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A PROPERTY OF OPERATORS WITH ORTHOGONAL RANGES

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ABSTRACT. Let $\mathcal{B}(\mathcal{H})$ be the space of linear and bounded operators on Hilbert space $\mathcal{H}$ and $\mathcal{B}^*(\mathcal{H})$ the space of linear and bounded operators which admit adjoint on $\mathcal{H}$, if $\mathcal{H}$ is a pseudo-Hilbert space. We will present an improvement of the equality

$$\sum_{i=1}^{n} r_i|A_i|^2 - |\sum_{i=1}^{n} A_i|^2 = \sum_{1\leq i<j\leq n} |\sqrt{\frac{r_i}{r_j}}A_i - \sqrt{\frac{r_j}{r_i}}A_j|^2,$$

where $r_i > 1$, $A_i \in \mathcal{B}(\mathcal{H})$ or $A_i \in \mathcal{B}^*(\mathcal{H})$ and $\sum_{i=1}^{n} \frac{1}{r_i} = 1$ for $i = 1, ..., n$, see [3]. Then we will prove some properties of linear and bounded operators on Hilbert spaces with orthogonal ranges.

1. Introduction

We can consider instead of gramian normal commutative operators only linear and bounded operators and the following result which generalize the identity (4) from Theorem 3, see [5], [3] and [2].

Theorem 1. *If $n \in N$, $n \geq 2$, $N_1, N_2, ..., N_n \in \mathcal{B}^*(\mathcal{H})$, where $\mathcal{H}$ is a Loynes Z-space and $a_1, a_2, ..., a_n \in R\setminus\{0\}$ with $a_1 + a_2 + ... + a_n \neq 0$ then,*

$$\frac{|N_1|^2}{a_1} + \frac{|N_2|^2}{a_2} + ... + \frac{|N_n|^2}{a_n} - \frac{|N_1 + N_2 + ... + N_n|^2}{a_1 + a_2 + ... + a_n} =$$

$$= \frac{1}{a_1 + a_2 + ... + a_n} \sum_{1\leq i<j\leq n} \frac{|a_iN_j - a_jN_i|^2}{a_ia_j}, \tag{1}$$

where $|N| = (N^*N)^{\frac{1}{2}}$ is the modulus of N.

Proof. The proof results by direct calculations, we don't need N_i to be normal operators.

$$\frac{|N_1|^2}{a_1} + \frac{|N_2|^2}{a_2} + ... + \frac{|N_n|^2}{a_n} - \frac{|N_1 + N_2 + ... + N_n|^2}{a_1 + a_2 + ... + a_n} =$$

$$= \frac{N_1^*N_1}{a_1} + \frac{N_2^*N_2}{a_2} + ... + \frac{N_n^*N_n}{a_n} - \frac{(N_1^* + N_2^* + ... + N_n^*)(N_1 + N_2 + ... + N_n)}{a_1 + a_2 + ... + a_n} =$$

$$= \frac{1}{a_1a_2...a_n(a_1 + a_2 + ... + a_n)}[(a_1+a_2+...+a_n)(a_2...a_nN_1^*N_1+a_1a_3...a_nN_2^*N_2+...+$$

$$+a_1...a_{n-1}N_n^*N_n) - a_1a_2...a_n(N_1^*N_1 + ... + N_n^*N_n) - a_1a_2...a_n(N_1^*N_2 + ... + N_1^*N_n+$$

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$$+N_2^*N_1+N_2^*N_3+...+N_n^*N_{n-1})] = \frac{1}{a_1a_2...a_n(a_1+a_2+...+a_n)}[a_3a_4...a_n(a_1^2N_2^*N_2+$$
$$+a_2^2N_1^*N_1 - a_1a_2N_2^*N_1 - a_1a_2N_1^*N_2) + a_2a_4...a_n(a_1^2N_3^*N_3 + a_3^2N_1^*N_1 - a_1a_3N_3^*N_1-$$
$$-a_1a_3N_1^*N_3) + ... + a_1a_2...a_{n-2}(a_{n-1}^2N_n^*N_n + a_n^2N_{n-1}^*N_{n-1} - a_na_{n-1}N_n^*N_{n-1}-$$
$$-a_na_{n-1}N_{n-1}^*N_n)] = \frac{1}{a_1a_2...a_n(a_1+a_2+...+a_n)}[a_3a_4...a_n|a_2N_1 - a_1N_2|^2+$$
$$+a_2a_4a_5...a_n|a_3N_1 - a_1N_3|^2 + ... + a_1a_2...a_{n-2}|a_{n-1}N_n - a_nN_{n-1}|^2] =$$
$$= \frac{1}{a_1+a_2+...+a_n}\sum_{1\le i<j\le n}\frac{|a_iN_j - a_jN_i|^2}{a_ia_j}.$$

∎

Consequence 1. *Using the above theorem where a_i will be replaced by $\frac{1}{a_i}$, $i = 1, 2, ..., n$, and take $a_i > 0$ such that $\frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n} = 1$ we obtain exactly Theorem 4.2 from* [3] *if we consider $\mathcal{H}$ as being a complex separable Hilbert space instead of Loynes space and $N_i \in \mathcal{B}(\mathcal{H})$*

$$a_1|N_1|^2 + a_2|N_2|^2 + ... + a_n|N_n|^2 - \frac{|N_1+N_2+...+N_n|^2}{\frac{1}{a_1}+\frac{1}{a_2}+...+\frac{1}{a_n}} =$$
$$= \frac{1}{\frac{1}{a_1}+\frac{1}{a_2}+...+\frac{1}{a_n}}\sum_{1\le i<j\le n} a_ia_j\left|\frac{N_j}{a_i} - \frac{N_i}{a_j}\right|^2 = \sum_{1\le i<j\le n}\left|\frac{\sqrt{a_j}}{\sqrt{a_i}}N_j - \frac{\sqrt{a_i}}{\sqrt{a_j}}N_i\right|^2.$$

If we don't consider above $a_i > 0$, $i = 1, ..., n$ then we obtain the inequality from Theorem 4.3, see [3].

Remark 1. *(i) If $n \in N$, $n \ge 2$, $N_1, N_2, ..., N_n \in \mathcal{B}^*(\mathcal{H})$ and $a_1, a_2, ..., a_n \in R - \{0\}$ with $a_1 + a_2 + ... + a_n > 0$ then,*

$$\frac{|N_1|^2}{a_1} + \frac{|N_2|^2}{a_2} + ... + \frac{|N_n|^2}{a_n} \ge \frac{1}{a_1+a_2+...+a_n}\sum_{1\le i<j\le n}\frac{|a_iN_j - a_jN_i|^2}{a_ia_j}.$$

(ii) Under the above conditions, if a_1, a_2, ..., $a_n \in (0, \infty)$ we also have,

$$\frac{|N_1|^2}{a_1} + \frac{|N_2|^2}{a_2} + ... + \frac{|N_n|^2}{a_n} \ge \frac{|N_1+N_2+...+N_n|^2}{a_1+a_2+...+a_n},$$

with equality if and only if $a_iN_j = a_jN_i$, for any $i, j \in \{1, 2, ..., n\}$.

Using Theorem 1 and that

$$|N_1 - N_2|^2 + |N_1 + N_2|^2 = 2|N_1|^2 + 2|N_2|^2,$$

where N_1, $N_2 \in \mathcal{B}^*(\mathcal{H})$ are arbitrary operators we obtain:

Theorem 2. *If $n \in \mathbb{N}$, $n \ge 2$, N_1, $N_2, ..., N_n \in \mathcal{B}^*(\mathcal{H})$ and $a_1, a_2, ..., a_n \in \mathbb{R} \setminus \{0\}$ with $\sum_{k=1}^n a_k \ne 0$, then*

$$\frac{|N_1+N_2+...+N_n|^2}{a_1+a_2+...+a_n} + \sum_{k=1}^n\frac{|N_k|^2}{a_k} - \frac{2}{a_1+a_2+...+a_n}\sum_{k=1}^n|N_k|^2$$
$$= \frac{1}{a_1+a_2+...+a_n}\sum_{1\le i<j\le n}\frac{|a_iN_j + a_jN_i|^2}{a_ia_j}.$$

Theorem 3. *If $n \in \mathbb{N}$, $n \geq 2$, $N_1, N_2, ..., N_n \in \mathcal{B}^*(\mathcal{H})$ with $N_i^* N_j = 0$, $(\forall)\ i, j = 1, ..., n$, $i \neq j$, i.e the operators have orthogonal ranges, and $a_1, a_2, ..., a_n \in \mathbb{R} \setminus \{0\}$, then*

$$\frac{1}{a_1^2 + ... + a_n^2} \sum_{1 \leq i < j \leq n} \frac{|a_i N_j - a_j N_i|^4}{a_i^2 a_j^2} = \frac{|N_1 + ... + N_n|^4}{a_1^2 + ... + a_n^2} + \sum_{k=1}^{n} (\frac{1}{a_k^2} - \frac{2}{a_1^2 + ... + a_n^2}) |N_k|^4 =$$

$$= \frac{1}{a_1^2 + ... + a_n^2} \sum_{1 \leq i < j \leq n} \frac{|a_i N_j + a_j N_i|^4}{a_i^2 a_j^2}.$$

Proof. Taking into account that the operators have orthogonal ranges and using then the previous theorem with a_i^2 instead of a_i and $|N_i|^2$ instead N_i, $i = 1, ..., n$, we have

$$\frac{1}{a_1^2 + ... + a_n^2} \sum_{1 \leq i < j \leq n} \frac{|a_i N_j - a_j N_i|^4}{a_i^2 a_j^2} = \frac{1}{a_1^2 + ... + a_n^2} \sum_{1 \leq i < j \leq n} \frac{(a_i^2 |N_j|^2 + a_j^2 |N_i|^2)^2}{a_i^2 a_j^2} =$$

$$= \frac{(|N_1|^2 + ... + |N_n|^2)^2}{a_1^2 + ... + a_n^2} + \sum_{k=1}^{n} (\frac{1}{a_k^2} - \frac{2}{a_1^2 + ... + a_n^2}) |N_k|^4 = \frac{|N_1 + ... + N_n|^4}{a_1^2 + ... + a_n^2} +$$

$$+ \sum_{k=1}^{n} (\frac{1}{a_k^2} - \frac{2}{a_1^2 + ... + a_n^2}) |N_k|^4.$$

∎

Consequence 2. *Using the above theorem and the Cauchy-Schwarz inequality we obtain:*

$$\frac{1}{a_1^2 + ... + a_n^2} \sum_{1 \leq i < j \leq n} \frac{|a_i N_j - a_j N_i|^4}{a_i^2 a_j^2} \leq \sum_{k=1}^{n} (\frac{1}{a_k^2} + \frac{n-2}{a_1^2 + ... + a_n^2}) |N_k|^4.$$

As a consequence of the Theorem 3 and Proposition 2.7, see [6], we have:

Consequence 3. *If $n \in N$, $n \geq 2$, $N_1, N_2, ..., N_n \in \mathcal{B}^*(\mathcal{H})$, where $\mathcal{H}$ is a Loynes Z- space with $N_i^* N_j = 0$, $(\forall)\ i, j = 1, ..., n$, $i \neq j$, i.e the operators have orthogonal ranges and $a_1, a_2, ..., a_n, \alpha_1, ..., \alpha_n \in R_+ \setminus \{0\}$ then,*

$$\sum_{1 \leq i < j \leq n} \frac{|\sqrt{a_i} N_j - \sqrt{a_j} N_i|^4}{a_i a_j} \leq \sum_{i=1}^{n} (a_i . \sum_{k=1}^{n} \frac{1}{a_k} + \frac{1}{a_i} \sum_{k=1}^{n} a_k - 2) |N_k|^4.$$

Proof. Taking in Proposition 2.7 $r = 4$ we have

$$|\sum_{i=1}^{n} N_i|^4 \leq \sum_{i=1}^{n} \frac{1}{\alpha_i} \sum_{i=1}^{n} \alpha_i |N_i|^4.$$

Using now Theorem 3 we obtain:

$$\sum_{1 \leq i < j \leq n} \frac{|\sqrt{a_i} N_j - \sqrt{a_j} N_i|^4}{a_i a_j} \leq \sum_{i=1}^{n} \frac{1}{\alpha_i} \sum_{i=1}^{n} \alpha_i |N_i|^4 + \sum_{k=1}^{n} (\frac{1}{a_k} \sum_{i=1}^{n} a_i - 2) |N_k|^4 =$$

$$= \sum_{i=1}^{n} [\alpha_i \sum_{k=1}^{n} \frac{1}{\alpha_k} + \frac{1}{a_i} \sum_{k=1}^{n} a_k - 2] |N_i|^4.$$

∎

Let $x = (x_1, ..., x_n) \in \mathbb{R}^n$, and we define, see [3], an $n \times n$ matrix $\Lambda(x) = x^* x = (x_i x_j)$ and $D(x) = diag(x_1, ..., x_n)$. Now we will give an analogue of Theorem 3.1 from [3] for operators with orthogonal ranges.

Proposition 1. *If $\Lambda(a^2) + \Lambda(b^2) \leq D(c^2)$ for $a,\ b,\ c \in \mathbb{R}^n$, then*

$$|\sum_{i=1}^{n} a_i A_i|^4 + |\sum_{i=1}^{n} b_i A_i|^4 \leq \sum_{i=1}^{n} c_i |A_i|^4$$

for arbitrary n-tuple (A_i) of operators with orthogonal ranges (i.e $A_j^* A_i = 0,\ (\forall)\ i \neq j,\ i, j \in \overline{1, n}$) in $\mathcal{B}^*(\mathcal{H})$ or $\mathcal{B}(\mathcal{H})$ respectively. In addition, if $\Lambda(a^2) + \Lambda(b^2) \geq D(c^2)$ for $a,\ b,\ c \in \mathbb{R}^n$, then

$$|\sum_{i=1}^{n} a_i A_i|^4 + |\sum_{i=1}^{n} b_i A_i|^4 \geq \sum_{i=1}^{n} c_i |A_i|^4$$

for arbitrary n-tuple (A_i) of operators with orthogonal ranges in $\mathcal{B}^*(\mathcal{H})$ or $\mathcal{B}(\mathcal{H})$ respectively.

Proof. We take into account that

$$|\sum_{i=1}^{n} a_i A_i|^4 + |\sum_{i=1}^{n} b_i A_i|^4 = |\sum_{i=1}^{n} a_i^2 |A_i|^2|^2 + |\sum_{i=1}^{n} b_i^2 |A_i|^2|^2$$

and then use Theorem 3.1, [3]. ∎

Using Theorem 28 from [1] we have:

Proposition 2. *Let $A_i \in \mathcal{B}^*(\mathcal{H})$ with $A_i^* A_j = 0,\ 1 \leq i \neq j \leq n$ and $\alpha_{ik},\ p_i \in \mathbb{R}$ for $i = 1, 2, ..., n$ and $k = 1, 2, ..., m$. Define $X = (x_{ij})$ where $x_{ij} = \sum_{k=1}^{m} \alpha_{ik}^4 - p_i$ if $i = j$ and $x_{ij} = \sum_{k=1}^{m} \alpha_{ik}^2 \alpha_{jk}^2$ if $i \neq j$.*

If $X \geq 0$ then

$$\sum_{k=1}^{m} |\sum_{i=1}^{n} \alpha_{ik} A_i|^4 \geq \sum_{i=1}^{n} p_i |A_i|^4.$$

If $X \leq 0$ then

$$\sum_{k=1}^{m} |\sum_{i=1}^{n} \alpha_{ik} A_i|^4 \leq \sum_{i=1}^{n} p_i |A_i|^4.$$

Proof. We use that

$$\sum_{k=1}^{m} |\sum_{i=1}^{n} \alpha_{ik} A_i|^4 = \sum_{k=1}^{m} (\sum_{i=1}^{n} \alpha_{ik}^2 |A_i|^2)^2 \geq \sum_{i=1}^{n} p_i |A_i|^4,$$

if $X \geq 0$, i.e. use Theorem 28, [1]. ∎

Proposition 3. *If $n \in N$, $n \geq 2$, $N_1, N_2, ..., N_n \in \mathcal{B}(\mathcal{H})$ and $\alpha_1, \alpha_2, ..., \alpha_n \in R \backslash \{0\}$ with $\alpha_1 + \alpha_2 + ... + \alpha_n \neq 0$ and $\Lambda(a) + \Lambda(b) \leq D(c)$ for $a, b, c \in \mathbb{R}^n$ then*

$$\sum_{1 \leq i < j \leq n} \frac{|\alpha_i a_j N_j - \alpha_j a_i N_i|^2 + |\alpha_i b_j N_j - \alpha_j b_i N_i|^2}{\alpha_i \alpha_j} \geq \sum_{i=1}^{n} (\frac{a_i + b_i}{\alpha_i} \sum_{k=1}^{n} \alpha_k - c_i) |N_i|^2.$$

Proof. Considering in Theorem 1, in

$$\sum_{i=1}^{n}\alpha_i\sum_{i=1}^{n}\frac{|N_i|^2}{\alpha_i}-\sum_{1\leq i<j\leq n}\frac{|\alpha_i N_j-\alpha_j N_i|^2}{\alpha_i\alpha_j}=|N_1+N_2+...+N_n|^2,$$

a_iN_i instead of N_i and then b_iN_i instead of N_i we obtain

$$|\sum_{i=1}^{n}a_iN_i|^2+|\sum_{i=1}^{n}b_iN_i|^2=\sum_{i=1}^{n}\alpha_i\sum_{i=1}^{n}\frac{(a_i+b_i)|N_i|^2}{\alpha_i}-$$
$$-\sum_{1\leq i<j\leq n}\frac{|\alpha_i a_j N_j-\alpha_j a_i N_i|^2+|\alpha_i b_j N_j-\alpha_j b_i N_i|^2}{\alpha_i\alpha_j}.$$

Using now Theorem 3.1, [3], we have

$$\sum_{1\leq i<j\leq n}\frac{|\alpha_i a_j N_j-\alpha_j a_i N_i|^2+|\alpha_i b_j N_j-\alpha_j b_i N_i|^2}{\alpha_i\alpha_j}\geq$$
$$\geq\sum_{i=1}^{n}\alpha_i\sum_{i=1}^{n}\frac{(a_i+b_i)|N_i|^2}{\alpha_i}-\sum_{i=1}^{n}c_i|N_i|^2=\sum_{i=1}^{n}(\frac{a_i+b_i}{\alpha_i}\sum_{k=1}^{n}\alpha_k-c_i)|N_i|^2.$$

∎

As in the case of real and complex numbers, see [7] it can be shown the following two inequalities, which also result from Theorem 1, if we take into consideration $n=2$:

Remark 2. *For any linear gramian bounded operators $N_1,\ N_2\in\mathcal{B}^*(\mathcal{H})$ we have*

$$\frac{|N_1+N_2|^2}{u+v}\leq\frac{|N_1|^2}{u}+\frac{|N_2|^2}{v},\tag{2}$$

if $u,\ v\neq 0,\ u+v\neq 0,\ uv(u+v)>0$,

and

$$\frac{|N_1+N_2|^2}{u+v}\geq\frac{|N_1|^2}{u}+\frac{|N_2|^2}{v},\tag{3}$$

if $u,\ v\neq 0,\ u+v\neq 0,\ uv(u+v)<0$.

In addition, if $N_1^*N_2=0$, then

$$\frac{|N_1|^4}{u}+\frac{|N_2|^4}{v}\geq\frac{|N_1+N_2|^4}{u+v},\tag{4}$$

if $u,\ v\neq 0,\ u+v\neq 0,\ uv(u+v)>0$,

or

$$\frac{|N_1+N_2|^4}{u+v}\geq\frac{|N_1|^4}{u}+\frac{|N_2|^4}{v},\tag{5}$$

if $u,\ v\neq 0,\ u+v\neq 0,\ uv(u+v)<0$.

Proof. Since $|vN_1-uN_2|^2\geq 0$ we have $(vN_1-uN_2)^*(vN_1-uN_2)=vN_1^*(vN_1-uN_2)+uN_2^*(uN_2-vN_1)>0$ i.e.

$uvN_1^*N_1+uvN_2^*N_2+uv(N_2^*N_1+N_1^*N_2)\leq uvN_1^*N_1+v^2N_1^*N_1+u^2N_2^*N_2+uvN_2^*N_2$

or $uv(N_1^*+N_2^*)(N_1+N_2)\leq(u+v)(vN_1^*N_1+uN_2^*N_2)$ which implies $uv|N_1+N_2|^2\leq(u+v)(v|N_1|^2+u|N_2|^2)$. For (4) we use the inequality (2) and knowing that N_1 and N_2 have the orthogonal ranges. ∎

We can obtain below another generalization of a theorem presented in [5].

Theorem 4. *(i) If* $n \in \mathbb{N}$, $n \geq 2$, $N_i \in \mathcal{B}^*(\mathcal{H})$, $i = \overline{1,n}$ *and* $a_1, a_2, ..., a_n \in R \setminus \{0\}$ *with* $a_1 + a_2 + ... + a_n \neq 0$, *and*

$$a_m(a_k + a_l) > 0,\ (\forall)\ m = \overline{1,n},\ m \neq k, l,\ a_k, a_l \neq 0,$$

then

$$\frac{|N_1|^2}{a_1} + \frac{|N_2|^2}{a_2} + ... + \frac{|N_n|^2}{a_n} - \frac{|N_1 + N_2 + ... + N_n|^2}{a_1 + a_2 + ... + a_n} \geq$$
$$\geq N_{k,l} + \frac{1}{a_1 + a_2 + ... + a_n} \sum_{1 \leq i < j \leq n,\ i,j \neq k,l} \frac{|a_i N_j - a_j N_i|^2}{a_i a_j},$$

where

$$N_{k,l} = \max_{1 \leq i < j \leq n} \frac{|a_i N_j - a_j N_i|^2}{a_i a_j (a_i + a_j)} = \frac{|a_k N_l - a_l N_k|^2}{a_k a_l (a_k + a_l)},\ 1 \leq k < l \leq n,$$

if there exist.

(ii) If $n \in \mathbb{N}$, $n \geq 2$, $N_i \in \mathcal{B}^*(\mathcal{H})$, $i = \overline{1,n}$ *and* $a_1, a_2, ..., a_n \in R \setminus \{0\}$ *with* $a_1 + a_2 + ... + a_n \neq 0$, *and*

$$a_m(a_k + a_l) < 0,\ (\forall)\ m = \overline{1,n},\ m \neq k, l,\ a_k, a_l \neq 0,$$

then

$$\frac{|N_1|^2}{a_1} + \frac{|N_2|^2}{a_2} + ... + \frac{|N_n|^2}{a_n} - \frac{|N_1 + N_2 + ... + N_n|^2}{a_1 + a_2 + ... + a_n} \leq$$
$$\leq N_{k,l} + \frac{1}{a_1 + a_2 + ... + a_n} \sum_{1 \leq i < j \leq n,\ i,j \neq k,l} \frac{|a_i N_j - a_j N_i|^2}{a_i a_j},$$

where

$$N_{k,l} = \min_{1 \leq i < j \leq n} \frac{|a_i N_j - a_j N_i|^2}{a_i a_j (a_i + a_j)} = \frac{|a_k N_l - a_l N_k|^2}{a_k a_l (a_k + a_l)},\ 1 \leq k < l \leq n,$$

if there exist.

Proof. The proof will be as in the case of real numbers, see [5]. Using the above remark, Remark 2

$$\frac{|N_1|^2}{u} + \frac{|N_2|^2}{v} \geq \frac{|N_1 + N_2|^2}{u + v},$$

where $u,\ v \neq 0,\ u + v \neq 0,\ uv(u + v) > 0$ and Theorem 1.

∎

We will present now a generalization of Theorem 4 when the operators N_i, $i = \overline{1, ..., n}$ have the orthogonal ranges.

Theorem 5. *If* $n \in \mathbb{N}$, $n \geq 2$, $N_i \in \mathcal{B}^*(\mathcal{H})$, $i = \overline{1,n}$ *with* $N_i^* N_j = 0$, $(\forall)\ i, j = 1, ..., n$, $i \neq j$, *i.e the operators have orthogonal ranges and* $a_1, a_2, ..., a_n \in R \setminus \{0\}$ *then*

$$\frac{|N_1 + ... + N_n|^4}{a_1^2 + ... + a_n^2} + \sum_{k=1}^{n} \left(\frac{1}{a_k^2} - \frac{2}{a_1^2 + ... + a_n^2}\right)|N_k|^4 \geq$$
$$\geq \frac{|a_k N_l - a_l N_k|^4}{a_k^2 a_l^2 (a_k^2 + a_l^2)} + \frac{1}{a_1^2 + ... + a_n^2} \sum_{1 \leq i < j \leq n} \frac{|a_i N_j - a_j N_i|^4}{a_i^2 a_l^2},$$

where

$$N_{k,l}=\max_{1\le i<j\le n}\frac{|a_iN_j-a_jN_i|^4}{a_i^2a_j^2(a_i^2+a_j^2)}=\frac{|a_kN_l-a_lN_k|^4}{a_k^2a_l^2(a_k^2+a_l^2)},\ 1\le k<l\le n,$$

if there exist.

Proof. The proof will be as in the case of real numbers, see [5]. Using Remark 2 from above

$$\frac{|N_1|^4}{u}+\frac{|N_2|^4}{v}\ge\frac{|N_1+N_2|^4}{u+v},$$

where $u,\ v\neq 0,\ u+v\neq 0,\ uv(u+v)>0$ and Theorem 3, we have

$$\frac{|N_1+...+N_n|^4}{a_1^2+...+a_n^2}+\sum_{k=1}^{n}(\frac{1}{a_k^2}-\frac{2}{a_1^2+...+a_n^2})|N_k|^4=$$

$$=\frac{1}{a_1^2+...+a_n^2}\sum_{1\le i<j\le n}\frac{|a_iN_j-a_jN_i|^4}{a_i^2a_j^2}=$$

$$=\frac{1}{a_1^2+...+a_n^2}(\frac{|a_kN_l-a_lN_k|^4}{a_k^2a_l^2}+\sum_{m=1,\ m\neq k,l}^{n}(\frac{|a_mN_l-a_lN_m|^4}{a_m^2a_l^2}+$$

$$+\frac{|a_mN_k-a_kN_m|^4}{a_m^2a_k^2}))+\frac{1}{a_1^2+...+a_n^2}\sum_{1\le i<j\le n,\ i,j\neq k,l}\frac{|a_iN_j-a_jN_i|^4}{a_i^2a_j^2}=$$

$$=\frac{1}{a_1^2+...+a_n^2}(\frac{|a_kN_l-a_lN_k|^4}{a_k^2a_l^2}+\sum_{m=1,\ m\neq k,l}^{n}(\frac{|a_kN_l-\frac{a_la_k}{a_m}N_m|^4}{\frac{a_l^2a_k^4}{a_m^2}}+$$

$$+\frac{|\frac{a_la_k}{a_m}N_m-a_lN_k|^4}{\frac{a_k^2a_l^4}{a_m^2}}))+\frac{1}{a_1^2+...+a_n^2}\sum_{1\le i<j\le n,\ i,j\neq k,l}\frac{|a_iN_j-a_jN_i|^4}{a_i^2a_j^2}\ge$$

$$\ge\frac{1}{a_1^2+...+a_n^2}(\frac{|a_kN_l-a_lN_k|^4}{a_k^2a_l^2}+\sum_{m=1,\ m\neq k,l}^{n}\frac{a_m^2|a_kN_l-a_lN_k|^4}{a_k^2a_l^2(a_k^2+a_l^2)})+$$

$$+\frac{1}{a_1^2+...+a_n^2}\sum_{1\le i<j\le n,\ i,j\neq k,l}\frac{|a_iN_j-a_jN_i|^4}{a_i^2a_j^2}=$$

$$=\frac{|a_kN_l-a_lN_k|^4}{a_k^2a_l^2(a_k^2+a_l^2)}+\frac{1}{a_1^2+...+a_n^2}\sum_{1\le i<j\le n,\ i,j\neq k,l}\frac{|a_iN_j-a_jN_i|^4}{a_i^2a_j^2}.$$

∎

Theorem 6. *If* $n\in N\ n\ge 2,\ N_i\in\mathcal{B}^*(\mathcal{H}),\ i=\overline{1,n}$ *and* $a_1,a_2,...,a_n\in R\setminus\{0\}$ *with* $a_1+a_2+...+a_n\neq 0$, *and*

$$a_m(a_k+a_l)>0,\ (\forall)\ m=\overline{1,n},\ m\neq k,l,\ a_k,a_l\neq 0,$$

then

$$\frac{1}{a_1+a_2+...+a_n}\sum_{1\le i<j\le n,\ i,j\neq k,l}\frac{|a_iN_j+a_jN_i|^2}{a_ia_j}\ge$$

$$\ge\frac{|N_1+N_2+...+N_n|^2}{a_1+a_2+...+a_n}+\sum_{k=1}^{n}\frac{|N_k|^2}{a_k}-$$

$$-2[\frac{|N_k|^2}{a_k}+\frac{|N_l|^2}{a_l}+\frac{a_k+a_l}{\sum_{i=1}^n a_i}\sum_{r=1,r\neq k,l}^{n}\frac{|N_r|^2}{a_r}+\frac{1}{\sum_{i=1}^n a_i}\sum_{t=2,t\neq k,l}^{n-1}|N_t|^2]+N_{k,l},$$

where

$$N_{k,l}=\max_{1\leq i<j\leq n}\frac{|a_iN_j-a_jN_i|^2}{a_ia_j(a_i+a_j)}=\frac{|a_kN_l-a_lN_k|^2}{a_ka_l(a_k+a_l)},\ 1\leq k<l\leq n,$$

if there exist.

Let $\mathcal{B}(\mathcal{H})$ denote the algebra of all bounded linear operators on a complex separable Hilbert space $\mathcal{H}$. We will consider as in [4], a unitarily invariant norm $|||.|||$ which is a norm on an ideal $C_{|||.|||}$ of $\mathcal{B}(\mathcal{H})$, making $C_{|||.|||}$ a Banach space and satisfying $|||UXV|||=|||X|||$ for all $X\in\mathcal{B}(\mathcal{H})$ and all unitary operators U and V in $\mathcal{B}(\mathcal{H})$.

Lemma 1 ([4])**.** *Let $A,B,X\in\mathcal{B}(\mathcal{H})$ where $\mathcal{H}$ is a Hilbert space with A and B self-adjoint and $X\geq\gamma_1 I$, γ_1 being a positive real number. Then*

$$\gamma_1|||A-B|||\leq|||AX-BX|||.$$

In the following we shall give a variant of the Theorem 2 of O. Hirzallah, see [4].

Consequence 4. *Let $A,B,X\in\mathcal{B}(\mathcal{H})$ where $\mathcal{H}$ is a Hilbert space as in* [4] *and $1+\beta\gamma<0$. If $X\geq\gamma_1 I$, γ_1 being a positive real number then*

$$\gamma_1|||\ |A-B|^2+|\beta A-\gamma B|^2|||\leq|\beta-\gamma||||\beta|A|^2X-\gamma X|B|^2|||.$$

Proof. We know that if $\alpha,\ \beta,\ \gamma$ are three real numbers which satisfies the conditions $\beta\neq 0,\ \gamma\neq 0,\ \alpha+\beta\gamma<0$ and $\alpha>0$ or $\alpha+\beta\gamma>0$ and $\alpha<0$ then

$$|\alpha A-B|^2+|\beta A-\gamma B|^2\leq\beta(\beta-\alpha\gamma)|A|^2+\gamma(\gamma-\frac{\beta}{\alpha})|B|^2.$$

We take $\alpha=1$ and then $1+\beta\gamma<0$, the inequality being

$$|A-B|^2\leq|A-B|^2+|\beta A-\gamma B|^2\leq(\beta-\gamma)(\beta|A|^2-\gamma|B|^2)$$

. Then

$$|||\ |A-B|^2|||\leq||||A-B|^2+|\beta A-\gamma B|^2|||\leq|\beta-\gamma||||\beta|A|^2-\gamma|B|^2|||.$$

By replacing in Lemma 1, A by $\beta|A|^2$ and B by $\gamma|B|^2$ which are obviously self-adjoint, we have

$$\gamma_1|||\beta|A|^2-\gamma|B|^2|||\leq|||\beta|A|^2X-\gamma X|B|^2|||$$

and then

$$\gamma_1|||\ |A-B|^2|||\leq\gamma_1|||\ |A-B|^2+|\beta A-\gamma B|^2|||\leq|\beta-\gamma||||\beta|A|^2X-\gamma X|B|^2|||.$$

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