}\right)\right|\left|f''(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}.$$

Since $|f''|^q$ is $m-$convex, we have

$$\int_0^{\frac{1}{2}}\left|\frac{t}{r}\left(\frac{1}{r+1}-t\right)\right|\left|f''(tb+(1-t)a)\right|^q dt \tag{2.9}$$
$$\leq \int_0^{\frac{1}{2}}\left|\frac{t}{r}\left(\frac{1}{r+1}-t\right)\right|\left[t\left|f''(b)\right|^q+m(1-t)\left|f''(\frac{a}{m})\right|^q\right]dt$$
$$= \int_0^{\frac{1}{r+1}}\left|\frac{t}{r}\left(\frac{1}{r+1}-t\right)\right|\left[t\left|f''(b)\right|^q+m(1-t)\left|f''(\frac{a}{m})\right|^q\right]dt$$
$$+\int_{\frac{1}{r+1}}^{\frac{1}{2}}\left|\frac{t}{r}\left(t-\frac{1}{r+1}\right)\right|\left[t\left|f''(b)\right|^q+m(1-t)\left|f''(\frac{a}{m})\right|^q\right]dt$$
$$= M_1\left|f''(b)\right|^q+M_2m\left|f''(\frac{a}{m})\right|^q$$

and

$$\int_{\frac{1}{2}}^{1}\left|(1-t)\left(\frac{t}{r}-\frac{1}{r+1}\right)\right|\left|f''(tb+(1-t)a)\right|^{q}dt \tag{2.10}$$
$$\leq \int_{\frac{1}{2}}^{1}\left|(1-t)\left(\frac{t}{r}-\frac{1}{r+1}\right)\right|\left[t\left|f''(b)\right|^{q}+m(1-t)\left|f''(\frac{a}{m})\right|^{q}\right]dt$$
$$= \int_{\frac{1}{2}}^{\frac{r}{r+1}}\left|(1-t)\left(\frac{t}{r}-\frac{1}{r+1}\right)\right|\left[t\left|f''(b)\right|^{q}+m(1-t)\left|f''(\frac{a}{m})\right|^{q}\right]dt$$
$$+\int_{\frac{r}{r+1}}^{1}\left|(1-t)\left(\frac{1}{r+1}-\frac{t}{r}\right)\right|\left[t\left|f''(b)\right|^{q}+m(1-t)\left|f''(\frac{a}{m})\right|^{q}\right]dt$$
$$= M_2\left|f''(b)\right|^{q}+M_1 m\left|f''(\frac{a}{m})\right|^{q}$$

where $M_1=\frac{3r^4+4r^3-6r^2-12r+27}{192r(r+1)^4}$ and $M_2=\frac{5r^4+4r^3-18r^2+36r+21}{192r(r+1)^4}$.
From (2.9) and (2.10) and by using the fact that

$$\int_{0}^{\frac{1}{2}}\left|\frac{t}{r}\left(\frac{1}{r+1}-t\right)\right|=\int_{\frac{1}{2}}^{1}\left|(1-t)\left(\frac{t}{r}-\frac{1}{r+1}\right)\right|=\frac{r^3-3r+6}{24r(r+1)^3}$$

we obtain

$$\left|\frac{1}{r(r+1)}\left[f(a)+f(b)\right]+\frac{2}{r+1}f\left(\frac{a+b}{2}\right)-\frac{2}{r(b-a)}\int_{a}^{b}f(x)\,dx\right| \tag{2.11}$$
$$\leq (b-a)^2\left(\frac{r^3-3r+6}{24r(r+1)^3}\right)^{1-\frac{1}{q}}\left\{\left(M_1\left|f''(b)\right|^{q}+M_2 m\left|f''(\frac{a}{m})\right|^{q}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(M_2\left|f''(b)\right|^{q}+M_1 m\left|f''(\frac{a}{m})\right|^{q}\right)^{\frac{1}{q}}\right\}.$$

which completes the proof. □

3. APPLICATIONS TO SPECIAL MEANS

Now, we consider some applications of our theorems to the following special means.
a) The arithmetic mean:

$$A=A(a,b):=\frac{a+b}{2},\quad a,b\geq 0,$$

b) The harmonic mean:

$$H=H(a,b):=\frac{2ab}{a+b},\quad a,b\geq 0,$$

c) The logarithmic mean:

$$L=L(a,b):=\begin{cases} a & \text{if} \quad a=b \\ \frac{b-a}{\ln b-\ln a} & \text{if} \quad a\neq b \end{cases},\quad a,b\geq 0,$$

d) The $p-$logarithmic mean:

$$L_p = L_p(a,b) := \begin{cases} \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{1/p} & \text{if} \quad a \neq b \\ a & \text{if} \quad a = b \end{cases}, \quad p \in \mathbb{R} \backslash \{-1, 0\}; \quad a,b > 0.$$

Proposition 1. *Let* $a,b,n \in \mathbb{R}$, *then we have*

$$|A(a^n,b^n) + A^n(a,b) - 2L_n^n(a,b)| \leq \frac{(b-a)^2}{24} n(n-1) A\left(|a|^{n-2}, |b|^{n-2}\right) \tag{3.1}$$

and

$$\left|\frac{1}{3}A(a^n,b^n) + \frac{2}{3}A^n(a,b) - L_n^n(a,b)\right| \leq \frac{(b-a)^2}{81} n(n-1) A\left(|a|^{n-2}, |b|^{n-2}\right)$$

Proof. The assertation follows from Theorem 4 applied for $f(x) = x^n$, $x \in [a,b]$ with $r = 1$ for the first inequality and $r = 2$ for the second inequality. □

And especially for $f(x) = \frac{1}{x}$, we can get

$$\left|H(a,b) + A^{-1}(a,b) - 2L(a,b)\right| \leq \frac{(b-a)^2}{12} A\left(|a|^{-3}, |b|^{-3}\right)$$

for $r = 1$ and

$$\left|\frac{1}{3}H(a,b) + \frac{2}{3}A^n(a,b) - L(a,b)\right| \leq \frac{(b-a)^2}{81} 2A\left(|a|^{-3}, |b|^{-3}\right)$$

for $r = 2$.

Proposition 2. *Let* $a,b,n \in \mathbb{R}$, *then we have*

$$|A(a^n,b^n) + A^n(a,b) - 2L_n^n(a,b)| \leq \frac{(b-a)^2}{24} n(n-1) A\left(m|a|^{n-2}, |b|^{n-2}\right) \tag{3.2}$$

and

$$\left|\frac{1}{3}A(a^n,b^n) + \frac{2}{3}A^n(a,b) - L_n^n(a,b)\right| \leq \frac{(b-a)^2}{216} n(n-1) A\left(m|a|^{n-2}, |b|^{n-2}\right)$$

Proof. The assertation follows from Theorem 6 applied for $f(x) = x^n$, $x \in [a,b]$ with $r = 1$ for the first inequality and $r = 2$ for the second inequality. □

References

[1] M.Z. Sarıkaya, E. Set and M.E. Özdemir, On new inequalities of Simpson's type for functions whose second derivatives absolute values are convex, RGMIA Res. Rep. Coll., 13 (1) (2010), Article 1.

[2] G.H. Toader, On a generalization of the convexity, Mathematica, 30 (53) (1988), 83-87.

[3] M. Alomari, M. Darus and S.S. Dragomir, New inequalities of Simpson's type for s-convex functions with applications, RGMIA Res. Rep. Coll., 12 (4) (2009), Article 9. [Online:http://www.staff.vu.edu.au/RGMIA/v12n4.asp].

[4] S.S. Dragomir, R.P. Agarwal and P. Cerone, On Simpson's inequality and applications, J. of Inequal. Appl., 5(2000), 533-579.

[5] B.Z. Liu, An inequality of Simpson type, Proc. R. Soc. A, 461 (2005), 2155-2158.

[6] M.E. Özdemir, E. Set, M. Z. Sarıkaya, Some new Hadamard's type inequalities for co-ordinated $m-$convex and $(\alpha, m)-$convex functions, Accepted.

[7] M. K. Bakula, M. E. Özdemir and J. Pecaric, Hadamard-type inequalities for $m-$convex and $(\alpha, m)-$convex functions, J. Inequal. Pure and Appl. Math., 9, (4), (2007), Article 96.

[8] M. K. Bakula, J. Pecaric and M. Ribibic, Companion inequalities to Jensen's inequality for $m-$convex and $(\alpha, m)-$convex functions, J. Inequal. Pure and Appl. Math., 7 (5) (2006), Article 194.

[9] S. S. Dragomir, G. Toader, Some inequalities for $m-$convex functions, Studia University Babes Bolyai, Mathematica, 38 (1) (1993), 21-28.

[10] M. E. Özdemir, M. Avcı and E. Set, On some inequalities of Hermite-Hadamard type via $m-$convexity, Applied Mathematics Letters, 23 (2010), 1065-1070.

■ Atatürk University, K.K. Education Faculty, Department of Mathematics, 25240, ERZURUM TURKEY
E-mail address: emos@atauni.edu.tr

♠ Ağri İbrahim Çeçen University, Faculty of Science and Letters, Department of Mathematics, 04100 AĞRI TURKEY
E-mail address: alperekinci@hotmail.com

E-mail address: ahmetakdemir@agri.edu.tr