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ONE STRENGTHEN FORM OF COPSON'S INEQUALITY

QIAN XU, BO-YAN XI, AND XIAO-MING ZHANG

ABSTRACT. In this paper, making use of monotonicity theorem (Independent Variables' Extremum Theorem) to the known Copson inequality, we get some improved results.

1. INTRODUCTION

Suppose $a_k > 0$, $p > 1$, then Hadry's inequality is

$$\left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty} a_n^p > \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} a_k\right)^p. \tag{1.1}$$

Hardy type inequality is playing a important role in analysis [1–4]. Near several decade, some generalizations and strengthens of Hardy type inequality have been very activity [1-5]. Reference [4,5] give the following inequality:

Copson's inequality[4,5]: Suppose that $a_n > 0\,(n = 1, 2, 3 \cdots)$. If $p > 1$, then

$$\sum_{n=1}^{\infty} \left(\sum_{k=n}^{\infty} \frac{a_k}{k}\right)^p < p^p \sum_{n=1}^{\infty} a_n^p. \tag{1.2}$$

If $0 < p < 1$, then

$$\sum_{n=1}^{\infty} \left(\sum_{k=n}^{\infty} \frac{a_k}{k}\right)^p > p^p \sum_{n=1}^{\infty} a_n^p. \tag{1.3}$$

About generalization of Copson's inequality, Book [4] and [7] carry out very much discussions.

In this paper, we make use of the Independent Variables' Extremum Theorem, strengthen inequalities (1.2) and (1.3).

2. RELATIONAL LEMMA AND DEFINITION

The following theorems A and B that are obtained by Zhang in [8, 9, 10] are called Independent Variables' Extremum Theorem.

Theorem A([8,Th.1.1]) Suppose that $a, b \in \mathbb{R}$ with $a < b$ and $c \in [a, b]$, $f : [a, b]^n \to \mathbb{R}$ has continuous partial derivatives, and

$$D_i = \left\{(x_1, x_2, \cdots, x_{n-1}, c) \,|\, \min_{1\le k\le n-1} \{x_k\} \ge c, x_i = \max_{1\le k\le n-1} \{x_k\} \ne c\right\}, i = 1, 2, \cdots, n-1,$$

If $\frac{\partial f(\mathbf{x})}{\partial x_i} > 0$ for all $\mathbf{x} \in D_i (i = 1, 2, \cdots, n-1)$, then

$$f(y_1, y_2, \cdots, y_{n-1}, c) \ge f(c, c, \cdots c, c),$$

for all $y_i \in [c, b]\,(i = 1, 2, \cdots, n-1)$.

Theorem B([8,Cor.1.3]) Suppose $a, b \in \mathbb{R}$ with $a < b$, $f : [a, b]^n \to \mathbb{R}$ has continuous partial derivatives and

$$D_i = \left\{(x_1, x_2, \cdots, x_n) \,|a \le \min_{1\le k\le n} \{x_k\} < x_i = \max_{1\le k\le n} \{x_k\} \le b\right\}, i = 1, 2, \cdots, n.$$

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If $\frac{\partial f(\mathbf{x})}{\partial x_i} > 0$ for all $\mathbf{x} \in D_i$ and $i = 1, 2, \cdots, n$, then

$$f(x_1, x_2, \cdots, x_n) \geq f(x_{\min}, x_{\min}, \cdots, x_{\min})$$

for all $x_i \in [a, b]$ $(i = 1, 2, \cdots, n)$ with $x_{\min} = \min_{1 \leq k \leq n} \{x_k\}$.

Definition1([1]). Let $G \subseteq \mathbb{R}^n$ be convex set, function $\phi : H \to \mathbb{R}$ be continuous, and

$$\phi(\alpha x + (1-\alpha)y) \leq (\geq) \alpha\phi(x) + (1-\alpha)\phi(y)$$

hold for any $x, y \in G, \alpha \in [0, 1]$.then φ is called convex (concave) function.

Lemma 1 (Hermite-Hadamard's Inequality). Suppose $\phi : [a, b] \to \mathbb{R}$ be convex (concave) function, then

$$\phi\left(\frac{a+b}{2}\right) \leq (\geq) \frac{1}{b-a} \int_a^b \phi(x)\, dx \leq (\geq) \frac{\phi(a) + \phi(b)}{2}, \tag{2.1}$$

equality hold only and only if ϕ is liner function

Lemma 2. Suppose $p > 0$.

(1) If $p \geq 1$, or $0 < p \leq \frac{1}{2}$, then

$$2p^p \geq 2^p; \tag{2.2}$$

(2) If $\frac{1}{2} < p < 1$, then

$$2^p > 2p^p; \tag{2.3}$$

(3) If $\frac{1}{4} < p < 1$, then

$$p^p + (p-1)2^p > 0. \tag{2.4}$$

proof Let $f_1 : p \in (0, +\infty) \to p \ln p + \ln 2 - p \ln 2$. Then it has

$$f_1{}'(p) = \ln p + 1 - \ln 2 = \ln\left(\frac{ep}{2}\right),$$

f is monotone decreasing on $\left(0, \frac{2}{e}\right)$ and monotone increasing on $\left(\frac{2}{e}, +\infty\right)$. Considering of $f_1(1) = 0$ and $f_1\left(\frac{1}{2}\right) = -\frac{1}{2}\ln 2 + \ln 2 - \frac{1}{2}\ln 2 = 0$, inequality (2.2) and inequality (2.3) holds

Let

$$f_2 : p \in (0, 1) \to p \ln p - p \ln 2 - \ln(1-p).$$

It has

$$f_2'(p) = \frac{1}{1-p}\left((1-p)\ln p + (1-p)(1-\ln 2) + 1\right) \stackrel{Def.}{=} \frac{1}{1-p} h(p),$$

$$h'(p) = -\ln p + \frac{1}{p} - 2 + \ln 2$$

and

$$h''(p) = -\frac{1}{p} - \frac{1}{p^2} < 0.$$

Therefore function h is concave function on $(0, 1)$. According to $h\left(\frac{1}{4}\right) > 0$ and $\lim_{p \to 1^-} h(p) > 0$, $h(p) > 0$ and $f_2'(p) > 0$ hold for $p \in \left[\frac{1}{4}, 1\right)$. Considering of $f_2\left(\frac{1}{4}\right) > 0$, we have $f_2(p) > 0$ holds for $\left[\frac{1}{4}, 1\right)$. Inequality (2.4) is proved.

Lemma 3. (1) If $p > 1$, then equation

$$p^p (1-x)\left(\frac{1}{2} - x\right)^{p-1} = 1 \tag{2.5}$$

of x has only a positive root within $\left(0, \frac{1}{2}\right)$.

(2) If $\frac{1}{2} < p < 1$, then equation

$$p^p (1+x) = \left(\frac{1}{2} + x\right)^{1-p} \tag{2.6}$$

of x has a only positive root within $\left(0, \frac{1}{2}\right)$.

Proof (1) Let $g_1 : x \in \left[0, \frac{1}{2}\right] \to p^p(1-x)\left(\frac{1}{2}-x\right)^{p-1} - 1$, then g_1 is monotone deceasing. Meanwhile, according to inequality (2.2),we have

$$g_1(0) = p^p\left(\frac{1}{2}\right)^{p-1} - 1 = \left(\frac{1}{2}\right)^{p-1}\left[p^p - 2^{p-1}\right] > 0$$

and $g_1\left(\frac{1}{2}\right) = -1$. So equation (2.5) has only a positive root within $\left(0, \frac{1}{2}\right)$.

(2) Let $g_2 : x \in \left[0, \frac{1}{2}\right] \to p^p\left(1+x\right) - \left(\frac{1}{2}+x\right)^{1-p}$. It has

$$g_2'\left(x\right) = p^p - (1-p)\left(\frac{1}{2}+x\right)^{-p} > \left(\frac{1}{2}+x\right)^{-p}\left[\frac{1}{2^p}\cdot p^p - (1-p)\right].$$

By inequality (2.4), g_2 is monotone increasing. According to inequality (2.3), we get

$$\lim_{x\to 0^+} g_2(x) = p^p - \frac{1}{2^{1-p}} < 0$$

and

$$\lim_{x\to (1/2)^+} g_2(x) = \frac{3}{2}p^p - 1 \geq \frac{3}{2}\cdot\left(\frac{1}{2}\right)^{1/2} - 1 > 0.$$

Therefore equation (2.6) has a only positive root within $\left(0, \frac{1}{2}\right)$.

Lemma 4. Let $p > 1$, $m > 0$, $m \in \mathrm{N}$, c be the only positive root of equation (2.3) within $\left(0, \frac{1}{2}\right)$. Then

$$\sum_{n=1}^{m}\left(\sum_{k=n}^{\infty}\frac{1}{(k-c)^{1+1/p}}\right)^{p-1} < p^p(m-c)^{1/p} \tag{2.7}$$

and

$$\sum_{n=1}^{m}\left(\sum_{k=n}^{m}\frac{1}{(k-c)^{1+1/p}}\right)^{p} < p^p\sum_{n=1}^{m}\frac{n^p}{(n-c)^{p+1}}. \tag{2.8}$$

Proof (1) If $m = 1$, by lemma 1, we get

$$\sum_{n=1}^{m}\left(\sum_{k=n}^{\infty}\frac{1}{(k-c)^{1+1/p}}\right)^{p-1} = \left(\sum_{k=1}^{\infty}\frac{1}{(k-c)^{1+1/p}}\right)^{p-1}$$

$$< \left(\int_{\frac{1}{2}}^{\infty}\frac{1}{(x-c)^{1+1/p}}dx\right)^{p-1} = p^{p-1}\left(\frac{1}{2}-c\right)^{-(p-1)/p}.$$

If $m \geq 2$, by lemma 1, we get

$$\sum_{n=1}^{m}\left(\sum_{k=n}^{\infty}\frac{1}{(k-c)^{1+1/p}}\right)^{p-1} < \sum_{n=1}^{m}\left(\int_{n-\frac{1}{2}}^{\infty}\frac{1}{(x-c)^{1+1/p}}dx\right)^{p-1}$$

$$= p^{p-1}\sum_{n=1}^{m}\left(n-\frac{1}{2}-c\right)^{-(p-1)/p}$$

$$= p^{p-1}\left(\left(\frac{1}{2}-c\right)^{-(p-1)/p} + \sum_{n=2}^{m}\left(n-\frac{1}{2}-c\right)^{-(p-1)/p}\right)$$

$$< p^{p-1}\left(\left(\frac{1}{2}-c\right)^{-(p-1)/p} + \int_{\frac{3}{2}}^{m+\frac{1}{2}}\left(x-\frac{1}{2}-c\right)^{-(p-1)/p}dx\right)$$

$$= p^{p-1}\left(\left(\frac{1}{2}-c\right)^{-(p-1)/p} + p(m-c)^{1/p} - p(1-c)^{1/p}\right).$$

So

$$\sum_{n=1}^{m}\left(\sum_{k=n}^{\infty}\frac{1}{(k-c)^{1+1/p}}\right)^{p-1}$$
$$< p^{p-1}\left(\left(\frac{1}{2}-c\right)^{-(p-1)/p}+p(m-c)^{1/p}-p(1-c)^{1/p}\right) \tag{2.9}$$

hold for any $m > 0$ and $m \in \mathrm{N}$. Because inequality (2.9) (2.10) and

$$\left(\frac{1}{2}-c\right)^{-(p-1)/p} = p(1-c)^{1/p}, \tag{2.10}$$

inequality (2.7) holds.

(2)

$$\sum_{n=1}^{m}\left(\sum_{k=n}^{m}\frac{1}{(k-c)^{1+1/p}}\right)^{p}$$
$$= p\sum_{n=1}^{m}\int_0^{\sum_{k=n}^{m}\frac{1}{(k-c)^{1+1/p}}} x^{p-1}dx$$
$$= p\sum_{n=1}^{m}$$
$$\left[\int_0^{\frac{1}{(m-c)^{1+1/p}}} x^{p-1}dx + \int_{\frac{1}{(m-c)^{1+1/p}}}^{\frac{1}{(m-c)^{1+1/p}}+\frac{1}{(m-c-1)^{1+1/p}}} x^{p-1}dx + \cdots + \int_{\sum_{k=n+1}^{m}\frac{1}{(k-c)^{1+1/p}}}^{\sum_{k=n}^{m}\frac{1}{(k-c)^{1+1/p}}} x^{p-1}dx\right]$$

$$< p\sum_{n=1}^{m}\left[\frac{1}{(m-c)^{1+1/p}}\cdot\left(\frac{1}{(m-c)^{1+1/p}}\right)^{p-1}+\frac{1}{(m-c-1)^{1+1/p}}\right.$$
$$\cdot\left(\frac{1}{(m-c)^{1+1/p}}+\frac{1}{(m-c-1)^{1+1/p}}\right)^{p-1}$$
$$\left.+\cdots+\frac{1}{(n-c)^{1+1/p}}\left(\sum_{k=n}^{m}\frac{1}{(k-c)^{1+1/p}}\right)^{p-1}\right]$$
$$= p\left[\frac{m}{(m-c)^{1+1/p}}\cdot\left(\frac{1}{(m-c)^{1+1/p}}\right)^{p-1}\right.$$
$$+\frac{m-1}{(m-c-1)^{1+1/p}}\left(\frac{1}{(m-c)^{1+1/p}}+\frac{1}{(m-c-1)^{1+1/p}}\right)^{p-1}$$
$$\left.+\cdots+\frac{1}{(1-c)^{1+1/p}}\left(\sum_{k=1}^{m}\frac{1}{k^{1+1/p}}\right)^{p-1}\right]$$
$$= p\sum_{n=1}^{m}\sum_{k=n}^{m}\frac{1}{(k-c)^{1+1/p}}\left(\sum_{i=k}^{m}\frac{1}{(i-c)^{1+1/p}}\right)^{p-1}$$
$$= p\sum_{n=1}^{m}\frac{n}{(n-c)^{1+1/p}}\left(\sum_{k=n}^{m}\frac{1}{(k-c)^{1+1/p}}\right)^{p-1}.$$

Let $q > 1$ and $1/p + 1/q = 1$. By Hölder-inequality,

$$\sum_{n=1}^{m}\left(\sum_{k=n}^{m}\frac{1}{(k-c)^{1+1/p}}\right)^{p}$$
$$< p\left[\sum_{n=1}^{m}\left(\frac{n}{(n-c)^{1+1/p}}\right)^{p}\right]^{1/p}\cdot\left[\sum_{n=1}^{m}\left(\sum_{k=n}^{m}\frac{1}{(k-c)^{1+1/p}}\right)^{(p-1)q}\right]^{1/q}$$
$$= p\left[\sum_{n=1}^{m}\frac{n^{p}}{(n-c)^{p+1}}\right]^{1/p}\cdot\left[\sum_{n=1}^{m}\left(\sum_{k=n}^{m}\frac{1}{(k-c)^{1+1/p}}\right)^{p}\right]^{1/q}.$$

Namely

$$\left[\sum_{n=1}^{m}\left(\sum_{k=n}^{m}\frac{1}{(k-c)^{1+1/p}}\right)^{p}\right]^{1/p} < p\left[\sum_{n=1}^{m}\frac{n^{p}}{(n-c)^{p+1}}\right]^{1/p}.$$

Thus inequality (2.8) holds.

Lemma 5. Let $m > 0$, $m \in \mathrm{N}$, d be the only positive root of equation (2.6) within $\left(0, \frac{1}{2}\right)$. Then

$$\sum_{n=1}^{m}\left(\sum_{k=n}^{m}\frac{1}{(k+d)^{1+1/p}}\right)^{p} > p^{p}\sum_{n=1}^{m}\frac{n^{p}}{(n+d)^{p+1}}. \tag{2.11}$$

Proof

$$\sum_{n=1}^{m}\left(\sum_{k=n}^{m}\frac{1}{(k+d)^{1+1/p}}\right)^{p}$$
$$= p\sum_{n=1}^{m}\int_{0}^{\sum_{k=n}^{m}\frac{1}{(k+d)^{1+1/p}}}x^{p-1}dx$$
$$= p\sum_{n=1}^{m}\left[\int_{0}^{\frac{1}{(m+d)^{1+1/p}}}x^{p-1}dx + \int_{\frac{1}{(m+d)^{1+1/p}}}^{\sum_{k=m-1}^{m}\frac{1}{(k+d)^{1+1/p}}}x^{p-1}dx + \int_{\sum_{k=m-1}^{m}\frac{1}{(k+d)^{1+1/p}}}^{\sum_{k=m-2}^{m}\frac{1}{(k+d)^{1+1/p}}}x^{p-1}dx\right.$$
$$\left. + \cdots + \int_{\sum_{k=n+1}^{m}\frac{1}{(k+d)^{1+1/p}}}^{\sum_{k=n}^{m}\frac{1}{(k+d)^{1+1/p}}}x^{p-1}dx\right]$$
$$> p\sum_{n=1}^{m}\left[\frac{1}{(m+d)^{1+1/p}}\left(\frac{1}{(m+d)^{1+1/p}}\right)^{-(1-p)}\right.$$
$$+\frac{1}{(m-1+d)^{1+1/p}}\left(\sum_{k=m-1}^{m}\frac{1}{(k+d)^{1+1/p}}\right)^{-(1-p)}$$
$$\left.+\cdots+\frac{1}{(n+d)^{1+1/p}}\left(\sum_{k=n}^{m}\frac{1}{(k+d)^{1+1/p}}\right)^{-(1-p)}\right]$$

(2.12)

$$= p\sum_{n=1}^{m}\frac{n}{(n+d)^{1+1/p}}\left(\sum_{k=n}^{m}\frac{1}{(k+d)^{1+1/p}}\right)^{-(1-p)}.$$

At the same time, by Hölder-inequality, it gets

$$\sum_{n=1}^{m}\left[\left(\frac{n}{(n+d)^{1+1/p}}\left(\sum_{k=n}^{m}\frac{1}{(k+d)^{1+1/p}}\right)^{-(1-p)}\right)^{p}\cdot\left(\sum_{k=n}^{m}\frac{1}{(k+d)^{1+1/p}}\right)^{p(1-p)}\right]$$
$$<\left[\sum_{n=1}^{m}\left(\frac{n}{(n+d)^{1+1/p}}\left(\sum_{k=n}^{m}\frac{1}{(k+d)^{1+1/p}}\right)^{-(1-p)}\right)\right]^{p}$$
$$\cdot\left[\sum_{n=1}^{m}\left(\sum_{k=n}^{m}\frac{1}{(k+d)^{1+1/p}}\right)^{p}\right]^{1-p}.$$

By inequality (2.12), it get

$$\sum_{n=1}^{m}\frac{n}{(n+d)^{1+1/p}}\left(\sum_{k=n}^{m}\frac{1}{(k+d)^{1+1/p}}\right)^{-(1-p)}\cdot\left(\sum_{n=1}^{m}\left(\sum_{k=n}^{m}\frac{1}{(k+d)^{1+1/p}}\right)^{p}\right)^{\frac{1-p}{p}}$$
$$>\left(\sum_{n=1}^{m}\left[\left(\frac{n}{(n+d)^{1+1/p}}\left(\sum_{k=n}^{m}\frac{1}{(k+d)^{1+1/p}}\right)^{-(1-p)}\right)^{p}\cdot\left(\sum_{k=n}^{m}\frac{1}{(k+d)^{1+1/p}}\right)^{p(1-p)}\right]\right)^{1/p}$$
$$=\left(\sum_{n=1}^{m}\left[\frac{n^{p}}{(n+d)^{p+1}}\left(\sum_{k=n}^{m}\frac{1}{(k+d)^{1+1/p}}\right)^{-(1-p)p}\cdot\left(\sum_{k=n}^{m}\frac{1}{(k+d)^{1+1/p}}\right)^{p(1-p)}\right]\right)^{1/p}$$
$$=\left(\sum_{n=1}^{m}\frac{n^{p}}{(n+d)^{p+1}}\right)^{1/p}. \tag{2.13}$$

By inequality (2.12) and inequality (2.13), it has

$$\sum_{n=1}^{m}\left(\sum_{k=n}^{m}\frac{1}{(k+d)^{1+1/p}}\right)^{p}>p\frac{\left(\sum_{n=1}^{m}\frac{n^{p}}{(n+d)^{p+1}}\right)^{1/p}}{\left(\sum_{n=1}^{m}\left(\sum_{k=n}^{m}\frac{1}{(k+d)^{1+1/p}}\right)^{p}\right)^{\frac{1-p}{p}}}$$

and

$$\left(\sum_{n=1}^{m}\left(\sum_{k=n}^{m}\frac{1}{(k+d)^{1+1/p}}\right)^{p}\right)^{1/p}>p\left(\sum_{n=1}^{m}\frac{n^{p}}{(n+d)^{p+1}}\right)^{1/p}.$$

Then inequality (2.11) holds.

3. Enhance of Copson's Inequality ($p>1$)

Theorem 1. Let $p>1$, $m>0$, $m\in \mathrm{N}$, $a_n>0\,(n=1,2,\cdots,m)$, c be the only positive root of equation (2.3) within $\left(0,\frac{1}{2}\right)$, $B_m=\min\limits_{1\le n\le m}\left\{(n-c)^{1/p}a_n\right\}$. Then

$$p^{p}\sum_{n=1}^{m}a_n^{p}-\sum_{n=1}^{m}\left(\sum_{k=n}^{m}\frac{a_k}{k-c}\right)^{p}$$
$$\ge B_m^{p}\left[p^{p}\sum_{n=1}^{m}\frac{1}{n-c}-\sum_{n=1}^{m}\left(\sum_{k=n}^{m}\frac{1}{(k-c)^{1+1/p}}\right)^{p}\right]. \tag{3.1}$$

Proof Let $b_n = (n-c)^{1/p} a_n \, (n = 1, 2, \cdots, m)$. Inequality (3.1) is equivalent to

$$p^p \sum\nolimits_{n=1}^{m} \frac{b_n^p}{n-c} - \sum\nolimits_{n=1}^{m} \left(\sum\nolimits_{k=n}^{m} \frac{b_k}{(k-c)^{1+1/p}} \right)^p$$

$$\geq B_m^p \left[p^p \sum\nolimits_{n=1}^{m} \frac{1}{n-c} - \sum\nolimits_{n=1}^{m} \left(\sum\nolimits_{k=n}^{m} \frac{1}{(k-c)^{1+1/p}} \right)^p \right] \tag{3.2}$$

with $B_m = \min\limits_{1 \leq n \leq N} \{b_n\}$. Let

$$f : b = (b_1, b_2, \cdots, b_m) \in [0, +\infty)^m \to p^p \sum\nolimits_{n=1}^{m} \frac{b_n^p}{n-c} - \sum\nolimits_{n=1}^{m} \left(\sum\nolimits_{k=n}^{m} \frac{b_k}{(k-c)^{1+1/p}} \right)^p$$

and $D_i = \left\{ (b_1, b_2, \cdots, b_n) \, | 0 \leq \min\limits_{1 \leq n \leq m} \{b_n\} < b_i = \max\limits_{1 \leq n \leq m} \{b_n\} \right\}$. If $(b_1, b_2, \cdots, b_n) \in D_i$,

$$\frac{\partial f}{\partial b_i} = p^p \frac{p b_i^{p-1}}{i-c} - \frac{p}{(i-c)^{1+1/p}} \sum\nolimits_{n=1}^{i} \left(\sum\nolimits_{k=n}^{m} \frac{b_k}{(k-c)^{1+1/p}} \right)^{p-1}$$

$$> \frac{p b_i^{p-1}}{i-c} \left[p^p (i-c)^{1/p} - \sum\nolimits_{n=1}^{i} \left(\sum\nolimits_{k=n}^{m} \frac{1}{(k-c)^{1+1/p}} \right)^{p-1} \right]$$

$$> \frac{p b_i^{p-1}}{i-c} \left[p^p (i-c)^{1/p} - \sum\nolimits_{n=1}^{i} \left(\sum\nolimits_{k=n}^{\infty} \frac{1}{(k-c)^{1+1/p}} \right)^{p-1} \right].$$

By inequality (2.7), we know $\frac{\partial f}{\partial b_i} > 0$. By theorem B, inequality (3.2) holds, theorem 1 is proved.

Corollary 1. Let $p > 1$, $m > 0$, $m \in \mathrm{N}$, $a_n > 0 \, (n = 1, 2, \cdots, m)$, c be the only positive root of equation (2.3) within $\left(0, \frac{1}{2}\right)$, $B_m = \min\limits_{1 \leq n \leq m} \left\{ (n-c)^{1/p} a_n \right\}$. Then

$$p^p \sum\nolimits_{n=1}^{m} a_n^p - \sum\nolimits_{n=1}^{m} \left(\sum\nolimits_{k=n}^{m} \frac{a_k}{k-c} \right)^p > -p^p B_m^p \sum\nolimits_{n=1}^{m} \frac{n^p - (n-c)^p}{(n-c)^{p+1}}. \tag{3.3}$$

Proof By (3.1) and (2.8), we know

$$p^p \sum\nolimits_{n=1}^{m} a_n^p - \sum\nolimits_{n=1}^{m} \left(\sum\nolimits_{k=n}^{m} \frac{a_k}{k-c} \right)^p > p^p B_m^p \left[\sum\nolimits_{n=1}^{m} \frac{1}{n-c} - \sum\nolimits_{n=1}^{m} \frac{n^p}{(n-c)^{p+1}} \right]$$

$$= -p^p B_m^p \sum\nolimits_{n=1}^{m} \frac{n^p - (n-c)^p}{(n-c)^{p+1}}.$$

Corollary 2 . Let $p > 1, a_n > 0 \, (n = 1, 2, \cdots)$, $\sum_{n=1}^{\infty} a_n^p < +\infty$, c be the only positive root of equation (2.3) within $\left(0, \frac{1}{2}\right)$. Then

$$\sum\nolimits_{n=1}^{\infty} \left(\sum\nolimits_{k=n}^{\infty} \frac{a_k}{k-c} \right)^p \leq p^p \sum\nolimits_{n=1}^{\infty} a_n^p. \tag{3.4}$$

Proof Because $\sum_{n=1}^{\infty} a_n^p < +\infty$, the infimum of $\left\{ (n-c)^{1/p} a_n \right\}_{n=1}^{\infty}$ is 0. Then it exists a series of $\{m_i | m_i \in \mathrm{N}\}$ thus that $\left\{ (m_i - c)^{1/p} a_{m_i} \right\}_{i=1}^{\infty}$ decrease to 0. By (3.3), it has

$$p^p \sum\nolimits_{n=1}^{m_i} a_n^p - \sum\nolimits_{n=1}^{m_i} \left(\sum\nolimits_{k=n}^{m_i} \frac{a_k}{k-c} \right)^p > -p^p \Big[(m_i - c)^{1/p} a_{m_i} \Big]^p \sum\nolimits_{n=1}^{m_i} \frac{n^p - (n-c)^{p+1}}{(n-c)^{p+1}}. \tag{3.5}$$

Let $i \to +\infty$ in inequality (3.5), it has $m_i \to +\infty$ and

$$\lim_{i\to+\infty} \left[(m_i - c)^{1/p} a_{m_i}\right]^p \sum_{n=1}^{m_i} \frac{n^p - (n-c)^{p+1}}{(n-c)^{p+1}} = 0.$$

By (3.5), it has

$$p^p \sum_{n=1}^{m_i} a_n^p - \sum_{n=1}^{m_i} \left(\sum_{k=n}^{m_i} \frac{a_k}{k-c}\right)^p \geq 0.$$

Therefore inequality (3.4) holds.

Remark: Inequality (3.4) strengthen inequality (1.2) obviously.

4. Strength of Copson's Inequality ($1/2 < p < 1$)

Theorem 2. Let $\frac{1}{2} < p < 1$, $m > 0$, $m \in$ m, $a_n > 0\,(n = 1, 2, \cdots, m)$, d be the only positive root of equation (2.6) within $\left(0, \frac{1}{2}\right)$, $B_m = \min\limits_{1\leq n\leq m} \left\{(n+d)^{1/p} a_n\right\}$. Then

$$\sum_{n=1}^{m} \left(\sum_{k=n}^{m} \frac{a_k}{k+d}\right)^p - p^p \sum_{n=1}^{m} a_n^p$$

$$\geq B_m^p \left[\sum_{n=1}^{m} \left(\sum_{k=n}^{m} \frac{1}{(k+d)^{1+1/p}}\right)^p - p^p \sum_{n=1}^{m} \frac{1}{n+d}\right]. \tag{4.1}$$

Proof Let $b_n = (n+d)^{1/p} a_n\,(n = 1, 2, \cdots, m)$. Inequality (4.1) is equivalent to

$$\sum_{n=1}^{m} \left(\sum_{k=n}^{m} \frac{b_k}{(k+d)^{1+1/p}}\right)^p - p^p \sum_{n=1}^{m} \frac{b_n^p}{n+d}$$

$$\geq B_m^p \left[p^p \sum_{n=1}^{m} \frac{1}{n} - \sum_{n=1}^{m} \left(\sum_{k=n}^{m} \frac{1}{k^{1+1/p}}\right)^p\right] \tag{4.2}$$

with $B_m = \min\limits_{1\leq n\leq m} \{b_n\}$. Let

$$f : b \in (0, +\infty)^m \to \sum_{n=1}^{N} \left(\sum_{k=n}^{N} \frac{b_k}{(k+d)^{1+1/p}}\right)^p - p^p \sum_{n=1}^{m} \frac{b_n^p}{n+d}$$

and $D_i = \left\{(b_1, b_2, \cdots, b_n)\,|0 \leq \min\limits_{1\leq n\leq m} \{b_n\} < b_i = \max\limits_{1\leq n\leq m} \{b_n\}\right\}$. If $(b_1, b_2, \cdots, b_n) \in D_i$, then

$$\begin{aligned}
\frac{\partial f}{\partial b_i} &= \frac{p}{(i+d)^{1+1/p}} \sum_{n=1}^{i} \left(\sum_{k=n}^{m} \frac{b_k}{(k+d)^{1+1/p}}\right)^{p-1} - p^{p+1} \frac{b_i^{p-1}}{i+d} \\
&= \frac{p b_i^{p-1}}{(i+d)^{1+1/p}} \left[\sum_{n=1}^{i} \left(\sum_{k=n}^{m} \frac{b_k}{(k+d)^{1+1/p} b_i}\right)^{-(1-p)} - p^p (i+d)^{1/p}\right] \\
&> \frac{p b_i^{p-1}}{(i+d)^{1+1/p}} \left[\sum_{n=1}^{i} \left(\sum_{k=n}^{m} \frac{1}{(k+d)^{1+1/p}}\right)^{-(1-p)} - p^p (i+d)^{1/p}\right] \\
&> \frac{p b_i^{p-1}}{(i+d)^{1+1/p}} \left[\sum_{n=1}^{i} \left(\sum_{k=n}^{\infty} \frac{1}{(k+d)^{1+1/p}}\right)^{-(1-p)} - p^p (i+d)^{1/p}\right].
\end{aligned}$$

By lemma 1, we have

$$\begin{aligned}\frac{\partial f}{\partial b_i} &> \frac{pb_i^{p-1}}{(i+d)^{1+1/p}}\left[\sum_{n=1}^{i}\left(\int_{n-\frac{1}{2}}^{\infty}\frac{1}{(x+d)^{1+1/p}}dx\right)^{-(1-p)} - p^p(i+d)^{1/p}\right]\\ &= \frac{pb_i^{p-1}}{(i+d)^{1+1/p}}\left[p^{-(1-p)}\sum_{n=1}^{i}\left(n-\frac{1}{2}+d\right)^{(1-p)/p} - p^p(i+d)^{1/p}\right],\end{aligned}$$

If $i=1$, by the definition of d, we have

$$\frac{\partial f}{\partial b_1} > \frac{pb_1^{p-1}}{(1+d)^{1+1/p}}\left[p^{-(1-p)}\left(\frac{1}{2}+d\right)^{(1-p)/p} - p^p(1+d)^{1/p}\right] = 0.$$

If $2 \le i \le m$, because $\frac{1}{2} < p < 1$, $0 < \frac{p-1}{p} \le 1$ and $g : x \in (0,+\infty) \to x^{(1-p)/p}$ is concave function, we have

$$\begin{aligned}\frac{\partial f}{\partial b_i} &> \frac{pb_i^{p-1}}{(i+d)^{1+1/p}}\left[p^{-(1-p)}\left(\left(\frac{1}{2}+d\right)^{(1-p)/p} + \sum_{n=2}^{i}\left(n-\frac{1}{2}+d\right)^{(1-p)/p}\right) - p^p(i+d)^{1/p}\right]\\ &> \frac{pb_i^{p-1}}{(i+d)^{1+1/p}}\left[p^{-(1-p)}\left(\left(\frac{1}{2}+d\right)^{(1-p)/p} + \int_{\frac{3}{2}}^{i+\frac{1}{2}}\left(x-\frac{1}{2}+d\right)^{(1-p)/p}dx\right) - p^p(i+d)^{1/p}\right]\\ &= \frac{pb_i^{p-1}}{(i+d)^{1+1/p}}\left[p^{-(1-p)}\left(\left(\frac{1}{2}+d\right)^{(1-p)/p} + p(i+d)^{1/p} - p(1+d)^{1/p}\right) - p^p(i+d)^{1/p}\right]\\ &= \frac{pb_i^{p-1}}{(i+d)^{1+1/p}}\left[p^{-(1-p)}\cdot p(i+d)^{1/p} - p^p(i+d)^{1/p}\right]\\ &= 0.\end{aligned}$$

Thus, in every D_i, $\partial f/\partial b_i > 0$. By theorem B, inequality (4.2) holds.

Corollary3. Let $\frac{1}{2} < p < 1$, $m > 0$, $m \in \mathrm{N}$, $a_n > 0\,(n = 1, 2, \cdots, m)$, d be the only positive root of equation (2.6) within $\left(0, \frac{1}{2}\right)$, $B_m = \min\limits_{1\le n\le m}\left\{(n+d)^{1/p}a_n\right\}$. Then

$$\sum_{n=1}^{m}\left(\sum_{k=n}^{m}\frac{a_k}{k+d}\right)^p - p^p\sum_{n=1}^{m}a_n^p \ge p^pB_m^p\sum_{n=1}^{m}\frac{n^p-(n+d)^p}{(n+d)^{p+1}}. \tag{4.3}$$

Proof By theorem 2 and lemma 5, it has

$$\sum_{n=1}^{m}\left(\sum_{k=n}^{m}\frac{a_k}{k+d}\right)^p - p^p\sum_{n=1}^{m}a_n^p \ge B_m^p\left[p^p\sum_{n=1}^{m}\frac{n^p}{(n+d)^{p+1}} - p^p\sum_{n=1}^{m}\frac{1}{n+d}\right].$$

Then inequality (4.3) holds.

Corollary4. Let $\frac{1}{2} < p < 1$, $a_n > 0\,(n = 1, 2, \cdots)$, d be the only positive root of equation (2.6) within $\left(0, \frac{1}{2}\right)$, and series $\sum_{n=1}^{\infty}\left(\sum_{k=n}^{\infty}\frac{a_k}{k+d}\right)^p < +\infty$. Then

$$\sum_{n=1}^{\infty}\left(\sum_{k=n}^{\infty}\frac{a_k}{k+d}\right)^p \ge p^p\sum_{n=1}^{\infty}a_n^p. \tag{4.4}$$

Proof According to inequality (4.3), we know

$$\sum_{n=1}^{m}\left(\sum_{k=n}^{m}\frac{a_k}{k+d}\right)^p + p^pB_m^p\sum_{n=1}^{m}\frac{(n+d)^p-n^p}{(n+d)^{p+1}} \ge p^p\sum_{n=1}^{m}a_n^p.$$

Then inequality (4.4) holds.

Remark: For $\frac{1}{2} < p < 1$, inequality (4.4) strengthen inequality (1.3) obviously.

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(Q. Xi) Jiaxing Broadcast & Radio University , Jiaxing City, Zhejiang Province, 314000, P. R. China
E-mail address: 906016807@qq.com

(B.-Y. Xi) Department of Mathematics, Inner Mongolia Nationalities 028000, China
E-mail address: baoyintu68@sohu.com

(X.-M. Zhang) Zhejiang Broadcast & Radio University Haining College, Haining City, Zhejiang Province, 314400, P. R. China
E-mail address: zjzxm79@126.com