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SOME NEW INEQUALITIES FOR AN INTERIOR POINT OF A TRIANGLE

JIAN LIU

Abstract. In this paper we establish three new inequalities involving an arbitrary point of a triangle. Some related conjectures and problems are put forward.

1. Introduction

Let P be an arbitrary point in the plane of triangle ABC and let D, E, F be the feet of the perpendiculars from P to BC, CA, AB, respectively. In [1], the author gave the following identity:

$$\vec{S}_{\triangle PBC} \cdot PA^2 + \vec{S}_{\triangle PCA} \cdot PB^2 + \vec{S}_{\triangle PAB} \cdot PC^2 = 4R^2 \vec{S}_{\triangle DEF}, \tag{1.1}$$

where R is the circumradius of $\triangle ABC$ and $\vec{S}_{\triangle PBC}, \vec{S}_{\triangle PCA}, \vec{S}_{\triangle PAB}$ denote directed areas of $\triangle PBC, \triangle PCA, \triangle PAB$.

In particular, when P lies inside triangle ABC, then

$$S_a R_1^2 + S_b R_2^2 + S_c R_3^2 = 4R^2 S_p, \tag{1.2}$$

where $R_1 = PA, R_2 = PB, R_3 = PC$ and S_a, S_b, S_c denote the areas of $\triangle PBC, \triangle PCA, \triangle PAB$ respectively, and S_p is the area of pedal triangle DEF.

It is well known that the following inequality holds between the area S of triangle ABC and the area S_p of pedal triangle DEF:

$$S_p \le \frac{1}{4} S, \tag{1.3}$$

with equality if and only if P is the circumradius of triangle ABC (see Figure 1). Therefore, it follows from (1.2) that (see Figure 2)

$$S_a R_1^2 + S_b R_2^2 + S_c R_3^2 \le S R^2. \tag{1.4}$$

This inequality inspires the author to find the similar conclusion:

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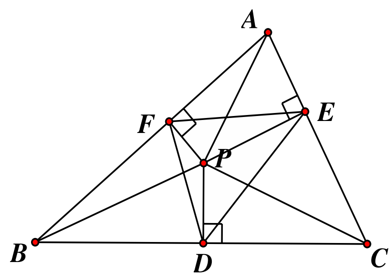

图1

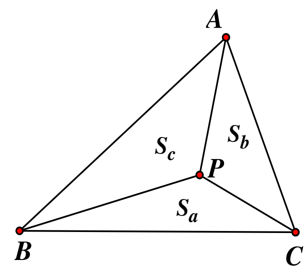

图2

Theorem 1.1. *Let P be an arbitrary interior point of triangle ABC. Then*

$$S_aR_1^3 + S_bR_2^3 + S_cR_3^3 \leq SR^3, \tag{1.5}$$

with equality if and only if P is the circumradius of triangle ABC.

At the same time when inequality (1.5) has been proven, we obtain the following two interesting geometric inequalities:

Theorem 1.2. *Let P be an arbitrary point of triangle ABC with the circumradius R and the inradius r. Let r_p be the inadius of pedal triangle of P with respect to triangle ABC. Then*

$$\frac{1}{2r_p} \geq \frac{1}{R} + \frac{1}{2r}, \tag{1.6}$$

with equality if and only if triangle ABC is equilateral and P is its center.

Theorem 1.3. *Let P be an arbitrary interior point of $\triangle ABC$ and let D, E, F denote the feet of the perpendiculars from P to BC, CA, AB respectively. Let r_p be the inradius of pedal triangle DEF and let $PA = R_1, PB = R_2, PC = R_3$, $PD = r_1, PE = r_2, PF = r_3$. Then*

$$R_1 + R_2 + R_3 \geq r_1 + r_2 + r_3 + 6r_p. \tag{1.7}$$

with equality if and only if $\triangle ABC$ is equilateral and P is its center.

In this note we will prove the above three theorems and propose some related conjectures and problems.

2. **Some Lemmas**

To prove our theorems, we need several lemmas.

Lemma 2.1. *Let P be an arbitrary point with barycentric coordinates (x, y, z) in the plane of triangle ABC. Then*

$$(x + y + z)R_1^2 = (x + y + z)(yc^2 + zb^2) - (yza^2 + zxb^2 + xyc^2). \tag{2.1}$$

where a, b, c are the lengths of side BC, CA, AB respectively.

The above formulae is well known (see e.g. [2]P_{278}).

Lemma 2.2. *For any point P inside triangle ABC, we have*

$$cr_2 + br_3 \leq aR_1, \tag{2.2}$$

If AO (O is the circumcenter of $\triangle ABC$) cuts BC at X, then the equality if and only if P lies on the segment AX.

Analogously to (2.2) we also have two inequalities. Lemma 2.1 is a simple important proposition and it has various proofs, see [3]-[9]. Next, we give a crucial lemma which is equivalent to Lemma 2.2 substantially.

Lemma 2.3. *For any point P inside triangle ABC, we have*

$$R_1 \geq \frac{R_1^2}{2R} + \frac{2RS_p}{S}, \tag{2.3}$$

with equality as in (2.2).

Proof. Note that $S_a = \frac{1}{2}ar_1, S_b = \frac{1}{2}br_2, S_c = \frac{1}{2}cr_3, S_a+S_b+S_c = S$, applying Lemma 2.1 and 2.2 we have

$$\begin{aligned}
&(S_a + S_b + S_c)R_1^2 \\
&= (S_a + S_b + S_c)(S_bc^2 + S_cb^2) - (S_bS_ca^2 + S_cS_ab^2 + S_aS_bc^2) \\
&= \frac{1}{2}(br_2c^2 + cr_3b^2)S - \frac{1}{4}(bcr_2r_3a^2 + car_3r_1b^2 + abr_1r_2c^2) \\
&= \frac{1}{2}bc(br_2 + cr_3)S - \frac{1}{4}abc(ar_2r_3 + br_3r_1 + cr_1r_2) \\
&\leq \frac{1}{2}abcR_1S - \frac{1}{2}abcR(r_2r_3 \sin A + r_3r_1 \sin B + r_1r_2 \sin C) \\
&= \frac{1}{2}abcR_1S - abcR(S_{\triangle PEF} + S_{\triangle PFD} + S_{\triangle PDE}) \\
&= \frac{1}{2}abc(R_1S - 2RS_p).
\end{aligned}$$

Then make use of $S_a + S_b + S_c = S$ and $abc = 4SR$, we get

$$R_1S - 2RS_p \geq \frac{SR_1^2}{2R},$$

this yields inequality (2.3). Clearly, the condition of the equality in (2.3) is the same as (2.2). □

Lemma 2.4. *For any point P inside triangle ABC, we have*

$$S_aR_1 + S_bR_2 + S_cR_3 \geq 4RS_p, \tag{2.4}$$

with equality if and only if P is the circumradius of triangle ABC.

Proof. Since the area of the quadrilateral is less than or equal to the half of product of two diagonals, so we have

$$S_b + S_c \leq \frac{1}{2}aR_1, S_c + S_a \leq \frac{1}{2}bR_2, S_a + S_b \leq \frac{1}{2}cR_3, \tag{2.5}$$

with equalities if and only if $PA \perp BC, PB \perp CA, PC \perp AB$ respectively. Adding up these inequalities and using identity $S_a+S_b+S_c = S$, one obtains

$$aR_1 + bR_2 + cR_3 \geq 4S. \tag{2.6}$$

Equality holds if and only if P is the orthocenter of $\triangle ABC$.

Applying inequality (2.6) to pedal $\triangle DEF$ (see Figure 2), we get

$$EF \cdot r_1 + FD \cdot r_2 + DE \cdot r_3 \geq 4S_p,$$

Observe that $EF = \frac{aR_1}{2R}, ar_1 = 2S_a$, etc., then inequality (2.4) follows at once. According to the equality condition of (2.6), we conclude easily that the equality in (2.4) holds if and only if P is the circumradius of triangle ABC. □

Remark 2.1. Inequality (2.4) can also be proven easily by using Lemma 2 and the identity (1.2).

Lemma 2.5. *Suppose that P is any point in the plane of triangle ABC. Then*

$$aR_1^2 + bR_2^2 + cR_3^2 \geq abc, \tag{2.7}$$

with equality holds if and only if P is the incenter of $\triangle ABC$.

Inequality (2.7) is given first by M.K.Lamkin (see [2]). The author ([10]) generalized its equivalent form:

$$R_1^2 \sin A + R_2^2 \sin B + R_3^2 \sin C \geq 2S \tag{2.8}$$

to the polygon. I proved the following result: For polygon $A_1A_2\cdots A_n$ and an arbitrary point P

$$\sum_{i=1}^{n} PA_i^2 \sin A_i \geq 2F, \tag{2.9}$$

where F is the area of the polygon. Later, the author further generalized inequality (2.9) into the case involving two arbitrary points (see [11]):

$$\sum_{i=1}^{n} PA_i \cdot QA_i \sin A_i \geq 2F. \tag{2.10}$$

Lemma 2.6. *For any arbitrary interior point P of $\triangle ABC$, we have*

$$\frac{R_1^2 + R_2^2 + R_3^2}{r_1 + r_2 + r_3} \geq 2R, \tag{2.11}$$

with equality if and only if $\triangle ABC$ is equilateral and P is its center.

Inequality (2.11) is first posed by the author and it is proven first by Xiao-Guang Chu and Zhen-Gang Xiao ([12]). Xue-Zhi Yang gave a simple proof in [13,P15]. We introduce curtly his proof as follows:

Applying the Cosine Law, one gets easily

$$4SR_1^2 = b^2c^2(r_2^2 + r_3^3) + bcr_2r_3(b^2 + c^2 - a^2). \tag{2.12}$$

Then we use $abc = 4SR$ and the identity:

$$ar_1 + br_2 + cr_3 = 2S, \tag{2.13}$$

we obtain

$$\begin{aligned}&4S^2\left(\sum R_1^2 - 2R\sum r_1\right)\\ &= \sum\left[b^2c^2(r_2^2 + r_3^3) + bcr_2r_3(b^2 + c^2 - a^2)\right] - abc\sum ar_1\sum r_1.\end{aligned} \tag{2.14}$$

where $\sum$ denotes cyclic sums. From this we can obtain the identity:

$$\begin{aligned}&4S^2\left(\sum R_1^2 - 2R\sum r_1\right)\\ &= \sum bcr_2r_3(b-c)^2 + \frac{1}{2}\sum a\left[c(r_3 + r_1) - b(r_1 + r_2)\right]^2,\end{aligned} \tag{2.15}$$

which implies inequality (2.11).

3. The Proofs of the Theorems

3.1. The Proof of Theorem 1.

Proof. We multiply both sides of inequality (2.3) by S_aR_1, then

$$\frac{S_aR_1^3}{2R} + \frac{2RS_pS_aR_1}{S} \leq S_aR_1^2.$$

Analogously, we have

$$\frac{S_bR_2^3}{2R} + \frac{2RS_pS_bR_2}{S} \leq S_bR_2^2, \quad \frac{S_cR_3^3}{2R} + \frac{2RS_pS_cR_3}{S} \leq S_cR_3^2.$$

By adding up three inequalities and then using identity (1.2) one has

$$\frac{S_aR_1^3 + S_bR_2^3 + S_cR_3^3}{2R} + \frac{2RS_p}{S}(S_aR_1 + S_bR_2 + S_cR_3) \leq 4R^2S_p.$$

So, it follows from Lemma 2.4 that

$$\frac{S_aR_1^3 + S_bR_2^3 + S_cR_3^3}{2R} + \frac{8R^2S_p^2}{S} \leq 4R^2S_p,$$

Namely

$$\begin{aligned}S_aR_1^3 + S_bR_2^3 + S_cR_3^3 &\leq 8R^3\left(S_p - \frac{2S_p^2}{S}\right)\\ &= SR^3 - \frac{(S-4S_p)^2R^3}{S}\\ &\leq SR^3.\end{aligned}$$

This completes the proof of (1.5). According to the conditions of equality (2.3) and (1.3), we conclude that the equality in (1.5) occurs if and only if P is the circumradius of triangle ABC. □

Remark 3.1. By applying the inequality of Theorem 1.1 and the weighted power mean inequality, we can get the following generalization of inequality (1.5):

$$S_aR_1^k + S_bR_2^k + S_cR_3^k \le SR^k. \tag{3.1}$$

where $0 < k \le 3$. In addition, by using Radon inequality, we can prove that if $k < 0$ then (3.1) holds inversely.

3.2. **The Proof of Theorem 2.**

Proof. From Lemma 2.3 and Lemma 2.5, we have

$$\begin{aligned} aR_1 + bR_2 + cR_3 &\ge \frac{1}{2R}(aR_1^2 + bR_2^2 + cR_3^2) + \frac{2RS_p}{S}(a+b+c) \\ &\ge \frac{abc}{2R} + \frac{2RS_p}{S}(a+b+c). \end{aligned}$$

Since $abc = 4SR, a+b+c = 2s$, it follows that

$$aR_1 + bR_2 + cR_3 \ge 2S + \frac{4R}{r}S_p. \tag{3.2}$$

Equality occurs if and only if the point P coincide with the circumcenter and the incenter of $\triangle ABC$. This means that $\triangle ABC$ is equilateral and P is its center.

$$\frac{aR_1 + bR_2 + cR_3}{8RS_p} \ge \frac{S}{4RS_p} + \frac{1}{2r}.$$

Noticing that

$$r_p = \frac{4RS_p}{aR_1 + bR_2 + cR_3}, \tag{3.3}$$

Thus we get

$$\frac{1}{2r_p} \ge \frac{S}{4RS_p} + \frac{1}{2r}. \tag{3.4}$$

Hence, the inequality of Theorem 2 follows from (3.4) and (1.3) rapidly. □

3.3. **The Proof of Theorem 3.**

Proof. First, by adding up the inequality of Lemma 2.3 and its analogues we get

$$R_1 + R_2 + R_3 \ge \frac{R_1^2 + R_2^2 + R_3^3}{2R} + \frac{6RS_p}{S}. \tag{3.5}$$

Form inequality (2.6) and identity (24), we know (2.6) is equivalent to

$$\frac{S_p}{S} \ge \frac{r_p}{R}, \tag{3.6}$$

with equality as in (2.6). Therefore, the inequality (1.7) of Theorem 3 follows form (3.5), (3.6) and the inequality (2.11) of Lemma 2.6 immediately. Clearly, the conditions of equality in (1.7) are just as like what have been mentioned as in Theorem 3. □

Remark 3.2. From inequality (3.6) and the following identity (we omit its proof)

$$\frac{R_1^2+R_2^2+R_3^3}{2R}+\frac{6RS_p}{S}=r_1\left(\frac{b}{c}+\frac{c}{b}\right)+r_2\left(\frac{c}{a}+\frac{a}{c}\right)+r_3\left(\frac{a}{b}+\frac{b}{a}\right), \quad (3.7)$$

we get

$$R_1+R_2+R_3\geq r_1\left(\frac{b}{c}+\frac{c}{b}\right)+r_2\left(\frac{c}{a}+\frac{a}{c}\right)+r_3\left(\frac{a}{b}+\frac{b}{a}\right). \quad (3.8)$$

Further, we get the following equality between R_1, R_2, R_3 and r_1, r_2, r_3:

$$R_1+R_2+R_3\geq 2(r_1+r_2+r_3) \quad (3.9)$$

This is the famous Erdös-Mordell inequality (see [4]-[9]). Recently, the author gave its absolute new proof as seen [14].

4. Several Conjectures and Problems

Euler inequality in the triangle is well known, it states that

$$R\geq 2r. \quad (4.1)$$

From this we consider the stronger inequality of Theorem 1.2. After being checked by the computer, we pose the following stronger conjecture:

Conjecture 1. *For any arbitrary interior point P, we have*

$$\frac{1}{2r_p}\geq\frac{1}{\sqrt{2Rr}}+\frac{1}{2r}. \quad (4.2)$$

It is easy to get the inequality from (1.6):

$$8r_p\leq R+2r.. \quad (4.3)$$

For this inequality, we have the following unsolved problem:

Problem 1. *Find the maximum value k such that the inequality*

$$2(k+2)r_p\leq R+kr \quad (4.4)$$

is valid for arbitrary interior point P of $\triangle ABC$.

Remark 4.1. From Euler inequality (4.1) we see that the inequality takes the maximum value k which is the strongest in all inequalities whose type is as (4.4). With the help of the computer, the author finds the maximum value k is about $7.88\cdots$.

Next, we denote the circumradius of pedal triangle DEF by R_p, note that $R_p\geq 2r_p$, we first suppose

$$R_p+6r_p\leq R+2r, \quad (4.5)$$

which is stronger than the inequality. Further, the equality with one parameter is put forward:

Conjecture 2. *If $1.8 \le k \le 7.8$, then the inequality*

$$R_p + 2kr_p \le R + (k-1)r. \tag{4.6}$$

holds for an arbitrary interior point P of $\triangle ABC$.

Also, we can put forward the following problem:

Problem 2. *Find the maximum and the minimum value of k such that the inequality (4.6) holds for an arbitrary interior point P of $\triangle ABC$.*

When $k = 2$, inequality (4.6) becomes

$$R_p + 4r_p \le R + r, \tag{4.7}$$

This equality has not been proven yet. The author thinks it has the following exponent generalization:

Conjecture 3. *If $k \ge \dfrac{3}{4}$, then the following inequality*

$$R_p^k + (4r_p)^k \le R^k + r^k \tag{4.8}$$

holds for an arbitrary interior point P of $\triangle ABC$.

Another similar difficult conjecture is

Conjecture 4. *If $k \ge \dfrac{1}{2}$, then the following inequality*

$$\frac{1}{R_p^k} + \frac{1}{r_p^k} \ge \frac{1}{r^k} + \frac{4^k}{R^k}. \tag{4.9}$$

holds for an arbitrary interior point P of $\triangle ABC$.

Considering the exponent generalization of Theorem 1.3, the following conjecture is brought forward:

Conjecture 5. *If $k > 0$, then the inequality:*

$$R_1^k + R_2^k + R_3^k - (r_1^k + r_2^k + r_3^k) \ge 3 \cdot 2^k (2^k - 1) r_p^k \tag{4.10}$$

holds for an arbitrary interior point P of $\triangle ABC$.

Finally, we propose a strict inequality which is similar to inequality of Theorem 1.3:

Conjecture 6. *For an arbitrary interior point P of $\triangle ABC$ holds:*

$$R_1 + R_2 + R_3 > r_1 + r_2 + r_3 + 2R_p. \tag{4.11}$$

References

[1] Jian Liu, *Several New Inequalities for the Thriangle*, Mathematics Competition (in Chinese), Hunan Education Press, **15**(1992)80-100.

[2] D.Mitrinovió, J. E. Peǒarić and V. Volenec, *Recent Advances in Geometric Inequalities*, Kluwer Academic Publishers, Dordrecht/Boston London, 1989.

[3] O. Bottema, R. Ž. Djordjević, R. R. Janić, D. S. Mitrinović, and P. M. Vasić, *Geometric Inequalities*, Groningen, 1969.

[4] V.Komornik, *A short proof of the Erdös-Mordell theorem*, Amer.Math.Monthly, **104**(1997)57-60.

[5] D.K.Kazarinoff, *A simple proof of the Erdös-Mordell inequality for triangles*, Michigan Mathematical Journal, **4**(1957)97-98.

[6] L.Bankoff, *An elementary proof of the Erdöos-Mordell theorem*, Amer.Math.Monthly, **65**(1958)521.

[7] A.Avez, *A short proof of the Erdös and Mordell theorem*, Amer.Math.Monthly, **100**(1993)60-62.

[8] H.Lee, *Another proof of the Erdös-Mordell theorem*, Forum Geom, **1**(2001)7-8.

[9] N.Dergiades, *Signed distances and the Erdös-Mordell inequality*, Forum Geom, **4**(2004)67-68.

[10] Jian Liu, *A Geometric Inequality*, Shuxue Tongxun (in Chinese), **9**(1988), 1-4.

[11] Jian Liu, *A Inequality for the polygon*, Hunan Shuxue Tongxun (in Chinese), **6**(1991), 36-37.

[12] Xiao-guang Chu, Zhen-gang Xiao, *The Proof of Some Geometric Inequality Conjectures*, Journal of Hunan Institute of Science Technology (Natural Sciences), **16(4)**(2003), 10-13.

[13] Xue-Zhi Yang, *The inequality study of maths Olympics (in Chinese)*, Harbin Institute of Technology Press, Harbin, China, 2009.

[14] Jian Liu, *A new proof of the Erdös-Mordell inequality*, RGMIA Research Report Collection, **14**(2011), Article 13, 6 pp. http://rgmia.org/v14.php.

[15] Z.Wang, *The proof of a geometric inequality*, Budengshi Yanjiutongxun (in Chinese), **16(1)**(2009), 66-68.

East China Jiaotong University
E-mail address: China99jian@163.com