Received 04/05/11

NEW PROOFS OF SCHUR-CONCAVITY FOR A CLASS OF SYMMETRIC FUNCTIONS

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Abstract. By properties of the Schur-convex function, Schur-concavity for a class of symmetric functions is simply proved.

1. Introduction

Throughout the paper, $\mathbb{R}$ denotes the set of real numbers, $\boldsymbol{x}=(x_1,x_2,\cdots,x_n)$ denotes n-tuple (n-dimensional real vector), the set of vectors can be written as

$$\mathbb{R}^n=\{\boldsymbol{x}=(x_1,\cdots,x_n): x_i\in\mathbb{R}, i=1,\dots,n\},$$
$$\mathbb{R}^n_+=\{\boldsymbol{x}=(x_1,\dots,x_n): x_i>0, i=1,\dots,n\}.$$

In particular, the notations $\mathbb{R}$ and $\mathbb{R}_+$ denote $\mathbb{R}^1$ and $\mathbb{R}^1_+$ respectively.

For convenience, we introduce some definitions as follows.

Definition 1. [1, 2] Let $\boldsymbol{x}=(x_1,\dots,x_n)$ and $\boldsymbol{y}=(y_1,\dots,y_n)\in\mathbb{R}^n$.

(i) $\boldsymbol{x}\geq\boldsymbol{y}$ means $x_i\geq y_i$ for all $i=1,2,\dots,n$.

(ii) let $\Omega\subset\mathbb{R}^n$, φ: $\Omega\to\mathbb{R}$ is said to be increasing if $\boldsymbol{x}\geq\boldsymbol{y}$ implies $\varphi(\boldsymbol{x})\geq\varphi(\boldsymbol{y})$. φ is said to be decreasing if and only if $-\varphi$ is increasing.

Definition 2. [1, 2] Let $\boldsymbol{x}=(x_1,\dots,x_n)$ and $\boldsymbol{y}=(y_1,\dots,y_n)\in\mathbb{R}^n$.

(i) $\boldsymbol{x}$ is said to be majorized by $\boldsymbol{y}$ (in symbols $\boldsymbol{x}\prec\boldsymbol{y}$) if $\sum_{i=1}^k x_{[i]}\leq\sum_{i=1}^k y_{[i]}$ for $k=1,2,\dots,n-1$ and $\sum_{i=1}^n x_i=\sum_{i=1}^n y_i$, where $x_{[1]}\geq\cdots\geq x_{[n]}$ and $y_{[1]}\geq\cdots\geq y_{[n]}$ are rearrangements of $\boldsymbol{x}$ and $\boldsymbol{y}$ in a descending order.

(ii) Let $\Omega\subset\mathbb{R}^n$, φ: $\Omega\to\mathbb{R}$ is said to be a Schur-convex function on Ω if $\boldsymbol{x}\prec\boldsymbol{y}$ on Ω implies $\varphi(\boldsymbol{x})\leq\varphi(\boldsymbol{y})$. φ is said to be a Schur-concave function on Ω if and only if $-\varphi$ is Schur-convex function on Ω.

Definition 3. [1, 2] Let $\boldsymbol{x}=(x_1,\dots,x_n)$ and $\boldsymbol{y}=(y_1,\dots,y_n)\in\mathbb{R}^n$.

(i) $\Omega\subseteq\mathbb{R}^n$ is said to be a convex set if $\boldsymbol{x},\boldsymbol{y}\in\Omega, 0\leq\alpha\leq 1$ implies $\alpha\boldsymbol{x}+(1-\alpha)\boldsymbol{y}=(\alpha x_1+(1-\alpha)y_1,\dots,\alpha x_n+(1-\alpha)y_n)\in\Omega$.

(ii) Let $\Omega\subset\mathbb{R}^n$ be convex set. A function φ: $\Omega\to\mathbb{R}$ is said to be a convex function on Ω if

$$\varphi(\alpha\boldsymbol{x}+(1-\alpha)\boldsymbol{y})\leq\alpha\varphi(\boldsymbol{x})+(1-\alpha)\varphi(\boldsymbol{y})$$

for all $\boldsymbol{x},\boldsymbol{y}\in\Omega$, and all $\alpha\in[0,1]$. φ is said to be a concave function on Ω if and only if $-\varphi$ is convex function on Ω.

2000 *Mathematics Subject Classification.* Primary 26D15, 05E05, 26B25.

Key words and phrases. majorization, Schur-concavity, inequality, symmetric functions, concave functions.

The first author was supported in part by the Scientific Research Common Program of Beijing Municipal Commission of Education (KM201111417006).

This paper was typeset using $\mathcal{A}\mathcal{M}\mathcal{S}$-LaTeX.

Recall that the following so-called Schurs condition is very useful for determining whether or not a given function is Schur-convex or Schur-concave.

Theorem A ([1, p. 5])**.** *Let $\Omega \subset \mathbb{R}^n$ is symmetric and has a nonempty interior convex set. Ω^0 is the interior of Ω. $\varphi : \Omega \to \mathbb{R}$ is continuous on Ω and differentiable in Ω^0. Then φ is the $Schur-convex(Schur-concave)function$, if and only if φ is symmetric on Ω and*

$$(x_1 - x_2)\left(\frac{\partial \varphi}{\partial x_1} - \frac{\partial \varphi}{\partial x_2}\right) \geq 0(\leq 0) \tag{1}$$

holds for any $\boldsymbol{x} \in \Omega^0$.

In recent years, by using Theorem A, many researchers have studied the Schur-convexity of some of symmetric functions.

Y.-M Chu etc. [3] defined the following symmetric functions

$$F_n(\boldsymbol{x}, k) = \prod_{1 \leq i_1 < \ldots < i_k \leq n} \frac{\sum_{j=1}^k x_{i_j}}{\sum_{j=1}^k (1 + x_{i_j})}, \ k = 1, \ldots, n, \tag{2}$$

and established the following results by using Theorem A.

Theorem B. *For $k = 1, \ldots, n$, $F_n(\boldsymbol{x}, k)$ is an Schur-concave function on $\mathbb{R}_+^n$.*

W.-D Jiang [4]are discussed the following symmetric functions

$$H_k^*(\boldsymbol{x}) = \prod_{1 \leq i_1 < \ldots < i_k \leq n} \sum_{j=1}^k x_{i_j}^{1/k}, \ k = 1, \ldots, n, \tag{3}$$

and established the following results by using Theorem A.

Theorem C. *For $k = 1, \ldots, n$, $H_k^*(\boldsymbol{x})$ is an Schur-concave function on $\mathbb{R}_+^n$.*

W.-F. Xia and Y.-M Chu [5] investigated the following symmetric functions

$$\phi_n(\boldsymbol{x}, k) = \prod_{1 \leq i_1 < \ldots < i_k \leq n} \sum_{j=1}^k \frac{x_{i_j}}{1 + x_{i_j}}, \ k = 1, \ldots, n, \tag{4}$$

and established the following results by using Theorem A.

Theorem D. *For $k = 1, \ldots, n$, $F_n(\boldsymbol{x}, k)$ is an Schur-concave function on $\mathbb{R}_+^n$.*

In this note, by properties of the Schur-convex function, we simply prove Theorem B, Theorem C and Theorem D.

2. New proofs three Theorems

To prove the above three theorems, we need the following lemmas.

Lemma 1. [2] *If φ is symmetric and convex (concave) on symmetric convex set Ω, then φ is Schur-convex $(Schur-concave)$ on Ω.*

Lemma 2. [2] *Let $\Omega \subset \mathbb{R}^n$, φ: $\Omega \to \mathbb{R}_+$. Then $\ln \varphi$ is Schur-convex (Schur-concave) if and only if φ is Schur-convex (Schur-concave).*

Lemma 3. [2] *Let $\Omega \subset \mathbb{R}^n$ be open convex set, $\varphi : \Omega \to \mathbb{R}$. For $\boldsymbol{x}, \boldsymbol{y} \in \Omega$, defined one variable function $g(t) = \varphi(t\boldsymbol{x} + (1-t)\boldsymbol{y})$ on interval $(0, 1)$. Then φ is convex (concave) on Ω if and only if g is convex (concave) on $(0, 1)$ for all $\boldsymbol{x}, \boldsymbol{y} \in \Omega$.*

Lemma 4. *Let* $\boldsymbol{x} = (x_1, \ldots, x_m)$ *and* $\boldsymbol{y} = (y_1, \ldots, y_m) \in \mathbb{R}^m$. *Then the following functions are concave on* $(0, 1)$.

(i) $f(t) = \ln \sum_{j=1}^{m} (tx_j + (1-t)y_j) - \ln \sum_{j=1}^{m} (1 + tx_j + (1-t)y_j)$,

(ii) $g(t) = \ln \sum_{j=1}^{m} (tx_j + (1-t)y_j)^{1/m}$,

(iii) $h(t) = \frac{1}{m} \ln \psi(t)$, *where*

$$\psi(t) = \sum_{j=1}^{m} \frac{tx_j + (1-t)y_j}{1 + tx_j + (1-t)y_j}.$$

Proof. (i) Directly calculating yields

$$f^{'}(t) = \sum_{j=1}^{m} (x_j - y_j) \left[\frac{1}{tx_j + (1-t)y_j} - \frac{1}{1 + tx_j + (1-t)y_j} \right]$$

and

$$\begin{aligned} f^{''}(t) &= -\sum_{j=1}^{m} (x_j - y_j)^2 \left[\frac{1}{(tx_j + (1-t)y_j)^2} - \frac{1}{(1 + tx_j + (1-t)y_j)^2} \right] \\ &= -\sum_{j=1}^{m} (x_j - y_j)^2 \frac{1 + 2tx_j + 2(1-t)y_j}{(tx_j + (1-t)y_j)^2 (1 + tx_j + (1-t)y_j)^2}. \end{aligned}$$

Since $f^{''}(t) \le 0$, $f(t)$ is concave on $(0, 1)$.

(ii) Directly calculating yields

$$g^{'}(t) = \frac{\frac{1}{m} \sum_{j=1}^{m} (x_j - y_j)^{\frac{1}{m} - 1}}{\sum_{j=1}^{m} (tx_j + (1-t)y_j)^{1/m}}$$

and

$$g^{''}(t) = -\frac{\left[\frac{1}{m} \sum_{j=1}^{m} (x_j - y_j)^{\frac{1}{m} - 1} \right]^2}{\sum_{j=1}^{m} (tx_j + (1-t)y_j)^{2/m}}.$$

Since $g^{''}(t) \le 0$, $f(t)$ is concave on $(0, 1)$.

(iii) By computing,

$$h^{'}(t) = \frac{1}{m} \frac{\psi^{'}(t)}{\psi(t)},$$

$$h^{''}(t) = \frac{1}{m} \frac{\psi^{''}(t)\psi(t) - (\psi^{'}(t))^2}{\psi^2(t)},$$

where

$$\psi^{'}(t) = \sum_{j=1}^{m} \frac{x_j - y_j}{(1 + tx_j + (1-t)y_j)^2}$$

and

$$\psi^{''}(t) = -\sum_{j=1}^{m} \frac{2(x_j - y_j)^2}{(1 + tx_j + (1-t)y_j)^3}.$$

Thus,

$$\psi^{''}(t)\psi(t)-(\psi^{'}(t))^2=-\sum_{j=1}^{m}\frac{2(x_j-y_j)^2}{(1+tx_j+(1-t)y_j)^3}\sum_{j=1}^{m}\frac{tx_j+(1-t)y_j}{1+tx_j+(1-t)y_j}$$
$$-\left[\sum_{j=1}^{m}\frac{x_j-y_j}{(1+tx_j+(1-t)y_j)^2}\right]^2\le 0,$$

and then $h^{''}(t)\le 0$, so $f(t)$ is concave on $(0,1)$.

The proof of Lemma 4 is completed. □

Proof of Theorem A: For any $1\le i_1<\ldots<i_k\le n$, by Lemma 3 and Lemma 4(i), it follows that $\ln\sum_{j=1}^{k}x_{i_j}-\ln\sum_{j=1}^{k}(1+x_{i_j})$ is concave on $\mathbb{R}_{+}^{n}$, and then $\ln F_n(\boldsymbol{x},k)=\prod_{1\le i_1<\ldots<i_k\le n}\left(\ln\sum_{j=1}^{k}x_{i_j}-\ln\sum_{j=1}^{k}(1+x_{i_j})\right)$ is concave on $\mathbb{R}_{+}^{n}$. Furthermore, it is clear that $\ln F_n(\boldsymbol{x},k)$ is symmetric on $\mathbb{R}_{+}^{n}$, by Lemma 1, it follows that $\ln F_n(\boldsymbol{x},k)$ is concave on $\mathbb{R}_{+}^{n}$, and then from Lemma 2 we conclude that $F_n(\boldsymbol{x},k)$ is also concave on $\mathbb{R}_{+}^{n}$.

The proof of Theorem A is completed.

Similar to the proof of Theorem A, by Lemma 4 (ii) and Lemma 4 (iii), we can prove Theorem B and Theorem C respectively. Omitted detailed process.

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