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REMARKS ON:INEQUALITIES: THEORY OF MAJORIZATION AND ITS APPLICATION BY A. W. MARSHALL ET AL

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ABSTRACT. This note points out two careless omissions in the monograph " Inequalities: theory of majorization and its application " (Second Edition) of A. W. Marshall et al

1. Introduction

In 1979, A.W. Marshall and I. Olkin [1] published the monograph entitled "Inequalities: Theory of Majorization and Its Application". This signifies theory of majorization become a emerging independent mathematics subject. In 1994, James Bondar [2] provides a retrospective view of majorization, and a clarification of some of the concepts. His section on complements adds some new material. An Errata Appendix helped the authors to prepare this new edition. In 2011, A. M. Marshall, I. Olkin and B. C. Arnold published second edition of this monograph[3].

The first version of the book has two omissions which have not been found James Bondar, the second edition is not for repair. In this note we will rectify the situation.

First, let us recall the definition of Schur-convex functions and convex functions.

Throughout the paper, $\mathbb{R}$ denotes the set of real numbers, $\boldsymbol{x} = (x_1, \cdots, x_n)$ denotes n-tuple (n-dimensional real vector), the set of vectors can be written as

$$\mathbb{R}^n = \{\boldsymbol{x} = (x_1, \cdots, x_n) : x_i \in \mathbb{R}, i = 1, \ldots, n\},$$

$$\mathbb{R}^n_+ = \{\boldsymbol{x} = (x_1, \ldots, x_n) : x_i > 0, i = 1, \ldots, n\}.$$

In particular, the notations $\mathbb{R}$ and $\mathbb{R}_+$ denote $\mathbb{R}^1$ and $\mathbb{R}^1_+$ respectively.

Definition 1. [1, 2] Let $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$.

(i) $\boldsymbol{x} \geq \boldsymbol{y}$ means $x_i \geq y_i$ for all $i = 1, 2, \ldots, n$.

(ii) let $\Omega \subset \mathbb{R}^n$, φ: $\Omega \to \mathbb{R}$ is said to be increasing if $\boldsymbol{x} \geq \boldsymbol{y}$ implies $\varphi(\boldsymbol{x}) \geq \varphi(\boldsymbol{y})$. φ is said to be decreasing if and only if $-\varphi$ is increasing.

Definition 2. [1, 3] Let $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$.

(i) $\boldsymbol{x}$ is said to be majorized by $\boldsymbol{y}$ (in symbols $\boldsymbol{x} \prec \boldsymbol{y}$) if $\sum_{i=1}^k x_{[i]} \leq \sum_{i=1}^k y_{[i]}$ for $k = 1, 2, \ldots, n-1$ and $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$, where $x_{[1]} \geq \cdots \geq x_{[n]}$ and $y_{[1]} \geq \cdots \geq y_{[n]}$ are rearrangements of $\boldsymbol{x}$ and $\boldsymbol{y}$ in a descending order.

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(*ii*) Let $\Omega \subset \mathbb{R}^n$, φ: $\Omega \to \mathbb{R}$ is said to be a Schur-convex function on Ω if $\boldsymbol{x} \prec \boldsymbol{y}$ on Ω implies $\varphi(\boldsymbol{x}) \le \varphi(\boldsymbol{y})$. φ is said to be a Schur-concave function on Ω if and only if $-\varphi$ is Schur-convex function on Ω.

Definition 3. [1, 3] Let $\boldsymbol{x} = (x_1, \dots, x_n)$ and $\boldsymbol{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$.

(*i*) $\Omega \subseteq \mathbb{R}^n$ is said to be a convex set if $\boldsymbol{x}, \boldsymbol{y} \in \Omega, 0 \le \alpha \le 1$ implies $\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y} = (\alpha x_1 + (1-\alpha)y_1, \dots, \alpha x_n + (1-\alpha)y_n) \in \Omega$.

(*ii*) Let $\Omega \subset \mathbb{R}^n$ be convex set. A function φ: $\Omega \to \mathbb{R}$ is said to be a convex function on Ω if

$$\varphi(\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y}) \le \alpha\varphi(\boldsymbol{x}) + (1-\alpha)\varphi(\boldsymbol{y})$$

for all $\boldsymbol{x}, \boldsymbol{y} \in \Omega$, and all $\alpha \in [0,1]$. φ is said to be a concave function on Ω if and only if $-\varphi$ is convex function on Ω.

The following so-called Schur's condition is very useful for determining whether or not a given function is Schur-convex or Schur-concave.

Lemma 1 ([1, p. 5])**.** *Let $\Omega \subset \mathbb{R}^n$ is symmetric and has a nonempty interior convex set. Ω^0 is the interior of Ω. $\varphi : \Omega \to \mathbb{R}$ is continuous on Ω and differentiable in Ω^0. Then φ is the $Schur-convex(Schur-concave)function$, if and only if φ is symmetric on Ω and*

$$(x_1 - x_2)\left(\frac{\partial\varphi}{\partial x_1} - \frac{\partial\varphi}{\partial x_2}\right) \ge 0 (\le 0) \tag{1}$$

holds for any $\boldsymbol{x} \in \Omega^0$.

Lemma 2. [4] *If φ is symmetric and convex (concave) on symmetric convex set Ω, then φ is Schur-convex $(Schur-concave)$ on Ω.*

2. On omission 1

In [1, p..84-85]and [3, p..122-123], the following proposition is proved.

G.1.h. If $g_j : \mathbb{R} \to \mathbb{R}_+$ and $\ln g_j$ is convex, $j = 1, \dots, n$, and if $h : \mathbb{R}^{n!} \to \mathbb{R}_+$ is symmetric, increasing, and convex, then

$$\psi(\boldsymbol{x}) = h\left(\prod_{j=1}^{n} g_j(x_{\pi_1(j)}), \dots, \prod_{j=1}^{n} g_j(x_{\pi_{n!}(j)})\right) \tag{2}$$

is symmetric and convex.

And by the proposition G.1.h, it follows the corollary

G.1.m. If $a > 0$,

$$\psi(\boldsymbol{x}) = \sum_{1 \le i_1 < \dots < i_k \le n} \prod_{j=1}^{k} \left(\frac{1}{x_{i_j}}\right)^a, \ k = 1, \dots, n \tag{3}$$

is symmetric decreasing and convex on $\mathbb{R}^n_+$.

Monographs [1] and [3]on the proof of the proposition G.1.m is the same, is reproduced as follows:

Proof of G.1.m.: In G.1.h take $g_j(x) = x^{-a}, j = 1, \dots, k, g_j(x) \equiv 1$ for $j = k+1, \dots, n$, and

$$h(z_1, \dots, z_{n!}) = \frac{(n-k)!}{n!k!} \sum_{i=1}^{n!} z_i \tag{4}$$

Thus ψ is symmetric and convex. The monotonicity is trivial.

The proof of G.1.m is completed.

Remark 1. In the above proof, the selected functions (4) is wrong. In fact, if taking $n = 3, k = 2$, then from (4) it follows that

$$\begin{aligned}\psi(\boldsymbol{x}) =& \frac{(3-2)!}{3!2!} \\ &\cdot \left[\left(\frac{1}{x_1x_2}\right)^a + \left(\frac{1}{x_1x_3}\right)^a + \left(\frac{1}{x_2x_3}\right)^a + \left(\frac{1}{x_2x_1}\right)^a + \left(\frac{1}{x_3x_1}\right)^a + \left(\frac{1}{x_3x_2}\right)^a\right] \\ =& \frac{1}{6}\left[\left(\frac{1}{x_1x_2}\right)^a + \left(\frac{1}{x_1x_3}\right)^a + \left(\frac{1}{x_2x_3}\right)^a\right].\end{aligned}$$

But from(3) it follows that

$$\psi(\boldsymbol{x}) = \left(\frac{1}{x_1x_2}\right)^a + \left(\frac{1}{x_1x_3}\right)^a + \left(\frac{1}{x_2x_3}\right)^a. \tag{5}$$

Two results are inconsistent. The function (4) should be amended to

$$h\left(z_1, \dots, z_{n!}\right) = \frac{n!}{(n-k)!k!} \sum_{i=1}^{n!} z_i. \tag{6}$$

3. On omission 2

In [1, p..89-91]and [3, p..127-129], the following Proposition G.3 and corollary G.3.a are proved.

G.3. Proposition. Let $\mathcal{A}$ be a symmetric convex subset of $\mathbf{R}^k$ and let φ be a Schur-convex function defined on $\mathcal{A}$ with the property that for each fixed $x_2, \dots, x_k$, $\varphi(z, x_2, \dots, x_k)$ is convex in z on $\{z : (z, x_2, \dots, x_k) \in \mathcal{A}\}$. Then for any $n > k$,

$$\psi(\boldsymbol{x}) = \psi\left(x_1, \dots, x_n\right) = \sum_{\pi} \varphi\left(x_{\pi(1)} \dots, x_{\pi(k)}\right) \tag{7}$$

is Schur-convex on

$$\mathcal{B} = \left\{(x_1, \dots, x_n) : \left(x_{\pi(1)}, \dots, x_{\pi(k)}\right) \in \mathcal{A} \; for\ all\ permutations\ \pi\right\}.$$

In most applications, $\mathcal{A}$ has the form I^k for some interval $I \subset \mathbf{R}$ and in this case $\mathcal{B} = I^n$. Notice that the convexity of φ in its first argument also implies that φ is convex in each argument, the other arguments being fixed, because φ is symmetric.

G.3.a. Notice that if

$$\tilde{\psi}(\boldsymbol{x}) = \psi(\boldsymbol{x})/k!(n-k)!$$

where ψ is the function defined in G.3, then

$$\tilde{\psi}(x_1, \dots, x_n) = \sum_{C(n,k)} \varphi(x_1, \dots, x_k),$$

where $(x_1, \dots, x_k)$ is generic notation for an arbitrary selection of k of the variables $x_1, \dots, x_n$ and $C(n, k)$ denotes summation over all C_n^k such selections. Of course, ψ is Schur-convex whenever ψ is Schur-convex.

In the monograph [1], by Proposition G.3 and corollary G.3.a, the following proposition is proved.

G.3.d. If $a > 0$, the function

$$\psi(\boldsymbol{x}) = \sum_{C(n,k)} \prod_{i=1}^{k} \left(\frac{1-x_i}{x_i} \right)^a \tag{8}$$

is Schur-convex on $\mathcal{B} = \{\boldsymbol{x} : \boldsymbol{x} \in \mathbb{R}_+^n \ and \ x_i + x_j \le 1 \ for \ all \ i \ne j\}$ and is Schur-concave on $\tilde{\mathcal{B}} = \{\boldsymbol{x} : \boldsymbol{x} \in \mathbb{R}_+^n \ and \ x_i + x_j \ge 1 \ for \ all \ i \ne j\}$.

The proposition G.3.d is G.3.e in [3]).

Remark 2. The first conclusion of the proposition G.3.d is right, and second conclusion is wrong. In fact, let $\varphi(\boldsymbol{z}) = \prod_{i=1}^{k}(1-z_i)/z_i$. Then

$$\frac{\partial \varphi(\boldsymbol{z})}{\partial z_1} = \varphi(\boldsymbol{z}) \left(\frac{-1}{1-z_1} - \frac{1}{z_1} \right), \ \frac{\partial \varphi(\boldsymbol{z})}{\partial z_2} = \varphi(\boldsymbol{z}) \left(\frac{-1}{1-z_2} - \frac{1}{z_2} \right),$$

$$\begin{aligned} \Delta &:= (z_1 - z_2) \left(\frac{\partial \varphi(\boldsymbol{z})}{\partial z_1} - \frac{\partial \varphi(\boldsymbol{z})}{\partial z_2} \right) \\ &= (z_1 - z_2)^2 \varphi(\boldsymbol{z}) \frac{1 - z_1 - z_2}{z_1 z_2 (1-z_1)(1-z_2)}. \end{aligned}$$

This shows $\Delta \ge 0$ for $\mathcal{A} = \{\boldsymbol{z} : \boldsymbol{z} \in (0,1)^k \ and \ z_i + z_j \le 1 \ for \ all \ i \ne j\}$ and $\Delta \le 0$ for $\tilde{\mathcal{A}} = \{\boldsymbol{z} : \boldsymbol{z} \in (0,1)^k \ and \ z_i + z_j \ge 1 \ for \ all \ i \ne j\}$. So by Lemma 1, it follows that $\psi(\boldsymbol{z})$ is Schur-convex and Schur-concave on $\mathcal{A}$ and $\tilde{\mathcal{A}}$ respectively. Further, $[\psi(\boldsymbol{z})]^a$ is Schur-convex on $\mathcal{A}$ for all $a > 0$.

Furthermore, let $g(t) = (1-t)/t$. Then $g''(t) = \frac{2}{t^3} > 0$ on (0,1). It follows that $\psi(\boldsymbol{z})$ is convex on (0,1) in each argument, by Proposition G.3 and corollary G.3.a, we conclude that $\psi(\boldsymbol{x})$ is Schur-convex on $\mathcal{B}$.

Now consider the case $a = 1$. Although $\psi(\boldsymbol{z})$ is Schur-concave on $\tilde{\mathcal{A}}$, but $\psi(\boldsymbol{z})$ is not concave on (0,1) in each argument, hence we can not conclude that $\psi(\boldsymbol{x})$ is Schur-concave on $\tilde{\mathcal{B}}$. Actually, for $n = 3, k = 2$, taking $\boldsymbol{x} = (2/3, 2/3, 2/3)$, $\boldsymbol{y} = (5/6, 2/3, 1/2)$, then $\boldsymbol{x}, \boldsymbol{y} \in \tilde{\mathcal{B}}$, and $\boldsymbol{x} \prec \boldsymbol{y}$. Suppose $\psi(\boldsymbol{x})$ is Schur-concave on $\tilde{\mathcal{B}}$, will lead to contradictory results as

$$\psi(\boldsymbol{x}) = \frac{3}{4} \ge \psi(\boldsymbol{y}) = \frac{4}{5}.$$

In 2009, Wei-Feng Xia and Yu-Ming Chu [5] established the following results:

Theorem A. For $2 \le k \le n$, $\psi(\boldsymbol{x})$ *is Schur-convex and Schur-concave on* $(0, \frac{2n-k-1}{2n-2}]^n$ *and* $[\frac{2n-k-1}{2n-2}, 1]^n$ *respectively.*

Theorem A also confirmed that the second conclusion of the corollary G.3.d is wrong. Since

$$D := \left[\frac{2n-k-2}{2n-2}, \frac{2n-k-1}{2n-2} \right]^n \subset \left(0, \frac{2n-k-1}{2n-2} \right]^n,$$

by Theorem A, $\psi(\boldsymbol{x})$ is Schur-convex on D. But when $k \le n-1$, for all $\boldsymbol{x} \in D$, we have

$$x_i + x_j \ge 2 \cdot \frac{2n-k-2}{2n-2} = \frac{2n-k-2}{n-1} \ge 1,$$

This means that $D \subset \tilde{\mathcal{B}}$.

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