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INEQUALITIES FOR MODULES AND UNITARY INVARIANT NORMS

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Abstract. Another elementary proof which doesn't use a matrix as in [9] and generalizations of the classical inequality of Bohr, see [3], Theorem 1, Corollary 1 and Theorem 2, are presented for linear bounded operators on Hilbert spaces or on pseudo-Hilbert spaces. Then several applications for unitary invariant norms and generalized derivative are given.

1. Introduction

We recall the classical Bohr′s inequality, see [3]. For any $z, w \in \mathbb{C}$ and $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$,

$$|z+w|^2 \leq p|z|^2 + q|w|^2,$$

with equality if and only if $w = (p-1)z$.

Many interesting generalizations of Bohr′s inequality have been obtained by Archbold, Makowski, Bergstrom, Mitrinovic, Mitrinovic et al., Vasic and Keckic, Rassias, Delbosco, and Pecaric and Janic, see [3]. Also in 1989, J. Pecaric and S.S. Dragomir generalized Bohr′ s inequality for normed vector spaces and O. Hirzallah [5], W.-S. Cheung, J. Pecaric [3], F. Zhang [9] and P. Chansangiam, P. Hemchote and P. Pantaragphong, see [2] and M. Fujii and H. Zuo [4] further generalized the inequality for operator algebras. S. Abramovich, J. Baric and J. Pecaric, [1] used superquadratic functions in order to obtain other interesting generalizations.

We need the following definition which was presented in [6]. Let Z be an admissible space in the Loynes sense. A linear topological space $\mathcal{H}$ is called pre-Loynes Z–space if it satisfies the following properties:

$\mathcal{H}$ is endowed with a Z–valued *inner product* (gramian), i.e. there exists an application $\mathcal{H} \times \mathcal{H} \ni (h,k) \to [h,k] \in Z$ having the properties: $[h,h] \geq 0$; $[h,h] = 0$ implies $h = 0$; $[h_1 + h_2, h] = [h_1, h] + [h_2, h]$; $[\lambda h, k] = \lambda[h,k]$; $[h,k]^* = [k,h]$; for all $h, k, h_1, h_2 \in \mathcal{H}$ and $\lambda \in \mathbb{C}$.

The topology of $\mathcal{H}$ is the weakest locally convex topology on $\mathcal{H}$ for which the application $\mathcal{H} \ni h \to [h,h] \in Z$ is continuous. Moreover, if $\mathcal{H}$ is a complete space with this topology, then $\mathcal{H}$ is called Loynes Z*–space* (pseudo-Hilbert space).

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2. The results

Lemma 2.2 ([7]) Let x and y be vectors in a normed linear space such that $||x+y|| = ||x|| + ||y||$, and let α and β be non-negative real numbers. Then $||\alpha x + \beta y|| = \alpha||x|| + \beta||y||$.

We can prove a similar result in pseudo-Hilbert spaces for example or Hilbert spaces for the modulus of a linear bounded operator.

Lemma 1. *Let $A,\ B \in \mathcal{B}^*(\mathcal{H})$ if $\mathcal{H}$ is a Loynes Z- space (or $A,\ B \in \mathcal{B}(\mathcal{H})$ if $\mathcal{H}$ is a Hilbert space).*

If $|A \pm B| = (|A|^2 + |B|^2)^{\frac{1}{2}}$ then $|\alpha A \pm \beta B| = (\alpha^2|A|^2 + \beta^2|B|^2)^{\frac{1}{2}}$, for all $\alpha,\ \beta$ non-negative real numbers.

Proof. By $|A \pm B|^2 = (A^* \pm B^*)(A \pm B) = |A|^2 + |B|^2 \pm B^*A \pm A^*B = |A|^2 + |B|^2$ it results that $B^*A + A^*B = 0$ and then $|\alpha A \pm \beta B|^2 = \alpha^2|A|^2 + \beta^2|B|^2$.
∎

We shall prove below another proof for a variant of Bohr's inequality that was given in [9] Theorem 6.

Theorem 1. *Let $A,\ B \in \mathcal{B}^*(\mathcal{H})$ where $\mathcal{H}$ is a Loynes Z- space or $A,\ B \in \mathcal{B}(\mathcal{H})$ where $\mathcal{H}$ is a complex separable Hilbert space.*

(i) If $\alpha,\ \beta,\ \gamma$ are three real numbers which satisfies the conditions $\beta \neq 0,\ \gamma \neq 0$, $\alpha + \beta\gamma > 0$ and $\alpha > 0$ or $\alpha + \beta\gamma < 0$ and $\alpha < 0$ then

$$|\alpha A - B|^2 + |\beta A - \gamma B|^2 \geq \beta(\beta - \alpha\gamma)|A|^2 + \gamma(\gamma - \frac{\beta}{\alpha})|B|^2.$$

(ii) If $\alpha,\ \beta,\ \gamma$ are three real numbers which satisfies the conditions $\alpha \neq 0,\ \beta \neq 0,\ \gamma \neq 0$, $\alpha + \beta\gamma > 0$ and $\frac{\beta}{\gamma} > 0$ or $\alpha + \beta\gamma < 0$ and $\frac{\beta}{\gamma} < 0$ then

$$|\alpha A - B|^2 + |\beta A - \gamma B|^2 \geq \alpha(\alpha - \frac{\beta}{\gamma})|A|^2 + (1 - \frac{\alpha\gamma}{\beta})|B|^2.$$

(iii) If $\alpha,\ \beta,\ \gamma$ are three real numbers which satisfies the conditions $\beta \neq 0,\ \gamma \neq 0$, $\alpha + \beta\gamma < 0$ and $\alpha > 0$ or $\alpha + \beta\gamma > 0$ and $\alpha < 0$ then

$$|\alpha A - B|^2 + |\beta A - \gamma B|^2 \leq \beta(\beta - \alpha\gamma)|A|^2 + \gamma(\gamma - \frac{\beta}{\alpha})|B|^2.$$

(iv) If $\alpha,\ \beta,\ \gamma$ are three real numbers which satisfies the conditions $\alpha \neq 0,\ \beta \neq 0,\ \gamma \neq 0$, $\alpha + \beta\gamma < 0$ and $\frac{\beta}{\gamma} > 0$ or $\alpha + \beta\gamma > 0$ and $\frac{\beta}{\gamma} < 0$ then

$$|\alpha A - B|^2 + |\beta A - \gamma B|^2 \leq \alpha(\alpha - \frac{\beta}{\gamma})|A|^2 + (1 - \frac{\alpha\gamma}{\beta})|B|^2.$$

Proof. Using the definition of the modulus of an operator we obtain,

$$|\alpha A - B|^2 + |\beta A - \gamma B|^2 = (\alpha^2 + \beta^2)|A|^2 + (\gamma^2 + 1)|B|^2 - (\alpha + \beta\gamma)(B^*A + A^*B) =$$

$$= [(\alpha + \beta\gamma)(\alpha + \frac{\beta}{\gamma}) - \alpha\beta(\gamma + \frac{1}{\gamma})]|A|^2 + [(\alpha + \beta\gamma)(\frac{\gamma}{\beta} + \frac{1}{\alpha}) - \gamma(\frac{\alpha}{\beta} + \frac{\beta}{\alpha})]|B|^2 -$$

$$-(\alpha + \beta\gamma)(B^*A + A^*B) = (\alpha + \beta\gamma)[(\alpha + \frac{\beta}{\gamma})|A|^2 + (\frac{\gamma}{\beta} + \frac{1}{\alpha})|B|^2 - (B^*A + A^*B)] -$$

$$-\alpha\beta(\gamma + \frac{1}{\gamma})|A|^2 - \gamma(\frac{\alpha}{\beta} + \frac{\beta}{\alpha})|B|^2 = (\alpha + \beta\gamma)[\alpha|A|^2 + \frac{\beta}{\gamma}|A|^2 + \frac{\gamma}{\beta}|B|^2 + \frac{1}{\alpha}|B|^2 - (B^*A +$$

$$+A^*B)]-\alpha\beta(\gamma+\frac{1}{\gamma})|A|^2-\gamma(\frac{\alpha}{\beta}+\frac{\beta}{\alpha})|B|^2.$$

For (i) we will consider the following continuation

$$|\alpha A-B|^2+|\beta A-\gamma B|^2=(\alpha+\beta\gamma)\{[\alpha|A|^2-(B^*A+A^*B)+\frac{1}{\alpha}|B|^2]+[\frac{\beta}{\gamma}|A|^2+\frac{\gamma}{\beta}|B|^2]\}-$$

$$-\alpha\beta(\gamma+\frac{1}{\gamma})|A|^2-\gamma(\frac{\alpha}{\beta}+\frac{\beta}{\alpha})|B|^2=(\alpha+\beta\gamma)|\sqrt{\alpha}A-\frac{1}{\sqrt{\alpha}}B|^2+[(\alpha+\beta\gamma)\frac{\beta}{\gamma}-\alpha\beta(\gamma+\frac{1}{\gamma})]|A|^2+$$

$$+[(\alpha+\beta\gamma)\frac{\gamma}{\beta}-\gamma(\frac{\alpha}{\beta}+\frac{\beta}{\alpha})]|B|^2\geq\beta(\beta-\alpha\gamma)|A|^2+\gamma(\gamma-\frac{\beta}{\alpha})|B|^2.$$

For the second part of (i) we will write

$$|\alpha A-B|^2+|\beta A-\gamma B|^2=(\alpha+\beta\gamma)\{[-|\alpha||A|^2-(B^*A+A^*B)-\frac{1}{|\alpha|}|B|^2]+[\frac{\beta}{\gamma}|A|^2+\frac{\gamma}{\beta}|B|^2]\}-$$

$$-\alpha\beta(\gamma+\frac{1}{\gamma})|A|^2-\gamma(\frac{\alpha}{\beta}+\frac{\beta}{\alpha})|B|^2=-(\alpha+\beta\gamma)|\sqrt{|\alpha|}A+\frac{1}{\sqrt{|\alpha|}}B|^2+\beta(\beta-\alpha\gamma)|A|^2+$$

$$+\gamma(\gamma-\frac{\beta}{\gamma})|B|^2\geq\beta(\beta-\alpha\gamma)|A|^2+\gamma(\gamma-\frac{\beta}{\alpha})|B|^2.$$

For (ii) we have,

$$|\alpha A-B|^2+|\beta A-\gamma B|^2=(\alpha+\beta\gamma)\{[\alpha|A|^2+\frac{1}{\alpha}|B|^2]+[\frac{\beta}{\gamma}|A|^2-(B^*A+A^*B)+\frac{\gamma}{\beta}|B|^2]\}-$$

$$-\alpha\beta(\gamma+\frac{1}{\gamma})|A|^2-\gamma(\frac{\alpha}{\beta}+\frac{\beta}{\alpha})|B|^2=(\alpha+\beta\gamma)|\sqrt{\frac{\beta}{\gamma}}A-\sqrt{\frac{\gamma}{\beta}}B|^2+[(\alpha+\beta\gamma)\alpha-$$

$$-\alpha\beta(\gamma+\frac{1}{\gamma})]|A|^2+[(\alpha+\beta\gamma)\frac{1}{\alpha}-\gamma(\frac{\alpha}{\beta}+\frac{\beta}{\alpha})]|B|^2\geq\alpha(\alpha-\frac{\beta}{\gamma})|A|^2+(1-\frac{\alpha\gamma}{\beta})|B|^2.$$

(iii) and (iv) will result from (i) and (ii). ∎

Consequence 1. *(i) Taking in Theorem 3 (i), first part,* $\alpha=1-p,\ \beta=\gamma=1$ *or in Theorem 3 (ii), first part,* $\alpha=1,\ \beta=1-p,\ \gamma=1$ *we obtain the inequality from* [3], *Theorem 2 for* $p<1$

$$|(1-p)A-B|^2+|A-B|^2\geq p|A|^2+q|B|^2,$$

with $\frac{1}{p}+\frac{1}{q}=1$.

(ii) Now taking in Theorem 3 (i), second part, $\alpha=1-p,\ \beta=1,\ \gamma=1$ *we obtain the inequality from* [3], *Corollary 1 for* $p>2$

$$|(1-p)A-B|^2+|A-B|^2\geq p|A|^2+q|B|^2,$$

with $\frac{1}{p}+\frac{1}{q}=1$.

(iii) Considering in Theorem 3 (iii), second part, again $\alpha=1-p,\ \beta=1,\ \gamma=1$ *we obtain* $1<p<2$ *and inequality*

$$|(1-p)A-B|^2+|A-B|^2\leq p|A|^2+q|B|^2,$$

with $\frac{1}{p}+\frac{1}{q}=1$, *as in* [3], *Theorem 1.*

(iv) If $\alpha=1$ *in Theorem 3 (i) first part then*

$$|A-B|^2+|\beta A-\gamma B|^2\geq(\beta-\gamma)(\beta|A|^2-\gamma|B|^2),$$

for $1+\beta\gamma>0$.

(v) If we take $\alpha = 1-p,\ \beta = 1,\ \gamma = -1$ in Theorem 3 (iii) first part then for $0 < p < 1$

$$|(1-p)A - B|^2 + |A+B|^2 \le (2-p)|A|^2 + \frac{2-p}{1-p}|B|^2.$$

(i) Indeed the conditions $\alpha + \beta\gamma > 0$ and $\alpha > 0$ become for example for $\alpha = 1-p,\ \beta = \gamma = 1,\ p < 2$ and $p < 1$ and then

$$|(1-p)A-B|^2+|A-B|^2 \ge (1-(1-p))|A|^2+(1-\frac{1}{1-p})|B|^2 = p|A|^2+\frac{p}{p-1}|B|^2 =$$

$$= p|A|^2 + q|B|^2.$$

Consequence 2. *Let $A,\ B \in \mathcal{B}^*(\mathcal{H})$ where $\mathcal{H}$ is a Loynes Z- space or $A,\ B \in \mathcal{B}(\mathcal{H})$ where $\mathcal{H}$ is a complex separable Hilbert space.*

(i) If $a,\ b,\ c,\ d \in \mathbb{R},\ \ a,\ b,\ c,\ d \ne 0$ and satisfies the conditions $ab + cd > 0$, and $\frac{a}{b} > 0$ or $ab + cd < 0$, and $\frac{a}{b} < 0$ then

$$|aA - bB|^2 + |cA - dB|^2 \ge c^2(1-\frac{ad}{cb})|A|^2 + d^2(1-\frac{cb}{ad})|B|^2,$$

or if $a,\ b,\ c,\ d \in \mathbb{R},\ a,\ b,\ c,\ d \ne 0$ and satisfies the conditions $ab + cd > 0$ and $\frac{c}{d} > 0$ or $ab + cd < 0$ and $\frac{c}{d} < 0$ then

$$|aA - bB|^2 + |cA - dB|^2 \ge a^2(1-\frac{bc}{ad})|A|^2 + b^2(1-\frac{ad}{bc})|B|^2.$$

(ii) If $a,\ b,\ c,\ d \in \mathbb{R},\ \ a,\ b,\ c,\ d \ne 0$ and satisfies the conditions $ab + cd < 0$, and $\frac{a}{b} > 0$ or $ab + cd > 0$, and $\frac{a}{b} < 0$ then

$$|aA - bB|^2 + |cA - dB|^2 \le c^2(1-\frac{ad}{cb})|A|^2 + d^2(1-\frac{cb}{ad})|B|^2,$$

or if $a,\ b,\ c,\ d \in \mathbb{R},\ a,\ b,\ c,\ d \ne 0$ and satisfies the conditions $ab + cd < 0$ and $\frac{c}{d} > 0$ or $ab + cd > 0$ and $\frac{c}{d} < 0$ then

$$|aA - bB|^2 + |cA - dB|^2 \le a^2(1-\frac{bc}{ad})|A|^2 + b^2(1-\frac{ad}{bc})|B|^2.$$

Proof. We write the expression $|aA-bB|^2+|cA-dB|^2$ as $b^2(|\frac{a}{b}A-B|^2+|\frac{c}{b}A-\frac{d}{b}B|^2)$ and use Theorem 1. ∎

Theorem 2. *(i) For any $r \ge 2,\ a,\ b \in \mathbb{R}_+$ and $u,\ v > 0$ with the property*

$$uv(u+v)^{r-2} = 1$$

we have

$$(1+\frac{u}{v})a^r + (1+\frac{v}{u})b^r \ge (ua+vb)^r + \frac{1}{2^{r-2}}|b-a|^r.$$

(ii) For any $1 \le r \le 2,\ a,\ b \in \mathbb{R}_+$ and $u,\ v > 0$ with the property

$$uv(u+v)^{r-2} = 1$$

we have

$$(1+\frac{u}{v})a^r + (1+\frac{v}{u})b^r \le (ua+vb)^r + \frac{1}{2^{r-2}}|b-a|^r.$$

Proof. Using that the functions $f(x) = x^r$, $x \geq 0$ are superquadratic for $r \geq 2$ and subquadratic for $0 \leq r \leq 2$, see [1] we obtain by Lemma 1.2, see [1], for $0 \leq \alpha_1 \leq 1$, a, $b \geq 0$ that

$$\alpha_1 f(a)+(1-\alpha_1)f(b)-f(\alpha_1 a+(1-\alpha_1 b)) \geq \alpha_1 f((1-\alpha_1)|b-a|)+(1-\alpha_1)f(\alpha_1|b-a|).$$

(i) As in the proof of Theorem 2.3 [1] we have

$$\alpha_1 a^r + \beta_1 b^r \geq (\alpha_1 a + \beta_1 b)^r + \alpha_1\beta_1(\beta_1^{r-1} + \alpha_1^{r-1})|b-a|^r,$$

where $\beta_1 = 1 - \alpha_1$.

For $\alpha_1 = \frac{\beta}{\beta+\gamma}$, $\beta_1 = \frac{\gamma}{\beta+\gamma}$, β, $\gamma > 0$ it results

$$\frac{\beta}{\beta+\gamma}a^r+\frac{\gamma}{\beta+\gamma}b^r \geq (\frac{\beta}{\beta+\gamma}a+\frac{\gamma}{\beta+\gamma}b)^r+\frac{\beta\gamma}{(\beta+\gamma)^2}[(\frac{\beta}{\beta+\gamma})^{r-1}+(\frac{\gamma}{\beta+\gamma})^{r-1}]|b-a|^r$$

Because $\frac{\beta}{\beta+\gamma} = 1 - \frac{\gamma}{\beta+\gamma}$, $\frac{\beta}{\beta+\gamma}$, $\frac{\gamma}{\beta+\gamma} \in [0,\ 1]$ and $r \geq 2$ we have $(\frac{\beta}{\beta+\gamma})^{r-1} + (\frac{\gamma}{\beta+\gamma})^{r-1} \geq \frac{1}{2^{r-2}}$ and this implies

$$\frac{(\beta a^r + \gamma b^r)}{\beta+\gamma} \geq \frac{(\beta a + \gamma b)^r}{(\beta+\gamma)^r} + \frac{\beta\gamma}{(\beta+\gamma)^2}\frac{1}{2^{r-2}}|b-a|^r$$

or

$$\frac{\beta+\gamma}{\beta\gamma}(\beta a^r + \gamma b^r) \geq \frac{(\beta a + \gamma b)^r}{\beta\gamma(\beta+\gamma)^{r-2}} + \frac{1}{2^{r-2}}|b-a|^r$$

which leads to

$$\frac{\beta+\gamma}{\gamma}a^r + \frac{\beta+\gamma}{\beta}b^r \geq (\frac{\beta}{(\beta\gamma)^{\frac{1}{r}}(\beta+\gamma)^{\frac{r-2}{r}}}a + \frac{\gamma}{(\beta\gamma)^{\frac{1}{r}}(\beta+\gamma)^{\frac{r-2}{r}}}b)^r + \frac{1}{2^{r-2}}|b-a|^r.$$

Now taking $\gamma = \frac{v}{u}\beta$ where $\beta > 0$ we have $\gamma > 0$, $\frac{\beta}{\gamma} = \frac{u}{v}$ and our inequality becomes

$$(1+\frac{u}{v})a^r + (1+\frac{v}{u})b^r \geq (ua+vb)^r + \frac{1}{2^{r-2}}|b-a|^r$$

because

$$\frac{\beta}{(\beta\gamma)^{\frac{1}{r}}(\beta+\gamma)^{\frac{r-2}{r}}} = \frac{\beta}{(\beta^2\frac{v}{u})^{\frac{1}{r}}(\beta+\frac{v}{u}\beta)^{\frac{r-2}{r}}} = \frac{u}{u^{\frac{1}{r}}v^{\frac{1}{r}}(u+v)^{1-\frac{2}{r}}} = u$$

and

$$\frac{\gamma}{(\beta\gamma)^{\frac{1}{r}}(\beta+\gamma)^{\frac{r-2}{r}}} = v$$

by hypothesis.

For (ii) we will use the same argument taking into account that $\alpha_1^{r-1} + \beta_1^{r-1} \leq \frac{1}{2^{r-2}}$ if $1 \leq r \leq 2$. ∎

Corollary 1. *If we take above $u = v(p-1)$ when $r \geq 2$ and $1 < p \leq 2$ then*

$$pa^r + qb^r \geq \frac{1}{2^{r-2}}((p-1)a+b)^r + \frac{1}{2^{r-2}}|b-a|^r,$$

for any a, $b \in \mathbb{R}_+$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. From $u = v(p-1)$ and $1 < p \leq 2$ it results $u > 0$ and

$$pa^r + qb^r = (1 + p - 1)a^r + (1 + \frac{1}{p-1})b^r \geq v^r((p-1)a + b)^r + \frac{1}{2^{r-2}}|b - a|^r.$$

Replacing u from $u = v(p-1)$ in $uv(u+v)^{r-2} = 1$ we have $v^2(p-1)[v(p-1)+v]^{r-2} = 1$ or $v^r(p-1)p^{r-2} = 1$ that means $v^r = \frac{1}{(p-1)p^{r-2}} \geq \frac{1}{2^{r-2}}$. Thus

$$pa^r + qb^r = (1 + p - 1)a^r + (1 + \frac{1}{p-1})b^r \geq \frac{1}{2^{r-2}}((p-1)a + b)^r + \frac{1}{2^{r-2}}|b - a|^r.$$

∎

3. Applications

Let $\mathcal{B}(\mathcal{H})$ denote the algebra of all bounded linear operators on a complex separable Hilbert space $\mathcal{H}$. We shall consider as in [5], a unitarily invariant norm $|||.|||$ which is a norm on an ideal $C_{|||.|||}$ of $\mathcal{B}(\mathcal{H})$, making $C_{|||.|||}$ a Banach space and satisfying $|||UXV||| = |||X|||$ for all $X \in \mathcal{B}(\mathcal{H})$ and all unitary operators U and V in $\mathcal{B}(\mathcal{H})$.

Lemma 2 ([5]). *Let $A, B, X \in \mathcal{B}(\mathcal{H})$ where $\mathcal{H}$ is a Hilbert space with A and B self-adjoint and $X \geq \gamma I$, γ being a positive real number. Then*

$$\gamma|||A - B||| \leq |||AX - BX|||.$$

In the following we shall give some generalizations of the Theorem 2 of O. Hirzallah, see [5].

Theorem 3. *Let $A,\ B,\ X \in \mathcal{B}(\mathcal{H})$ where $\mathcal{H}$ is a complex separable Hilbert space and $X \geq \gamma I$, for positive real number γ.*

If $a,\ b,\ c,\ d \in \mathbb{R}$, $a,\ b,\ c,\ d \neq 0$ and satisfies the conditions $ab + cd < 0$ and $\frac{c}{d} > 0$ or $ab + cd > 0$ and $\frac{c}{d} < 0$ then

$$\gamma|||\ |aA - bB|^2 + |cA - dB|^2||| \leq |||a^2(1 - \frac{bc}{ad})|A|^2X + b^2(1 - \frac{ad}{bc})X|B|^2|||.$$

Proof. We will replace in Lemma 2, A by $a^2(1 - \frac{bc}{ad})|A|^2$ and B by $-b^2(1 - \frac{ad}{bc})|B|^2$ and we will use Consequence 3, (ii). ∎

Remark 1. *(a) Let $A, B, X \in \mathcal{B}(\mathcal{H})$ where $\mathcal{H}$ is a Hilbert space as in* [5] *and $1 + \beta\gamma < 0$. If $X \geq \gamma_1 I$, γ_1 being a positive real number then*

$$\gamma_1|||\ |A - B|^2 + |\beta A - \gamma B|^2||| \leq |\beta - \gamma||||\beta|A|^2X - \gamma X|B|^2|||.$$

(b) If we take in Theorem 3, (i), $a = 1 - p$, $b = c = d = 1$ we obtain

$$\gamma_1|||\ |(1 - p)A - B|^2 + |A - B|^2||| \leq |||\ p|A|^2X + qX|B|^2|||,$$

when $1 < p < 2$ and $\frac{1}{p} + \frac{1}{q} = 1$.

(c) If we take in Theorem 3, (ii), $a = b = c = 1$ and $d = 1 - q$ we obtain

$$\gamma_1|||\ |A - B|^2 + |A - (1 - q)B|^2||| \leq |||\ p|A|^2X + qX|B|^2|||,$$

when $p > 2$ and $\frac{1}{p} + \frac{1}{q} = 1$.

We recall that the generalized derivation $\delta_{A,B}$ of two operators $A,\ B \in \mathcal{B}(\mathcal{H})$ is defined by $\delta_{A,B}(X) = AX - XB$ for all $X \in \mathcal{B}(\mathcal{H})$. Also $\delta^2_{A,B}(X) = \delta_{A,B}(\delta_{A,B}(X))$.

We shall give a generalization of the Theorem 4 from [5]. The constants p and q from Theorem 4 can be replaced by $a,\ b,\ c,\ d$ as below.

Proposition 1. *Let $A,\ B,\ X \in \mathcal{B}(\mathcal{H})$ where $\mathcal{H}$ is a complex separable Hilbert space and $A,\ B$ are normal operators.*

(i) If $a,\ b,\ c,\ d \in \mathbb{R},\ \ a,\ b,\ c,\ d \neq 0$ and satisfies the conditions $ab + cd < 0$, and $\frac{a}{b} > 0$ or $ab + cd > 0$, and $\frac{a}{b} < 0$ then

$$||\delta^2_{aA,bB}(X)||^2_2 + ||\delta^2_{cA,dB}(X)||^2_2 \leq ||\ c^2(1 - \frac{ad}{bc})|A|^2X + d^2(1 - \frac{cb}{ad})X|B|^2||^2_2.$$

(ii) If $a,\ b,\ c,\ d \in \mathbb{R},\ \ a,\ b,\ c,\ d \neq 0$ and satisfies the conditions $ab + cd > 0$, and $\frac{a}{b} > 0$ or $ab + cd < 0$, and $\frac{a}{b} < 0$ then

$$||\delta^2_{aA,bB}(X)||^2_2 + ||\delta^2_{cA,dB}(X)||^2_2 \geq \frac{1}{2}||\ c^2(1 - \frac{ad}{bc})|A|^2X + d^2(1 - \frac{cb}{ad})X|B|^2||^2_2.$$

Proof. It will be as in [5], Theorem 4, but we will use Consequence 3 (ii), (i) instead of the inequality (7), see [5]. Also we use the elementary inequalities $(x + y)^2 \leq 2(x^2 + y^2)$ and $x^2 + y^2 \leq (x + y)^2$, for all $x,\ y \geq 0$. ∎

We know, see [8] for example, that for $uv(u + v) > 0$ and $r > 1,\ z_1,\ z_2 \in \mathbb{C}$ we have

$$\frac{|z_1 + z_2|^r}{u + v} \leq \frac{|z_1|^r}{u} + \frac{|z_2|^r}{v}$$

and for $uv(u + v) < 0$ and $r > 1,\ z_1,\ z_2 \in \mathbb{C}$ we have

$$\frac{|z_1 + z_2|^r}{u + v} \geq \frac{|z_1|^r}{u} + \frac{|z_2|^r}{v}.$$

Considering $r \in \mathbb{N},\ r \neq 0$ we can define $\delta^r_{A,B}(X)$ by $\delta^r_{A,B}(X) = \delta_{A,B}(\delta^{r-1}_{A,B}(X))$ and by induction we can show that

$$\delta^r_{A,B}(X) = A^rX - C^1_rA^{r-1}XB + ... + (-1)^{r-1}C^{r-1}_rAXB^{r-1} + (-1)^rXB^r.$$

We will give two similar properties for the generalized derivation $\delta_{A,B}$ of two normal operators $A,\ B \in \mathcal{B}(\mathcal{H})$.

Proposition 2. *Let $A,\ B,\ X \in \mathcal{B}(\mathcal{H})$ where $\mathcal{H}$ is a complex separable Hilbert space and $A,\ B$ are normal operators.*

(i) If $uv > 0$ and $r \geq 1,\ r \in \mathbb{N}$ then

$$\frac{1}{(u + v)^2}||\delta^r_{A,-B}(X)||^2_2 \leq ||\frac{1}{u}|A|^rX + \frac{1}{v}X|B|^r||^2_2.$$

for every $X \in \mathcal{B}(\mathcal{H})$.

(ii) If $r \in \mathbb{N},\ r \geq 2$ and $w_1,\ w_2 \in \mathbb{R}_+$ then

$$||\delta^r_{A,-B}(X)||^2_2 \leq ||\frac{w_1}{w_1^{\frac{1}{1-r}} + w_2^{\frac{1}{1-r}}}|A|^rX + \frac{w_2}{w_1^{\frac{1}{1-r}} + w_2^{\frac{1}{1-r}}}X|B|^r||^2_2.$$

(iii) If $r \geq 3,\ r \in \mathbb{N}$ and $D_1,\ D_2$ are two diagonal positive operators in $\mathcal{B}(\mathcal{H})$ with $u,\ v$ as in Theorem 2 then

$$||\delta^r_{uD_1,-vD_2}(X)||^2_2 + \frac{1}{2^{2(r-2)}}||\delta^r_{D_1,D_2}(X)||^2_2 \leq ||(1 + \frac{u}{v})D^r_1X + (1 + \frac{v}{u})XD^r_2||^2_2.$$

Proof. We use the same method as in [5]. We use Voiculescu's Theorem. For a given $\varepsilon > 0$ there are diagonal operators $D_1,\ D_2$ and Hilbert-Schmidt operators $K_1,\ K_2$ such that $A = D_1 + K_1,\ B = D_2 + K_2,\ ||K_1|| < \varepsilon,\ ||K_2|| < \varepsilon,\ D_1 e_i = \lambda_i e_i$ and $D_2 f_i = \mu_i f_i,\ i \in \mathbb{N}$ for some orthonormal bases $\{e_i\}$ and $\{f_i\}$ for $\mathcal{H}$ and some sequences $\{\lambda_i\}$ and $\{\mu_i\}$ of complex numbers. If we show that the inequality is true for $D_1,\ D_2$ then by a limit argument the inequality from (i), (ii) will be true.

For (i), we calculate

$$\frac{1}{(u+v)^2}||\delta^r_{D_1,-D_2}(X)||_2^2 = \frac{1}{(u+v)^2}\sum_{i,j=1}^{\infty} |<\delta^r_{D_1,-D_2}(X)f_j, e_i>|^2 =$$

$$= \sum_{i,j=1}^{\infty}(\frac{|\lambda_i+\mu_j|^r}{u+v})^2|<Xf_j,e_i>|^2 \leq \sum_{i,j=1}^{\infty}(\frac{|\lambda_i|^r}{u}+\frac{|\mu_j|^r}{v})^2|<Xf_j,e_i>|^2 =$$

$$= ||\frac{1}{u}|D_1|^r X + \frac{1}{v}X|D_2|^r||^2.$$

For (ii), we have

$$||\delta^r_{D_1,-D_2}(X)||_2^2 = \sum_{i,j=1}^{\infty} |<\delta^r_{D_1,-D_2}(X)f_j,e_i>|^2 = \sum_{i,j=1}^{\infty}(|\lambda_i+\mu_j|^r)^2|<Xf_j,e_i>|^2 \leq$$

$$\leq \sum_{i,j=1}^{\infty}(\frac{w_1}{w_1^{\frac{1}{1-r}}+w_2^{\frac{1}{1-r}}}|\lambda_i|^r + \frac{w_2}{w_1^{\frac{1}{1-r}}+w_2^{\frac{1}{1-r}}}|\mu_j|^r)^2|<Xf_j,e_i>|^2 =$$

$$= ||\frac{w_1}{w_1^{\frac{1}{1-r}}+w_2^{\frac{1}{1-r}}}|D_1|^r X + \frac{w_2}{w_1^{\frac{1}{1-r}}+w_2^{\frac{1}{1-r}}}X|D_2|^r||_2^2,$$

taking into account that we used the inequality (1.1) from Theorem 1, see [8].

For (iii), we use Theorem 2 (i) and we obtain:

$$||\delta^r_{uD_1,-vD_2}(X)||_2^2 + \frac{1}{2^{2(r-2)}}||\delta^r_{D_1,D_2}(X)||_2^2 = \sum_{i,j=1}^{\infty}|<\delta^r_{uD_1,-vD_2}(X)f_j,e_i>|^2 +$$

$$+\frac{1}{2^{2(r-2)}}\sum_{i,j=1}^{\infty}|<\delta^r_{D_1,D_2}(X)f_j,e_i>|^2 = \sum_{i,j=1}^{\infty}|<|u\lambda_i+v\mu_j|^r Xf_j,e_i>|^2 +$$

$$+\frac{1}{2^{2(r-2)}}\sum_{i,j=1}^{\infty}|<|\lambda_i-\mu_j|^r Xf_j,e_i>|^2 = \sum_{i,j=1}^{\infty}((u\lambda_i+v\mu_j)^{2r}+\frac{1}{2^{2(r-2)}}|\lambda_i-\mu_j|^{2r})$$

$$\cdot|<Xf_j,e_i>|^2 \leq \sum_{i,j=1}^{\infty}((u\lambda_i+v\mu_j)^r + \frac{1}{2^{(r-2)}}|\lambda_i-\mu_j|^r)^2|<Xf_j,e_i>|^2 \leq$$

$$\leq \sum_{i,j=1}^{\infty}((1+\frac{u}{v})\lambda_i^r + (1+\frac{v}{u})\mu_j^r)^2|<Xf_j,e_i>|^2 = ||(1+\frac{u}{v})D_1^r X + (1+\frac{v}{u})XD_2^r||_2^2.$$

∎

We also can replace in Theorem 3 from [5] the constants p and q with c and d as below.

Proposition 3. *Let* $A,\ B,\ X \in \mathcal{B}(\mathcal{H})$ *where* $\mathcal{H}$ *is a complex separable Hilbert space and* $c,\ d \in \mathbb{R} - \{0\}$ *with* $cd < -1$. *If* $X \in \mathcal{B}(\mathcal{H})$ *with* $X \geq (c-d)(c|A|^2 - d|B|^2)$, *then*

$$||| \, |A - B|^4 ||| \leq |c-d| \cdot |||c|A|^2X - dX|B|^2|||.$$

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