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BESSEL AND GRÜSS TYPE INEQUALITIES IN INNER PRODUCT MODULES OVER BANACH $*$-ALGEBRAS

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ABSTRACT. We give an analogue of the Bessel inequality and we state a simple formulation of the Grüss type inequality in inner product C^*-modules, which is a refinement of it. We obtain some further generalization of the Grüss type inequalities in inner product modules over, proper H^*-algebras and unital Banach $*$-algebras for C^*-seminorms and positive linear functionals.

1. INTRODUCTION

A proper H^*-algebra is a complex Banach $*$-algebra $(\mathcal{A}, \|.\|)$ where the underlying Banach space is a Hilbert space with respect to the inner product $\langle ., . \rangle$ satisfying the properties $\langle ab, c\rangle = \langle b, a^*c\rangle$ and $\langle ba, c\rangle = \langle b, ca^*\rangle$ for all $a, b, c \in \mathcal{A}$. A C^*-algebra is a complex Banach $*$-algebra $(\mathcal{A}, \|.\|)$ such that $\|a^*a\| = \|a\|^2$ for every $a \in \mathcal{A}$. If $\mathcal{A}$ is a proper H^*-algebra or a C^*-algebra and $a \in \mathcal{A}$ is such that $\mathcal{A}a = 0$ or $a\mathcal{A} = 0$ then $a = 0$.

For a proper H^*-algebra $\mathcal{A}$, the trace class associated with $\mathcal{A}$ is $\tau(\mathcal{A}) = \{ab : a, b \in \mathcal{A}\}$. For every positive $a \in \tau(\mathcal{A})$ there exists the square root of a, that is, a unique positive $a^{\frac{1}{2}} \in \mathcal{A}$ such that $(a^{\frac{1}{2}})^2 = a$, the square root of a^*a is denoted by $|a|$. There are a positive linear functional tr on $\tau(\mathcal{A})$ and a norm τ on $\tau(\mathcal{A})$, related to the norm of A by the equality $tr(a^*a) = \tau(a^*a) = \|a\|^2$ for every $a \in \mathcal{A}$.

Let $\mathcal{A}$ be a proper H^*-algebra or a C^*-algebra. A semi-inner product module over $\mathcal{A}$ is a right module X over $\mathcal{A}$ together with a generalized semi-inner product, that is with a mapping $\langle ., . \rangle$ on $X \times X$, which is $\tau(\mathcal{A})$-valued if $\mathcal{A}$ is a proper H^*-algebra, or $\mathcal{A}$-valued if $\mathcal{A}$ is a C^*-algebra, having the following properties:

(i) $\langle x, y + z\rangle = \langle x, y\rangle + \langle x, z\rangle$ for all $x, y, z \in X$,
(ii) $\langle x, ya\rangle = \langle x, y\rangle\, a$ for $x, y \in X, a \in \mathcal{A}$,
(iii) $\langle x, y\rangle^* = \langle y, x\rangle$ for all $x, y \in X$,
(iv) $\langle x, x\rangle \geq 0$ for $x \in X$.

We shall say that X is a semi-inner product H^*-module if $\mathcal{A}$ is a proper H^*-algebra and that X is a semi-inner product C^*-module if $\mathcal{A}$ is a C^*-algebra.
If, in addition,

(v) $\langle x, x\rangle = 0$ implies $x = 0$,

then X is called an inner product module over $\mathcal{A}$. The absolute value of $x \in X$ is defined as the square root of $\langle x, x\rangle$ and it is denoted by $|x|$.

Let $\mathcal{A}$ be a $*$-algebra. A seminorm γ on $\mathcal{A}$ is a real-valued function on $\mathcal{A}$ such that for $a, b \in \mathcal{A}$ and $\lambda \in \mathbb{C}$: $\gamma(a) \geq 0$, $\gamma(\lambda a) = |\lambda|\gamma(a)$, $\gamma(a + b) \leq \gamma(a) + \gamma(b)$. A

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seminorm γ on $\mathcal{A}$ is called a C^*-seminorm if it satisfies the C^*-condition: $\gamma(a^*a) = (\gamma(a))^2 \quad (a \in \mathcal{A})$. By Sebestyen's theorem [4, Theorem 38.1] every C^*-seminorm γ on a $*$-algebra $\mathcal{A}$ is submultiplicative, i.e., $\gamma(ab) \leq \gamma(a)\gamma(b) \quad (a, b \in \mathcal{A})$, and by [3, Section 39, Lemma 2 (i)] $\gamma(a) = \gamma(a^*)$. For every $a \in \mathcal{A}$, the spectral radius of a is defined to be $r(a) = \sup\{|\lambda| : \lambda \in \sigma_{\mathcal{A}}(a)\}$.

The Pták function ρ on $*$-algebra $\mathcal{A}$ is defined to be $\rho : \mathcal{A} \to [0, \infty)$, where $\rho(a) = (r(a^*a))^{1/2}$. This function has important roles in Banach $*$-algebras, for example, on C^*-algebras, ρ is equal to the norm and on hermitian Banach $*$-algebras ρ is the greatest C^*-seminorm. By utilizing properties of the spectral radius and the Pták function, V. Pták [9] showed in 1970 that an elegant theory for Banach $*$-algebras arises from the inequality $r(a) \leq \rho(a)$.
This inequality characterizes hermitian (and symmetric) Banach $*$-algebras, and further characterizations of C^*-algebras follow as a result of Pták theory.

Let $\mathcal{A}$ be a $*$-algebra. We define $\mathcal{A}^+$ by

$$\mathcal{A}^+ = \left\{ \sum_{k=1}^{n} a_k^* a_k : n \in \mathbb{N}, a_k \in \mathcal{A} \ \text{ for } k = 1, 2, ..., n \right\},$$

and call the elements of $\mathcal{A}^+$ positive.
The set $\mathcal{A}^+$ of positive elements is obviously a convex cone (i.e., it is closed under convex combinations and multiplication by positive constants). Hence we call $\mathcal{A}^+$ the positive cone. By definition, zero belongs to $\mathcal{A}^+$. It is also clear that each positive element is hermitian.

We recall that a Banach $*$-algebra $(\mathcal{A}, \|.\|)$ is said to be an A^*-algebra provided there exists on $\mathcal{A}$ a second norm $|.|$, not necessarily complete, which is a C^*-norm. The second norm will be called an auxiliary norm.

Definition 1. Let $\mathcal{A}$ be a $*$-algebra. A semi-inner product $\mathcal{A}$-module (or semi-inner product $*$-module) is a complex vector space which is also a right $\mathcal{A}$-module X with a sesquilinear semi-inner product $\langle \cdot, \cdot \rangle : X \times X \to \mathcal{A}$, fulfilling

$$\langle x, ya \rangle = \langle x, y \rangle a \qquad \text{(right linearity)}$$
$$\langle x, x \rangle \in \mathcal{A}^+ \qquad \text{(positivity)}$$

for $x, y \in X,\ a \in \mathcal{A}$. Furthermore, if X satisfies the strict positivity condition

$$x = 0 \quad \text{if } \langle x, x \rangle = 0, \quad \text{(strict positivity)}$$

then X is called an inner product $\mathcal{A}$-module (or inner product $*$-module).

Let γ be a seminorm or a positive linear functional on $\mathcal{A}$ and $\Gamma(x) = (\gamma(\langle x, x \rangle))^{1/2}$ $(x \in X)$. If Γ is a seminorm on a semi-inner product $\mathcal{A}$-module X, then (X, Γ) is said to be a semi-Hilbert $\mathcal{A}$-module.

If Γ is a norm on an inner product $\mathcal{A}$-module X, then (X, Γ) is said to be a pre-Hilbert $\mathcal{A}$-module.

A pre-Hilbert $\mathcal{A}$-module which is complete with respect to its norm is called a Hilbert $\mathcal{A}$-module.

Since $\langle x + y, x + y \rangle$ and $\langle x + iy, x + iy \rangle$ are self adjoint, therefore we get the following Corollary.

Corollary 1. *If X is a semi-inner product $*$-module then the following symmetry condition holds:*

$$\langle x, y \rangle^* = \langle y, x \rangle \quad \textit{for } x, y \in X. \qquad \textit{(symmetry)}$$

Example 1.

(a) Let $\mathcal{A}$ be a $*$-algebra and γ a positive linear functional or a C^*-seminorm on $\mathcal{A}$. It is known that $(\mathcal{A}, \gamma)$ is a semi-Hilbert $\mathcal{A}$-module over itself with the inner product defined by $\langle a, b\rangle := a^*b$, in this case $\Gamma = \gamma$.

(b) Let $\mathcal{A}$ be a hermitian Banach $*$-algebra and ρ be the Pták function on $\mathcal{A}$. If X is a semi-inner product $\mathcal{A}$-module and $P(x) = (\rho(\langle x, x\rangle))^{1/2}(x \in X)$, then (X, P) is a semi-Hilbert $\mathcal{A}$-module.

(c) Let $\mathcal{A}$ be a A^*-algebra and $|\cdot|$ be the auxiliary norm on $\mathcal{A}$. If X is an inner product $\mathcal{A}$-module and $|x| = |\langle x, x\rangle|^{1/2}(x \in X)$, then $(X, |\cdot|)$ is a pre-Hilbert $\mathcal{A}$-module.

(d) Let $\mathcal{A}$ be a H^*-algebra and X (a semi-inner product) an inner product $\mathcal{A}$-module. Since tr is a positive linear functional on $\tau(\mathcal{A})$ and for every $x \in X$ we have $tr(\langle x, x\rangle) = \| \, |x| \, \|^2$ therefore $(X, \| \, |.| \, \|)$ is a (semi-Hilbert) pre-Hilbert $\mathcal{A}$-module.

In the present note we give an analogue of the Bessel inequality (2.7) and we obtain some further generalization and a simple form for the Grüss type inequalities in inner product modules over C^*-algebras, proper H^*-algebras and unital Banach $*$-algebras.

2. Schwarz and Bessel Inequality

If X is a semi-inner product C^*-module, then the following Schwarz inequality holds:

$$\|\langle x, y\rangle\|^2 \leq \|\langle x, x\rangle\|\|\langle y, y\rangle\| \quad (x, y \in X). \tag{2.1}$$

(e.g. [11, Lemma 15.1.3]).

If X is a semi-inner product H^*-module, then there are two forms of the Schwarz inequality: for every $x, y \in X$

$$tr(\langle x, y\rangle)^2 \leq tr(\langle x, x\rangle)tr(\langle y, y\rangle) \qquad \text{(the weak Schwarz inequality)}, \tag{2.2}$$

$$\tau(\langle x, y\rangle)^2 \leq tr(\langle x, x\rangle)tr(\langle y, y\rangle) \qquad \text{(the strong Schwarz inequality)}. \tag{2.3}$$

First Saworotnow in [10] proved the strong Schwarz inequality, but the direct proof of that for a semi-inner product H^*-module can be found in [8].

Now let $\mathcal{A}$ be a $*$-algebra, φ a positive linear functional on $\mathcal{A}$ and X be a semi-inner $\mathcal{A}$-module. We can define a sesquilinear form on $X \times X$ by $\sigma(x, y) = \varphi(\langle x, y\rangle)$; the Schwarz inequality for σ implies that

$$|\varphi(\langle x, y\rangle)|^2 \leq \varphi(\langle x, x\rangle)\varphi(\langle y, y\rangle). \tag{2.4}$$

In [7, Proposition 1, Remark 1] the authors present two other forms of the Schwarz inequality in semi-inner $\mathcal{A}$-module X, one for positive linear functional φ on $\mathcal{A}$:

$$\varphi(\langle x, y\rangle\langle x, y\rangle) \leq \varphi(\langle x, x\rangle)r(\langle y, y\rangle), \tag{2.5}$$

another one for C^*-seminorm γ on $\mathcal{A}$:

$$\gamma(\langle x, y\rangle)^2 \leq \gamma(\langle x, x\rangle)\gamma(\langle y, y\rangle). \tag{2.6}$$

The classical Bessel inequality states that if $\{e_i\}_{i \in I}$ is a family of orthonormal vectors in a Hilbert space $(H, \langle ., .\rangle)$, then

$$\sum_{i\in I}|\langle x,e_i\rangle|^2\le\|x\|^2\quad(x\in H). \tag{2.7}$$

Furthermore, some results concerning upper bounds for the expression

$$\|x\|^2-\sum_{i\in I}|\langle x,e_i\rangle|^2\quad(x\in H)$$

and for the expression related to the Grüss-type inequality

$$\left|\langle x,y\rangle-\sum_{i\in I}\langle x,e_i\rangle\langle e_i,y\rangle\right|\quad(x,y\in H)$$

have been proved in [5]. A version of the Bessel inequality for inner product H^*-modules and inner product C^*-modules can be found in [2], also there is a version of it for Hilbert C^*-modules in [6, Theorem 3.1.]. We provide here an analogue of the Bessel inequality for inner product $*$-modules.

Lemma 1. *Let $\mathcal{A}$ be a $*$-algebra, X an inner product $\mathcal{A}$-module and $\{e_1,...,e_n\}$ be a finite set of orthogonal elements in X such that $\langle e_i,e_i\rangle\,(i=1,...,n)$ are idempotent. Then*

$$\langle x,x\rangle-\sum_{i=1}^n\langle x,e_i\rangle\langle e_i,x\rangle\ge 0. \tag{2.8}$$

Proof. By [1, Lemma 1] or a straightforward calculation shows that

$$\begin{aligned}0&\le\left\langle x-\sum_{i=1}^n e_i\langle e_i,x\rangle,\quad x-\sum_{i=1}^n e_i\langle e_i,x\rangle\right\rangle\\&=\langle x,x\rangle-\sum_{i=1}^n\langle x,e_i\rangle\langle e_i,x\rangle-\sum_{i=1}^n\langle x,e_i\rangle\langle e_i,x\rangle+\sum_{i=1}^n\langle x,e_i\rangle\langle e_i,e_i\rangle\langle e_i,x\rangle\\&=\langle x,x\rangle-\sum_{i=1}^n\langle x,e_i\rangle\langle e_i,x\rangle-\sum_{i=1}^n\langle x,e_i\rangle\langle e_i,x\rangle+\sum_{i=1}^n\langle x,e_i\rangle\langle e_i,x\rangle\\&=\langle x,x\rangle-\sum_{i=1}^n\langle x,e_i\rangle\langle e_i,x\rangle.\end{aligned}$$

□

3. Grüss Type Inequalities

Before stating the main results let us fix the rest of our notation. We assume, unless stated otherwise, throughout this section that $\mathcal{A}$ is a unital Banach $*$-algebra. Also if X is a semi-inner product $\mathcal{A}$-module and γ is a C^*-seminorm on $\mathcal{A}$ we put $\Gamma(x)=(\gamma(\langle x,x\rangle))^{1/2}(x\in X)$ and if φ is a positive linear functional on $\mathcal{A}$ we put $\Phi(x)=(\varphi(\langle x,x\rangle))^{1/2}(x\in X)$. Let $\{e_1,...,e_n\}$ be a finite set of orthogonal elements in X such that $\langle e_i,e_i\rangle\,(i=1,...,n)$ be idempotent, we set $G_{x,y}:=\langle x,y\rangle-\sum_{i=1}^n\langle x,e_i\rangle\langle e_i,y\rangle$ and $G_x:=\langle x,x\rangle-\sum_{i=1}^n\langle x,e_i\rangle\langle e_i,x\rangle$.

Dragomir in [5, Lemma 4] shows that in a Hilbert space H the condition

$$\operatorname{Re}\left\langle\sum_{i=1}^n\alpha_ie_i-x,x-\sum_{i=1}^n\beta_ie_i\right\rangle\ge 0,$$

is equivalent to the condition

$$\left\| x - \sum_{i=1}^{n} \left(\frac{\alpha_i + \beta_i}{2} \right) e_i \right\| \leq \frac{1}{2} \left(\sum_{i=1}^{n} \|\alpha_i - \beta_i\|^2 \right)^{\frac{1}{2}},$$

where $x, e_1, ..., e_n \in H$ and $\alpha_1, ..., \alpha_n, \beta_1, ..., \beta_n \in \mathbb{C}$. But for semi-inner product $\mathcal{A}$-modules we have the following lemma, which is a generalization of [7, Lemma 1].

Lemma 2. *Let X be a semi-inner product $\mathcal{A}$-module and $a_1, ..., a_n, b_1, ..., b_n \in \mathcal{A}$ $x, y_1, ..., y_n \in X$. Then*

$$\operatorname{Re} \left\langle \sum_{i=1}^{n} y_i a_i - x, x - \sum_{i=1}^{n} y_i b_i \right\rangle \geq 0 \tag{3.1}$$

if and only if

$$\left\langle x - \sum_{i=1}^{n} y_i \left(\frac{a_i + b_i}{2} \right), x - \sum_{i=1}^{n} y_i \left(\frac{a_i + b_i}{2} \right) \right\rangle \leq \frac{1}{4} \sum_{i=1}^{n} (a_i - b_i)^* \langle y_i, y_i \rangle (a_i - b_i).$$

Proof. Follows from the equalities:

$$\begin{aligned}
&\operatorname{Re} \left\langle \sum_{i=1}^{n} y_i a_i - x, x - \sum_{i=1}^{n} y_i b_i \right\rangle \\
&\quad = \frac{1}{2} \left(\left\langle \sum_{i=1}^{n} y_i a_i - x, x - \sum_{i=1}^{n} y_i b_i \right\rangle + \left\langle x - \sum_{i=1}^{n} y_i b_i, \sum_{i=1}^{n} y_i a_i - x \right\rangle \right) \\
&\quad = \sum_{i=1}^{n} \frac{a_i^* + b_i^*}{2} \langle y_i, x \rangle - \frac{1}{2} \sum_{i=1}^{n} (a_i^* \langle y_i, y_i \rangle b_i + b_i^* \langle y_i, y_i \rangle a_i) \\
&\qquad - \langle x, x \rangle + \sum_{i=1}^{n} \langle x, y_i \rangle \frac{a_i + b_i}{2} \\
&\quad = \frac{1}{4} \sum_{i=1}^{n} (a_i - b_i)^* \langle y_i, y_i \rangle (a_i - b_i) \\
&\qquad - \left\langle x - \sum_{i=1}^{n} y_i \left(\frac{a_i + b_i}{2} \right), x - \sum_{i=1}^{n} y_i \left(\frac{a_i + b_i}{2} \right) \right\rangle.
\end{aligned}$$

□

Remark 1. By making use of previous Lemma 2 we may conclude the following statements.

(i) Let X be an inner product C^*-module and $\{e_1, ..., e_n\}$ be a finite set of orthogonal elements in X such that $\langle e_i, e_i \rangle \, (i = 1, ..., n)$ are idempotent, then inequality (3.1) implies that

$$\left\| x - \sum_{i=1}^{n} e_i \left(\frac{a_i + b_i}{2} \right) \right\| \leq \frac{1}{2} \left(\sum_{i=1}^{n} \|a_i - b_i\|^2 \right)^{\frac{1}{2}} \|\langle e_i, e_i \rangle\| = \frac{1}{2} \left(\sum_{i=1}^{n} \|a_i - b_i\|^2 \right)^{\frac{1}{2}}.$$

(ii) Let X be an inner product $\mathcal{A}$-module and $\{e_1, ..., e_n\}$ be a finite set of orthogonal elements in X such that $\langle e_i, e_i\rangle\,(i = 1, ..., n)$ are idempotent. If γ is a C^*-seminorm on $\mathcal{A}$ then inequality (3.1) implies that

$$\Gamma\left(x-\sum_{i=1}^{n}e_i\left(\frac{a_i+b_i}{2}\right)\right)\le\frac{1}{2}\left(\sum_{i=1}^{n}\gamma(a_i-b_i)^2\right)^{\frac{1}{2}}\Gamma(e_i)\le\frac{1}{2}\left(\sum_{i=1}^{n}\gamma(a_i-b_i)^2\right)^{\frac{1}{2}},$$

and if φ is a positive linear functional on $\mathcal{A}$ from inequality (3.1) and [3, Section 37 Lemma 6 (iii)] we get

$$\begin{aligned}\Phi\left(x-\sum_{i=1}^{n}e_i\left(\frac{a_i+b_i}{2}\right)\right)^2 &\le \frac{1}{4}\sum_{i=1}^{n}\varphi\left((a_i-b_i)^*\langle e_i,e_i\rangle(a_i-b_i)\right)\\ &\le \frac{1}{4}\sum_{i=1}^{n}\varphi\left((a_i-b_i)^*(a_i-b_i)\right)r(\langle e_i,e_i\rangle).\end{aligned}$$

(iii) Let $\mathcal{A}$ be a proper H^*-algebra, X an inner product $\mathcal{A}$-module and $\{e_1, ..., e_n\}$ be a finite set of orthogonal elements in X such that $\langle e_i, e_i\rangle\,(i = 1, ..., n)$ are idempotent. Since for every $a \in H$, $\quad tr(a^*a) = \|a\|^2$ inequality (3.1) is valid only if

$$\left|\left|\left|x-\sum_{i=1}^{n}e_i\left(\frac{a_i+b_i}{2}\right)\right|\right|\right|\le\frac{1}{2}\left(\sum_{i=1}^{n}\|a_i-b_i\|^2\right)^{\frac{1}{2}}.$$

We are able now to state our first main result:

Theorem 1. *Let X be an inner product C^*-module and $\{e_1, ..., e_n\}$ be a finite set of orthogonal elements in X such that $\langle e_i, e_i\rangle\,(i = 1, ..., n)$ are idempotent. If $a_1, ..., a_n, b_1, ..., b_n \in \mathcal{A}$, r, s are real numbers and $x, y \in X$ such that*

$$\left\|x-\sum_{i=1}^{n}e_ia_i\right\|\le r,\qquad \left\|y-\sum_{i=1}^{n}e_ib_i\right\|\le s \tag{3.2}$$

hold, then one has the inequality

$$\|G_{x,y}\|\le rs-\sqrt{r^2-\|G_x\|}\sqrt{s^2-\|G_y\|}. \tag{3.3}$$

Proof. By [1, Lemma 2] or, a straightforward calculation shows that for every $a_1, ..., a_n \in \mathcal{A}$

$$\begin{aligned}G_x=\langle x,x\rangle-\sum_{i=1}^{n}\langle x,e_i\rangle\langle e_i,x\rangle&=\left\langle x-\sum_{i=1}^{n}e_ia_i,\quad x-\sum_{i=1}^{n}e_ia_i\right\rangle\\ &-\left\langle\sum_{i=1}^{n}e_i(a_i-\langle e_i,x\rangle),\quad\sum_{i=1}^{n}e_i(a_i-\langle e_i,x\rangle)\right\rangle.\end{aligned}\tag{3.4}$$

Therefore

$$G_x\le\left\langle x-\sum_{i=1}^{n}e_ia_i,\quad x-\sum_{i=1}^{n}e_ia_i\right\rangle. \tag{3.5}$$

Analogously, for every $b_1, ..., b_n \in \mathcal{A}$ we have

$$G_y\le\left\langle y-\sum_{i=1}^{n}e_ib_i,\quad y-\sum_{i=1}^{n}e_ib_i\right\rangle. \tag{3.6}$$

The equalities (3.2), (3.5) and (3.6) imply that

$$\|G_x\| \leq \left\| x - \sum_{i=1}^{n} e_i a_i \right\|^2 \leq r^2, \tag{3.7}$$

and

$$\|G_y\| \leq \left\| y - \sum_{i=1}^{n} e_i b_i \right\|^2 \leq s^2. \tag{3.8}$$

Since

$$G_{x,y} = \left\langle x - \sum_{i=1}^{n} e_i \langle e_i, x\rangle, y - \sum_{i=1}^{n} e_i \langle e_i, y\rangle \right\rangle,$$

therefore the Schwarz's inequality (2.1) holds, i.e.,

$$\|G_{x,y}\|^2 \leq \|G_x\| \|G_y\|.$$

Finally, using the elementary inequality for real numbers

$$(m^2 - n^2)(p^2 - q^2) \leq (mp - nq)^2 \tag{3.9}$$

on

$$m = r, \quad n = \sqrt{r^2 - \|G_x\|}, \quad p = s, \quad q = \sqrt{s^2 - \|G_y\|},$$

we get

$$\|G_{x,y}\|^2 \leq \|G_x\| \|G_y\| \leq \left(rs - \sqrt{r^2 - \|G_x\|}\sqrt{s^2 - \|G_y\|} \right)^2.$$

□

Remark 2.

(i) Let X be an inner product C^*-module and $\{e_1, ..., e_n\}$ a finite set of orthogonal elements in X such that $\langle e_i, e_i\rangle\,(i = 1, ..., n)$ are idempotent. If $a_i, b_i, c_i, d_i \in \mathcal{A}(i = 1, ..., n)$ and $x, y \in X$ are such that

$$\left\| x - \sum_{i=1}^{n} e_i \left(\frac{a_i + b_i}{2} \right) \right\| \leq \frac{1}{2} \left(\sum_{i=1}^{n} \|a_i - b_i\|^2 \right)^{\frac{1}{2}},$$

$$\left\| y - \sum_{i=1}^{n} e_i \left(\frac{c_i + d_i}{2} \right) \right\| \leq \frac{1}{2} \left(\sum_{i=1}^{n} \|c_i - d_i\|^2 \right)^{\frac{1}{2}}$$

and if we put $r = \frac{1}{2}\left(\sum_{i=1}^{n} \|a_i - b_i\|^2\right)^{\frac{1}{2}}$ and $s = \frac{1}{2}\left(\sum_{i=1}^{n} \|c_i - d_i\|^2\right)^{\frac{1}{2}}$ then, by (3.7) and (3.8) we have

$$\|G_x\| \leq \left\| x - \sum_{i=1}^{n} e_i \left(\frac{a_i + b_i}{2} \right) \right\|^2 \leq r^2, \quad \|G_y\| \leq \left\| y - \sum_{i=1}^{n} e_i \left(\frac{c_i + d_i}{2} \right) \right\|^2 \leq s^2.$$

These and (3.3) imply that

$$\begin{aligned}\|G_{x,y}\| &\leq rs-\sqrt{r^2-\|G_x\|}\sqrt{s^2-\|G_y\|}\\ &\leq \frac{1}{4}\left(\sum_{i=1}^{n}\|a_i-b_i\|^2\sum_{i=1}^{n}\|c_i-d_i\|^2\right)^{\frac{1}{2}}\\ &\quad-\left(\frac{1}{4}\sum_{i=1}^{n}\|a_i-b_i\|^2-\left\|x-\sum_{i=1}^{n}e_i\left(\frac{a_i+b_i}{2}\right)\right\|^2\right)^{\frac{1}{2}}\\ &\quad\times\left(\frac{1}{4}\sum_{i=1}^{n}\|c_i-d_i\|^2-\left\|y-\sum_{i=1}^{n}e_i\left(\frac{c_i+d_i}{2}\right)\right\|^2\right)^{\frac{1}{2}}\\ &\leq \frac{1}{4}\left(\sum_{i=1}^{n}\|a_i-b_i\|^2\sum_{i=1}^{n}\|c_i-d_i\|^2\right)^{\frac{1}{2}}=rs.\end{aligned}$$

Therefore (3.3) is a refinement and a simple formulation of [2, Theorem 4.1].

(ii) If for $i=1,...,n$ we set

$$a_i=\alpha_i\langle e_i,e_i\rangle\,,\quad b_i=\beta_i\langle e_i,e_i\rangle\,,$$
$$c_i=\lambda_i\langle e_i,e_i\rangle\,,\quad d_i=\mu_i\langle e_i,e_i\rangle\,,$$

then similarly (3.3) is a refinement and a simple form of [2, Corollary 4.3].

Corollary 2. *Let $\mathcal{A}$ be a Banach $*$-algebra, X an inner product $\mathcal{A}$-module, and $\{e_1,...,e_n\}$ be a finite set of orthogonal elements in X such that $\langle e_i,e_i\rangle$ $(i=1,...,n)$ are idempotent. If $a_1,...a_n,b_1,...,b_n\in\mathcal{A}$, r,s are real numbers and $x,y\in X$ such that*

$$\Gamma\left(x-\sum_{i=1}^{n}e_ia_i\right)\leq r,\quad \Gamma\left(y-\sum_{i=1}^{n}e_ib_i\right)\leq s \tag{3.10}$$

hold, then one has the inequality

$$\gamma(G_{x,y})\leq rs-\sqrt{r^2-\gamma(G_x)}\sqrt{s^2-\gamma(G_y)}. \tag{3.11}$$

Proof. Using the schwarz's inequality (2.6) we have

$$\gamma(G_{x,y})^2\leq\gamma(G_x)\gamma(G_y).$$

The assumptions (3.10) and the elementary inequality for real numbers (3.9), will provide the desired result (3.11). □

Example 2. Let $\mathcal{A}$ be a hermitian Banach $*$-algebra and ρ be the Pták function on $\mathcal{A}$. If X is a semi-inner product $\mathcal{A}$-module and $P(x)=(\rho(\langle x,x\rangle))^{1/2}(x\in X)$ with the properties that

$$P\left(x-\sum_{i=1}^{n}e_ia_i\right)\leq r,\quad P\left(y-\sum_{i=1}^{n}e_ib_i\right)\leq s,$$

then we have

$$\rho(G_{x,y}) \leq rs - \sqrt{r^2 - \rho(G_x)}\sqrt{s^2 - \rho(G_y)}.$$

That is interesting in its own right.

Corollary 3. *Let $\mathcal{A}$ be a proper H^*-algebra, X an inner product $\mathcal{A}$-module, and $\{e_1, ..., e_n\}$ be a finite set of orthogonal elements in X such that $\langle e_i, e_i\rangle$ $(i = 1, ..., n)$ are idempotent. If $a_1, ...a_n, b_1, ..., b_n \in \mathcal{A}$, r, s are real numbers and $x, y \in X$ such that*

$$\left|\left|\left| x - \sum_{i=1}^{n} e_i a_i \right|\right|\right| \leq r, \quad \left|\left|\left| y - \sum_{i=1}^{n} e_i b_i \right|\right|\right| \leq s \tag{3.12}$$

hold, then one has the inequality

$$\tau(G_{x,y}) \leq rs - \sqrt{r^2 - tr(G_x)}\sqrt{s^2 - tr(G_y)}. \tag{3.13}$$

Proof. Using the strong Schwarz's inequality (2.3) we have

$$\tau(G_{x,y})^2 \leq tr(G_x)tr(G_y).$$

The assumptions (3.12) and the elementary inequality for real numbers (3.9), will provide (3.13). □

The following companion of the Grüss inequality for positive linear functionals holds:

Theorem 2. *Let X be an inner product $\mathcal{A}$-module, φ a positive linear functional on $\mathcal{A}$ and $\{e_1, ..., e_n\}$ be a finite set of orthogonal elements in X such that $\langle e_i, e_i\rangle$ $(i = 1, ..., n)$ are idempotent. If $a_1, ...a_n, b_1, ..., b_n \in \mathcal{A}$, r, s are real numbers and $x, y \in X$ such that*

$$\Phi\left(x - \sum_{i=1}^{n} e_i a_i\right) \leq r, \quad \Phi\left(y - \sum_{i=1}^{n} e_i b_i\right) \leq s$$

hold, then one has the inequality

$$|\varphi(G_{x,y})| \leq rs - \sum_{i=1}^{n} \Phi(e_i a_i - e_i\langle e_i, x\rangle)\Phi(e_i b_i - e_i\langle e_i, y\rangle). \tag{3.14}$$

Proof. By taking φ on both sides of (3.4) we have

$$\begin{aligned}\varphi(G_x) &= \Phi\left(x - \sum_{i=1}^{n} e_i a_i\right)^2 - \sum_{i=1}^{n} \Phi(e_i a_i - e_i\langle e_i, x\rangle)^2 \\ &\leq r^2 - \sum_{i=1}^{n} \Phi(e_i a_i - e_i\langle e_i, x\rangle)^2.\end{aligned}$$

Analogously

$$\begin{aligned}\varphi(G_y) &= \Phi\left(y - \sum_{i=1}^{n} e_i b_i\right)^2 - \sum_{i=1}^{n} \Phi(e_i b_i - e_i\langle e_i, y\rangle)^2 \\ &\leq s^2 - \sum_{i=1}^{n} \Phi(e_i b_i - e_i\langle e_i, y\rangle)^2.\end{aligned}$$

Now using Aczél's inequality for real numbers, i.e., we recall that

$$\left(a^2 - \sum_{i=1}^{n} a_i^2\right)\left(b^2 - \sum_{i=1}^{n} b_i^2\right) \le \left(ab - \sum_{i=1}^{n} a_i b_i\right)^2,$$

and the Schwarz's inequality for positive linear functionals, i.e.,

$$\varphi(G_{x,y})^2 \le \varphi(G_x)\varphi(G_y)$$

we deduce (3.14). □

4. Some Related Results

Theorem 3. *Let X be an inner product C^*-module and $\{e_1, ..., e_n\}$ be a finite set of orthogonal elements in X such that $\langle e_i, e_i\rangle\,(i = 1, ..., n)$ are idempotent. Let $x, y \in X$ and if we define*

$$r_0 = \inf\left\{\left\|x - \sum_{i=1}^{n} e_i a_i\right\| : (a_1, ..., a_n) \in \mathcal{A}^n\right\},$$

and

$$s_0 = \inf\left\{\left\|y - \sum_{i=1}^{n} e_i a_i\right\| : (a_1, ..., a_n) \in \mathcal{A}^n\right\},$$

then we have

$$\|G_{x,y}\| \le r_0 s_0 - \sqrt{r_0^2 - \|G_x\|}\sqrt{s_0^2 - \|G_y\|}.$$

Proof. For every $a_1, ..., a_n, b_1, ..., b_n \in \mathcal{A}$, by (3.5) and (3.6) we have

$$\|G_x\| \le \left\|x - \sum_{i=1}^{n} e_i a_i\right\|^2, \quad \|G_y\| \le \left\|y - \sum_{i=1}^{n} e_i b_i\right\|^2.$$

Therefore

$$\|G_x\| \le r_0^2, \quad \|G_y\| \le s_0^2.$$

Now, using the elementary inequality for real numbers

$$(m^2 - n^2)(p^2 - q^2) \le (mp - nq)^2$$

on

$$m = r_0, \quad n = \sqrt{r_0^2 - \|G_x\|}, \quad p = s_0, \quad q = \sqrt{s_0^2 - \|G_y\|},$$

we get

$$\|G_{x,y}\|^2 \le \|G_x\|\|G_y\| \le \left(r_0 s_0 - \sqrt{r_0^2 - \|G_x\|}\sqrt{s_0^2 - \|G_y\|}\right)^2.$$

□

Corollary 4. *Let $\mathcal{A}$ be a Banach $*$-algebra, X an inner product $\mathcal{A}$-module, and $\{e_1, ..., e_n\}$ be a finite set of orthogonal elements in X such that $\langle e_i, e_i\rangle\,(i = 1, ..., n)$ are idempotent. Let $x, y \in X$ and put*

$$r_0 = \inf\left\{\Gamma\left(x - \sum_{i=1}^{n} e_i a_i\right) : (a_1, ..., a_n) \in \mathcal{A}^n\right\},$$

$$s_0 = \inf\left\{\Gamma\left(y - \sum_{i=1}^{n} e_i a_i\right) : (a_1, ..., a_n) \in \mathcal{A}^n\right\},$$

then

$$\gamma(G_{x,y}) \leq r_0 s_0 - \sqrt{r_0^2 - \gamma(G_x)}\sqrt{s_0^2 - \gamma(G_y)}.$$

Corollary 5. *Let $\mathcal{A}$ be a proper H^*-algebra, X an inner product $\mathcal{A}$-module, and $\{e_1, ..., e_n\}$ be a finite set of orthogonal elements in X such that $\langle e_i, e_i \rangle$ $(i = 1, ..., n)$ are idempotent. Let $x, y \in X$ and if we consider*

$$r_0 = \inf \left\{ \left\| \left| \left(x - \sum_{i=1}^{n} e_i a_i \right) \right| \right\| : (a_1, ..., a_n) \in \mathcal{A}^n \right\},$$

$$s_0 = \inf \left\{ \left\| \left| \left(y - \sum_{i=1}^{n} e_i a_i \right) \right| \right\| : (a_1, ..., a_n) \in \mathcal{A}^n \right\},$$

then

$$\tau(G_{x,y}) \leq r_0 s_0 - \sqrt{r_0^2 - tr(G_x)}\sqrt{s_0^2 - tr(G_y)}.$$

From a different perspective, we can state the following result as well:

Theorem 4. *Let X be an inner product C^*-module and $\{e_1, ..., e_n\}$ be a finite set of orthogonal elements in X such that $\langle e_i, e_i \rangle$ $(i = 1, ..., n)$ are idempotent. If $a_1, ...a_n \in \mathcal{A}$, $r \in \mathbb{R}$, $\lambda \in (0, 1)$ and $x, y \in X$ such that*

$$\left\| \lambda x + (1 - \lambda)y - \sum_{i=1}^{n} e_i a_i \right\| \leq r, \tag{4.1}$$

then we have the inequality

$$\|Re(G_{x,y})\| \leq \frac{1}{4} . \frac{1}{\lambda(1 - \lambda)} r^2. \tag{4.2}$$

Proof. We know that for any $a, b \in X$ and $\lambda \in (0, 1)$ one has

$$Re \langle a, b \rangle = \frac{1}{2}(\langle a, b \rangle + \langle b, a \rangle) \leq \frac{1}{4\lambda(1 - \lambda)} \langle \lambda a + (1 - \lambda)b, \lambda a + (1 - \lambda)b \rangle . \tag{4.3}$$

Put $a = x - \sum_{i=1}^{n} e_i \langle e_i, x \rangle$, $\quad b = y - \sum_{i=1}^{n} e_i \langle e_i, y \rangle$ and since

$$G_{x,y} = \left\langle x - \sum_{i=1}^{n} e_i \langle e_i, x \rangle, y - \sum_{i=1}^{n} e_i \langle e_i, y \rangle \right\rangle = \langle a, b \rangle$$

using (4.3), we have

$$\begin{aligned} \|Re(G_{x,y})\| &= \|Re(\langle a, b \rangle)\| \leq \frac{1}{4\lambda(1 - \lambda)} \|\lambda a + (1 - \lambda)b\|^2 \\ &\leq \frac{1}{4\lambda(1 - \lambda)} \|\lambda x + (1 - \lambda)y - \sum_{i=1}^{n} e_i \langle e_i, \lambda x + (1 - \lambda)y \rangle \|^2 \\ &= \frac{1}{4\lambda(1 - \lambda)} \|G_{\lambda x + (1 - \lambda)y}\|^2. \end{aligned} \tag{4.4}$$

Now the inequality (4.2) follows from inequalities (3.7) and (4.4). □

The following companion of the Grüss inequality for positive linear functionals holds:

Theorem 5. *Let X be an inner product $\mathcal{A}$-module, φ a positive linear functional on $\mathcal{A}$ and $\{e_1,...,e_n\}$ be a finite set of orthogonal elements in X such that $\langle e_i, e_i\rangle$ $(i = 1,...,n)$ are idempotent. If $a_1,...a_n \in \mathcal{A}$, $r \in \mathbb{R}$, $\lambda \in (0,1)$ and $x, y \in X$ are such that*

$$\Phi\left(\lambda x + (1-\lambda)y - \sum_{i=1}^{n} e_i a_i\right) \le r, \tag{4.5}$$

then we have the inequality

$$|\varphi(Re(G_{x,y}))| \le \frac{1}{4}.\frac{1}{\lambda(1-\lambda)}\left(r^2 - \sum_{i=1}^{n}\Phi\left(e_i a_i - e_i\left\langle e_i, \lambda x + (1-\lambda)y\right\rangle\right)^2\right). \tag{4.6}$$

Proof. The inequality (4.3) for $a = x - \sum_{i=1}^{n} e_i\langle e_i, x\rangle$, $b = y - \sum_{i=1}^{n} e_i\langle e_i, y\rangle$ implies that

$$\begin{aligned} |\varphi(Re(G_{x,y}))| &= |\varphi(Re(\langle a, b\rangle))| \le \frac{1}{4\lambda(1-\lambda)}\Phi(\lambda a + (1-\lambda)b)^2 \\ &\le \frac{1}{4\lambda(1-\lambda)}\Phi\left(\lambda x + (1-\lambda)y - \sum_{i=1}^{n} e_i\left\langle e_i, \lambda x + (1-\lambda)y\right\rangle\right)^2 \\ &= \frac{1}{4\lambda(1-\lambda)}\varphi(G_{\lambda x+(1-\lambda)y})^2. \end{aligned} \tag{4.7}$$

By making use of the inequality (3.4) for $\lambda x + (1-\lambda)y$ instead of x and taking φ on both sides, we have

$$\begin{aligned} &\varphi(G_{\lambda x+(1-\lambda)y}) \\ &= \Phi\left(\lambda x + (1-\lambda)y - \sum_{i=1}^{n} e_i a_i\right)^2 - \sum_{i=1}^{n}\Phi\left(e_i a_i - e_i\left\langle e_i, \lambda x + (1-\lambda)y\right\rangle\right)^2 \\ &\le r^2 - \sum_{i=1}^{n}\Phi\left(e_i a_i - e_i\left\langle e_i, \lambda x + (1-\lambda)y\right\rangle\right)^2. \end{aligned} \tag{4.8}$$

From (4.7) and (4.8) we easily deduce (4.6). □

Remark 3.

(i) The constant 1 coefficient of rs in (3.3) is sharp, in the sense that it cannot be replaced by a smaller quantity. If the submodule of H generated by $e_1,...,e_n$ is not equal to X, then there exists $t \in X$ such that $t \ne \sum_{i=1}^{n} e_i\langle e_i, t\rangle$. We put $z = t - \sum_{i=1}^{n} e_i\langle e_i, t\rangle$ then $0 \ne z \in X$ and for any $j \in \{1, 2, ..., n\}$ we have

$$\begin{aligned} \langle z, e_j\rangle &= \langle t, e_j\rangle - \sum_{i=1}^{n}\langle t, e_j\rangle\langle e_i, e_j\rangle \\ &= \langle t, e_j\rangle - \langle t, e_j\rangle\langle e_j, e_j\rangle = 0. \end{aligned}$$

For every $\epsilon > 0$, if we put

$$x_\epsilon = \frac{rz}{\|z\| + \epsilon} + \sum_{i=1}^{n} e_i a_i, \quad y_\epsilon = \frac{sz}{\|z\| + \epsilon} + \sum_{i=1}^{n} e_i b_i \tag{4.9}$$

then

$$\begin{aligned} G_{x_\epsilon,y_\epsilon} &= \langle x_\epsilon, y_\epsilon\rangle - \sum_{j=1}^{n}\langle x_\epsilon, e_j\rangle\langle e_j, y_\epsilon\rangle \\ &= \frac{rs}{(\|z\|+\epsilon)^2}\langle z,z\rangle + \sum_{i=1}^{n} a_i^*\langle e_i, e_i\rangle b_i - \sum_{j=1}^{n} a_i^*\langle e_j, e_j\rangle\langle e_j, e_j\rangle b_i \\ &= \frac{rs}{(\|z\|+\epsilon)^2}\langle z,z\rangle, \end{aligned}$$

therefore

$$\|G_{x_\epsilon,y_\epsilon}\| = \frac{rs}{(\|z\|+\epsilon)^2}\|z\|^2.$$

Now if c is a constant such that $0 < c < 1$ then there is a $\epsilon > 0$ such that $\frac{\|z\|^2}{\|z\|+\epsilon)^2} > c$ therefore

$$\|G_{x_\epsilon,y_\epsilon}\| > crs.$$

(ii) Similarly the constant 1 coefficient of rs in (3.13) is best possible, it is sufficient instead of (4.9) to put

$$x_\epsilon = \frac{rz}{\|\,|z|^2\| + \epsilon} + \sum_{i=1}^{n} e_i a_i, \quad y_\epsilon = \frac{sz}{\|\,|z|^2\| + \epsilon} + \sum_{i=1}^{n} e_i b_i.$$

(iii) If there is a non zero element z in X such that $z \perp \{e_1, ..., e_n\}$ and $\Gamma(z) \neq 0$ (resp. $\Phi(z) \neq 0$) then the constant 1 coefficient of rs in (3.11) (resp. (3.14)) is best possible. Also similarly the inequalities in Theorem 3, Corollary 4, Corollary 5, Theorem 4 and Theorem 5 are sharp. However, the details are omitted.

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