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APPROXIMATING THE RIEMANN-STIELTJES INTEGRAL OF n-TIME DIFFERENTIABLE INTEGRANDS AND OF BOUNDED VARIATION INTEGRATORS WITH APPLICATIONS

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Abstract. In the present paper we investigate the problem of approximating the Riemann-Stieltjes integral $\int_a^b f(\lambda)\,du(\lambda)$ in the case when the integrand f is n-time differentiable and the derivative $f^{(n)}$ is either of locally bounded variation, or Lipschitzian on an interval incorporating $[a,b]$. A priory error bounds for several classes of integrators u and applications in approximating the finite Laplace-Stieltjes transform and the finite Fourier-Stieltjes sine and cosine transforms are provided as well.

1. Introduction

The concept of *Riemann-Stieltjes integral* $\int_a^b f(t)\,du(t)$,where f is called *the integrand*, u is called *the integrator,* plays an important role in Mathematics, for instance in the definition of complex integral, the representation of bounded linear functionals on the Banach space of all continuous functions on an interval $[a,b]$, in the spectral representation of selfadjoint operators on complex Hilbert spaces and other classes of operators such as the unitary operators etc...

However the *Numerical Analysis* of this integral is quite poor as pointed out by the seminal paper due to Michael Tortorella from 1990, [42]. Earlier results in this direction, however, were provided by Dubuc and Todor in their 1984 and 1987 papers [31] and [32], respectively. For recent results concerning the approximation of the Riemann-Stieltjes integral, see the work of Diethelm [16], Liu [34], Mercer [35], Munteanu [38], Mozyrska et al. [37] and the references therein. For other recent results obtained in the same direction by the first author and his colleagues from RGMIA, see [7], [6], [8], [15], [13], [14], [24] and [21]. A comprehensive list of preprints related to this subject may be found at http://rgmia.org.

In order to approximate the Riemann-Stieltjes integral $\int_a^b p(t)dv(t)$, where $p,v:[a,b]\to\mathbb{R}$ are functions for which the above integral exists, S.S. Dragomir established in [18] the following integral identity:

$$\begin{aligned}(1.1)\qquad &[u(b)-u(a)]\,f(x)-\int_a^b f(t)du(t)\\ &=\int_a^x [u(t)-u(a)]\,df(t)+\int_x^b [u(t)-u(b)]\,df(t),\quad x\in[a,b]\end{aligned}$$

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provided that the involved integrals exist. In the particular case when $u(t)=t$, $t\in[a,b]$, the above identity reduces to the celebrated *Montgomery identity*(see [36, p. 565]) that has been extensively used by many authors in obtaining various *inequalities of Ostrowski type.* For a comprehensive recent collection of works related to Ostrowski's inequality, see the book [30], the papers [2] – [11], [33], [39], [41] and [43].

It has been shown in [18] that, if $f:[a,b]\to\mathbb{R}$ is a function of *bounded variation* and $u:[a,b]\to\mathbb{R}$ is of *r-H-Hölder type*, i.e.,

$$|u(x)-u(y)|\leq H|x-y|^{r}\quad\text{for any}\quad x,y\in[a,b]\ , \tag{1.2}$$

where $r\in(0,1]$ and $H>0$ are given, then

$$\begin{aligned}\left|[u(b)-u(a)]\,f(x)-\int_a^b f(t)du(t)\right| &\leq H\left[(x-a)^{r}\bigvee_a^x(f)+(b-x)^{r}\bigvee_x^b(f)\right]\\ &\leq H\times\begin{cases}[(x-a)^{r}+(b-x)^{r}]\left[\frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|\right];\\ \\ [(x-a)^{qr}+(b-x)^{qr}]^{\frac{1}{q}}\left[\left(\bigvee_a^x(f)\right)^{p}+\left(\bigvee_x^b(f)\right)^{p}\right]^{\frac{1}{p}}\\ \qquad\qquad\text{if}\quad p>1,\frac{1}{p}+\frac{1}{q}=1\ ;\\ \\ \left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^{r}\bigvee_a^b(f)\ ;\end{cases}\end{aligned} \tag{1.3}$$

for any $x\in[a,b]$, where $\bigvee_c^d(f)$ denotes the *total variation*of f on $[c,d]$. Out of (1.3) we can obtain the following *mid-point inequality*

$$\left|[u(b)-u(a)]\,f\left(\frac{a+b}{2}\right)-\int_a^b f(t)du(t)\right|\leq\frac{H(b-a)^{r}}{2^{r}}\cdot\bigvee_a^b(f). \tag{1.4}$$

The dual result (see [20]), can be stated as follows:

If $u:[a,b]\to\mathbb{R}$ is of bounded variation on $[a,b]$ and $f:[a,b]\to\mathbb{R}$ is of q-K-Hölder type, then

$$\left|[u(b)-u(a)]f(x)-\int_a^b f(t)du(t)\right|\leq K\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^{q}\bigvee_a^b(u) \tag{1.5}$$

for any $x\in[a,b]$. In particular, for $x=\frac{a+b}{2}$, we get the mid-point inequality

$$\left|[u(b)-u(a)]\,f\left(\frac{a+b}{2}\right)-\int_a^b f(t)du(t)\right|\leq\frac{K(b-a)^{q}}{2^{q}}\cdot\bigvee_a^b(u). \tag{1.6}$$

In [26], in order to approximate the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ in a different manner, the authors considered the following *generalized trapezoid formula*

$$[u(b)-u(x)]\,f(b)+[u(x)-u(a)]\,f(a)\,,\quad x\in[a,b]\,.$$

They proved the error estimate

$$\left|\int_a^b f(t)\,du(t) - [u(b) - u(x)] f(b) - [u(x) - u(a)] f(a)\right| \le H\left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]^r \bigvee_a^b (f) \tag{1.7}$$

for any $x \in [a,b]$, provided that $f : [a,b] \to \mathbb{R}$ is of bounded variation on $[a,b]$ and u is of r-H-Hölder type.

The case $x = \frac{a+b}{2}$ provides the simpler result

$$\left|\int_a^b f(t)\,du(t) - \left[u(b) - u\left(\frac{a+b}{2}\right)\right] f(b) - \left[u\left(\frac{a+b}{2}\right) - u(a)\right] f(a)\right| \le H\frac{1}{2^r}(b-a)^r \bigvee_a^b (f). \tag{1.8}$$

In [12], the following dual result has been obtained as well:

$$\begin{aligned}
&\left|\int_a^b f(t)\,du(t) - [u(b) - u(x)] f(b) - [u(x) - u(a)] f(a)\right| \\
&\le K\left[(x-a)^q \bigvee_a^x (u) + (b-x)^q \bigvee_x^b (u)\right] \\
&\le K \times \begin{cases} [(x-a)^q + (b-x)^q]\left[\frac{1}{2}\bigvee_a^b (u) + \frac{1}{2}\left|\bigvee_a^x (u) - \bigvee_x^b (u)\right|\right]; \\ \left[(x-a)^{\alpha q} + (b-x)^{\beta q}\right]^{\frac{1}{\alpha}}\left[\left[\bigvee_a^x (u)\right]^\beta + \left[\bigvee_x^b (u)\right]^\beta\right]^{\frac{1}{\beta}} \\ \qquad\qquad \text{if } \alpha > 1,\ \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]^q \bigvee_a^b (u); \end{cases}
\end{aligned} \tag{1.9}$$

for any $x \in [a,b]$, provided that f is of q-K-Hölder type and u is of bounded variation.

In particular we have

$$\left|\int_a^b f(t)\,du(t) - \left[u(b) - u\left(\frac{a+b}{2}\right)\right] f(b) - \left[u\left(\frac{a+b}{2}\right) - u(a)\right] f(a)\right| \le K\frac{1}{2^q}(b-a)^q \bigvee_a^b (u) \tag{1.10}$$

For other inequalities of this type, see the recent papers [9], [6] and [15].

Motivated by the above results, in the present paper we investigate the problem of approximating the Riemann-Stieltjes integral $\int_a^b f(\lambda)\,du(\lambda)$ in the case when the integrand f is n-time differentiable and the derivative $f^{(n)}$ is either of locally bounded variation, or Lipschitzian on an interval incorporating $[a,b]$. A priory error bounds for several classes of integrators u and applications in approximating the finite Laplace-Stieltjes transform and the finite Fourier-Stieltjes sine and cosine transforms are provided as well.

2. Some Representation Results

In this section we establish some representation results for the Riemann-Stieltjes integral when the integrand is n-time differentiable and the integrator is of locally bounded variation. Several particular cases of interest are considered as well.

Theorem 1. *Assume that the function $f : I \to \mathbb{C}$ is n-time differentiable on the interior $\mathring{I}$ of the interval I $(n \geq 1)$ and the n-th derivative $f^{(n)}$ is of locally bounded variation on $\mathring{I}$. If $a, b \in \mathring{I}$ with $a < b, c \in [a, b]$ and $u : [a, b] \to \mathbb{C}$ is of bounded variation on $[a, b]$, then the Riemann-Stieltjes integral $\int_a^b f(\lambda)\, du(\lambda)$ exists, we have the identity*

$$\int_a^b f(\lambda)\, du(\lambda) = T_n(f, u, a, c, b) + R_n(f, u, a, c, b) \tag{2.1}$$

where

$$T_n(f, u, a, c, b) := \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(c) \left[(b-c)^k u(b) + (-1)^{k+1} (c-a)^k u(a) \right] - \sum_{k=0}^{n-1} \frac{1}{k!} f^{(k+1)}(c) \int_a^b (\lambda - c)^k u(\lambda)\, d\lambda \tag{2.2}$$

and the remainder $R_n(f, u, a, c, b)$ can be represented as

$$R_n(f, u, a, c, b) := \frac{1}{n!} \int_a^b \left(\int_c^\lambda (\lambda - t)^n\, df^{(n)}(t) \right) du(\lambda). \tag{2.3}$$

Both integrals in (2.3) are taken in the Riemann-Stieltjes sense.

Proof. Under the assumption of the theorem, we utilize the following Taylor's representation

$$f(\lambda) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(c)(\lambda - c)^k + \frac{1}{n!} \int_c^\lambda (\lambda - t)^n\, df^{(n)}(t) \tag{2.4}$$

that holds for any $c \in [a, b]$ and $n \geq 0$. The integral in (2.4) is taken in the Riemann-Stieltjes sense.

We can prove this equality by induction.

Indeed for $n = 0$ we have

$$f(\lambda) = f(c) + \int_c^\lambda df(t)$$

that holds for any function of locally bounded variation on $\mathring{I}$.

Now, assume that (2.4) is true for a given $n \geq 0$ and let us prove that it holds for "$n + 1$", namely

$$f(\lambda) = \sum_{k=0}^{n+1} \frac{1}{k!} f^{(k)}(c)(\lambda - c)^k + \frac{1}{(n+1)!} \int_c^\lambda (\lambda - t)^{n+1}\, df^{(n+1)}(t) \tag{2.5}$$

provided that the function $f : I \to \mathbb{C}$ is $(n + 1)$-time differentiable on the interior $\mathring{I}$ of the interval I and the $(n + 1)$-th derivative $f^{(n+1)}$ is of locally bounded variation on $\mathring{I}$.

Utilizing the integration by parts formula for the Riemann-Stieltjes integral and the reduction of the Riemann-Stieltjes integral to a Riemann integral (see for instance [1]) we have:

$$\begin{aligned} (2.6) \quad & \int_c^\lambda (\lambda - t)^{n+1} df^{(n+1)}(t) \\ &= (\lambda - t)^{n+1} f^{(n+1)}(t)\Big|_c^\lambda + (n+1)\int_c^\lambda (\lambda - t)^n f^{(n+1)}(t)\,dt \\ &= -(\lambda - c)^{n+1} f^{(n+1)}(c) + (n+1)\int_c^\lambda (\lambda - t)^n df^{(n)}(t). \end{aligned}$$

From (2.4) we have that

$$\int_c^\lambda (\lambda - t)^n df^{(n)}(t) = \left[f(\lambda) - \sum_{k=0}^n \frac{1}{k!} f^{(k)}(c)(\lambda - c)^k \right] n!$$

which inserted in the last part of (2.6) provides the equality

$$\begin{aligned} (2.7) \quad \int_c^\lambda (\lambda - t)^{n+1} df^{(n+1)}(t) &= -(\lambda - c)^{n+1} f^{(n+1)}(c) \\ &+ (n+1)! \left[f(\lambda) - \sum_{k=0}^n \frac{1}{k!} f^{(k)}(c)(\lambda - c)^k \right]. \end{aligned}$$

We observe that, by division with $(n+1)!$, the equality (2.7) becomes the desired representation (2.5).

Further on, by integrating the identity (2.4) over $du(t)$ we get

$$\begin{aligned} (2.8) \quad \int_a^b f(\lambda)\,du(\lambda) &= \sum_{k=0}^n \frac{1}{k!} f^{(k)}(c) \int_a^b (\lambda - c)^k du(\lambda) \\ &+ \frac{1}{n!}\int_a^b \left(\int_c^\lambda (\lambda - t)^n df^{(n)}(t) \right) du(\lambda). \end{aligned}$$

Utilizing the integration by parts formula we have for $k \geq 1$ that

$$\begin{aligned} (2.9) \quad \int_a^b (\lambda - c)^k du(\lambda) &= (\lambda - c)^k u(\lambda)\Big|_a^b - k\int_a^b (\lambda - c)^{k-1} u(\lambda)\,d\lambda \\ &= (b - c)^k u(b) + (-1)^{k+1}(c - a)^k u(a) \\ &- k\int_a^b (\lambda - c)^{k-1} u(\lambda)\,d\lambda. \end{aligned}$$

For $k = 0$ we have $\int_a^b du(\lambda) = u(b) - u(a)$.

Therefore, by (2.9) we get

$$\begin{aligned}(2.10)\quad & \sum_{k=0}^{n}\frac{1}{k!}f^{(k)}(c)\int_a^b(\lambda-c)^k\,du(\lambda)\\ &=\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}(c)\left[(b-c)^k u(b)+(-1)^{k+1}(c-a)^k u(a)\right]\\ &-\sum_{k=0}^{n-1}\frac{1}{k!}f^{(k+1)}(c)\int_a^b(\lambda-c)^k u(\lambda)\,d\lambda\\ &=T_n(f,u,a,c,b)\end{aligned}$$

and by (2.8) the representation (2.1) is thus obtained.

This completes the proof. □

Remark 1. *Assume that the function $f : I \to \mathbb{C}$ is n-time differentiable on the interior $\mathring{I}$ of the interval I $(n \geq 1)$ and the n-th derivative $f^{(n)}$ is of locally bounded variation on $\mathring{I}$. If $a, b \in \mathring{I}$ with $a < b$ and $u : [a, b] \to \mathbb{C}$ is of bounded variation on $[a, b]$, then, by choosing $c = a$ in the formulae above we have*

$$\begin{aligned}(2.11)\quad {}_dD_n(f,u,a,b) &:= T_n(f,u,a,a,b)=\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}(a)(b-a)^k u(b)\\ &-\sum_{k=0}^{n-1}\frac{1}{k!}f^{(k+1)}(a)\int_a^b(\lambda-a)^k u(\lambda)\,d\lambda\end{aligned}$$

and

$$(2.12)\quad {}_dR_n(f,u,a,b) := R_n(f,u,a,a,b)=\frac{1}{n!}\int_a^b\left(\int_a^\lambda(\lambda-t)^n\,df^{(n)}(t)\right)du(\lambda).$$

This give the representation

$$(2.13)\quad \int_a^b f(\lambda)\,du(\lambda) ={}_d D_n(f,u,a,b) +{}_d R_n(f,u,a,b).$$

Now, if we choose $c = \frac{a+b}{2}$, then we have

$$\begin{aligned}(2.14)\quad M_n(f,u,a,b) &:= T_n\left(f,u,a,\frac{a+b}{2},b\right)\\ &=\sum_{k=0}^{n}\frac{1}{k!2^k}f^{(k)}\left(\frac{a+b}{2}\right)(b-a)^k\left[u(b)+(-1)^{k+1}u(a)\right]\\ &-\sum_{k=0}^{n-1}\frac{1}{k!}f^{(k+1)}\left(\frac{a+b}{2}\right)\int_a^b\left(\lambda-\frac{a+b}{2}\right)^k u(\lambda)\,d\lambda\end{aligned}$$

and

$$\begin{aligned}(2.15)\quad {}_MR_n(f,u,a,b) &:= R_n\left(f,u,a,\frac{a+b}{2},b\right)\\ &=\frac{1}{n!}\int_a^b\left(\int_{\frac{a+b}{2}}^\lambda(\lambda-t)^n\,df^{(n)}(t)\right)du(\lambda),\end{aligned}$$

which provide the representation

$$\int_a^b f(\lambda)\,du(\lambda) = M_n(f,u,a,b) +_M R_n(f,u,a,b). \tag{2.16}$$

Finally, if we choose $c=b$, then we have

$$\begin{aligned} {}_uD_n(f,u,a,b) &:= T_n(f,u,a,b,b) \\ &= \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(b)(-1)^{k+1}(b-a)^k u(a) \\ &+ \sum_{k=0}^{n-1} \frac{(-1)^{k+1}}{k!} f^{(k+1)}(b) \int_a^b (b-\lambda)^k u(\lambda)\,d\lambda \end{aligned} \tag{2.17}$$

and the remainder

$$\begin{aligned} {}_uR_n(f,u,a,b) &:= R_n(f,u,a,b,b) \\ &= \frac{(-1)^{n+1}}{n!} \int_a^b \left(\int_\lambda^b (t-\lambda)^n df^{(n)}(t) \right) du(\lambda). \end{aligned} \tag{2.18}$$

Making use of (2.1) we get

$$\int_a^b f(\lambda)\,du(\lambda) =_u D_n(f,u,a,b) +_u R_n(f,u,a,b). \tag{2.19}$$

3. Error Bounds

In order to provide sharp error bounds in the approximation rules outlined above, we need the following well known lemma concerning sharp estimates for the Riemann-Stieltjes integral for various pairs of integrands and integrators (see for instance [1]).

Lemma 1. *Let $p,v:[a,b]\to\mathbb{C}$ two bounded functions on the compact interval $[a,b]$.*

(i) *If p is continuous and v is of bounded variation, then the Riemann-Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and*

$$\left| \int_a^b p(t)\,dv(t) \right| \le \max_{t\in[a,b]} |p(t)| \bigvee_a^b (v), \tag{3.1}$$

where $\bigvee_a^b (v)$ denotes the total variation of v on the interval $[a,b]$.

(ii) *If p is Riemann integrable and v is Lipschitzian with the constant $L>0$, i.e.,*

$$|v(t)-v(s)| \le L|t-s| \text{ for each } t,s\in[a,b],$$

then the Riemann-Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and

$$\left| \int_a^b p(t)\,dv(t) \right| \le L\int_a^b |p(t)|\,dt \left(\le L \sup_{t\in[a,b]} |p(t)|(b-a) \right). \tag{3.2}$$

All the above inequalities are sharp in the sense that there are examples of functions for which each equality case is realized.

Utilizing this result concerning bounds for the Riemann-Stieltjes integral, we can provide the following error bounds in approximating the integral $\int_a^b f(\lambda)\,du(\lambda)$.

Theorem 2. *Assume that the function $f : I \to \mathbb{C}$ is n-time differentiable on the interior $\mathring{I}$ of the interval I $(n \geq 1)$ and the n-th derivative $f^{(n)}$ is of locally bounded variation on $\mathring{I}$. If $a, b \in \mathring{I}$ with $a < b, c \in [a,b]$ and $u : [a,b] \to \mathbb{C}$ is of bounded variation on $[a,b]$, then we have the representation (2.1) where the approximation term $T_n(f,u,a,c,b)$ is given by (2.2) and the remainder $R_n(f,u,a,c,b)$ satisfies the inequality*

$$|R_n(f,u,a,c,b)| \leq \frac{1}{n!}\left[\frac{1}{2}(b-a) + \left|c - \frac{a+b}{2}\right|\right]^n \bigvee_a^b \left(f^{(n)}\right) \bigvee_a^b (u), \tag{3.3}$$

for any $c \in [a,b]$.

If the n-th derivative $f^{(n)}$ is Lipschitzian with the constant $L_n > 0$ on $[a,b]$, then we have

$$|R_n(f,u,a,c,b)| \leq \frac{1}{(n+1)!} L_n \left[\frac{1}{2}(b-a) + \left|c - \frac{a+b}{2}\right|\right]^{n+1} \bigvee_a^b (u), \tag{3.4}$$

for any $c \in [a,b]$.

Proof. Utilizing the property (i) from Lemma 1 we have successively

$$\begin{aligned} |R_n(f,u,a,c,b)| &= \frac{1}{n!}\left|\int_a^b \left(\int_c^\lambda (\lambda-t)^n df^{(n)}(t)\right) du(\lambda)\right| \\ &\leq \frac{1}{n!}\max_{\lambda\in[a,b]}\left|\int_c^\lambda (\lambda-t)^n df^{(n)}(t)\right| \bigvee_a^b (u) \end{aligned} \tag{3.5}$$

for any $c \in [a,b]$.

For $c, \lambda \in [a,b]$, denote

$$B(\lambda,c) := \left|\int_c^\lambda (\lambda-t)^n df^{(n)}(t)\right|. \tag{3.6}$$

By the property (i) from Lemma 1 applied for $f^{(n)}$ we have for $c < \lambda$ that

$$\begin{aligned} B(\lambda,c) &\leq \max_{t\in[c,\lambda]} |\lambda-t|^n \bigvee_c^\lambda \left(f^{(n)}\right) \\ &= (\lambda-c)^n \bigvee_c^\lambda \left(f^{(n)}\right) \leq (\lambda-c)^n \bigvee_a^b \left(f^{(n)}\right) \\ &\leq (b-c)^n \bigvee_a^b \left(f^{(n)}\right) \end{aligned}$$

and for $c > \lambda$ that

$$\begin{aligned} B(\lambda, c) &\leq \max_{t \in [\lambda, c]} |\lambda - t|^n \bigvee_{\lambda}^{c} \left(f^{(n)}\right) \\ &= (c-\lambda)^n \bigvee_{\lambda}^{c} \left(f^{(n)}\right) \leq (c-\lambda)^n \bigvee_{a}^{b} \left(f^{(n)}\right) \\ &\leq (c-a)^n \bigvee_{a}^{b} \left(f^{(n)}\right) \end{aligned}$$

Therefore

$$\begin{aligned} \max_{\lambda \in [a,b]} B(\lambda, c) &\leq \max\left\{(b-c)^n, (c-a)^n\right\} \bigvee_{a}^{b} \left(f^{(n)}\right) \\ &= \left[\max\left\{b-c, c-a\right\}\right]^n \bigvee_{a}^{b} \left(f^{(n)}\right) \\ &= \left[\frac{1}{2}(b-a) + \left|c - \frac{a+b}{2}\right|\right]^n \bigvee_{a}^{b} \left(f^{(n)}\right), \end{aligned} \tag{3.7}$$

for any $c \in [a, b]$.

Utilizing (3.5) and (3.7) we deduce the desired inequality (3.3).

By the property (ii) from Lemma 1 applied for $f^{(n)}$ we have that

$$B(\lambda, c) \leq L_n \left| \int_c^{\lambda} |\lambda - t|^n \, dt \right| = \frac{L_n}{n+1} |\lambda - c|^{n+1}$$

$c, \lambda \in [a, b]$, which produces the bound

$$\begin{aligned} \max_{\lambda \in [a,b]} B(\lambda, c) &\leq \frac{L_n}{n+1} \max_{\lambda \in [a,b]} |\lambda - c|^{n+1} \\ &= \frac{L_n}{n+1} \max\left\{(b-c)^{n+1}, (c-a)^{n+1}\right\} \\ &= \frac{L_n}{n+1} \left[\max\left\{b-c, c-a\right\}\right]^{n+1} \\ &= \frac{L_n}{n+1} \left[\frac{1}{2}(b-a) + \left|c - \frac{a+b}{2}\right|\right]^{n+1} \end{aligned} \tag{3.8}$$

for any $c \in [a, b]$.

Utilizing (3.5) and (3.8) we deduce the desired inequality (3.4). □

The best error bounds we can get from Theorem 2 are as follows:

Corollary 1. *Under the assumptions of Theorem 2 we have the representation*

$$\int_a^b f(\lambda) \, du(\lambda) = M_n(f, u, a, b) +_M R_n(f, u, a, b) \tag{3.9}$$

where $M_n(f, u, a, b)$ *is defined in (2.14) and the error* ${}_M R_n(f, u, a, b)$ *satisfies the bound*

$$\left|{}_M R_n(f, u, a, b)\right| \leq \frac{1}{2^n n!} (b-a)^n \bigvee_{a}^{b} \left(f^{(n)}\right) \bigvee_{a}^{b} (u). \tag{3.10}$$

Moreover, if the n-th derivative $f^{(n)}$ is Lipschitzian with the constant $L_n > 0$ on $[a,b]$, then we have

$$|{}_M R_n(f,u,a,b)| \leq \frac{1}{2^{n+1}(n+1)!} L_n (b-a)^{n+1} \bigvee_a^b (u). \tag{3.11}$$

The case of Lipschitzian integrators may be of interest as well and will be considered in the following:

Theorem 3. *Assume that the function $f : I \to \mathbb{C}$ is n-time differentiable on the interior $\mathring{I}$ of the interval I $(n \geq 1)$ and the n-th derivative $f^{(n)}$ is of locally bounded variation on $\mathring{I}$. If $a, b \in \mathring{I}$ with $a < b, c \in [a,b]$ and $u : [a,b] \to \mathbb{C}$ is Lipschitzian on $[a,b]$ with the constant $K > 0$ then we have the representation (2.1) where the approximation term $T_n(f,u,a,c,b)$ is given by (2.2) and the remainder $R_n(f,u,a,c,b)$ satisfies the inequality*

$$\begin{aligned} |R_n(f,u,a,c,b)| &\leq \frac{1}{n!} K \int_a^b |\lambda - c|^n \left| \bigvee_c^\lambda \left(f^{(n)}\right) \right| d\lambda \\ &\leq \frac{1}{(n+1)!} K \left[(b-c)^{n+1} + (c-a)^{n+1}\right] \bigvee_a^b \left(f^{(n)}\right) \end{aligned} \tag{3.12}$$

for any $c \in [a,b]$.

If the n-th derivative $f^{(n)}$ is Lipschitzian with the constant $L_n > 0$ on $[a,b]$, then we have

$$|R_n(f,u,a,c,b)| \leq \frac{1}{(n+2)!} K L_n \left[(b-c)^{n+2} + (c-a)^{n+2}\right] \tag{3.13}$$

for any $c \in [a,b]$.

Proof. Utilizing the property (ii) from Lemma 1 we have successively

$$\begin{aligned} |R_n(f,u,a,c,b)| &= \frac{1}{n!} \left| \int_a^b \left(\int_c^\lambda (\lambda - t)^n \, df^{(n)}(t) \right) du(\lambda) \right| \\ &\leq \frac{1}{n!} K \int_a^b \left| \int_c^\lambda (\lambda - t)^n \, df^{(n)}(t) \right| d\lambda \\ &= \frac{1}{n!} K \int_a^b B(\lambda, c) \, d\lambda \end{aligned} \tag{3.14}$$

for any $c \in [a,b]$, where as above $B(\lambda, c) := \left| \int_c^\lambda (\lambda - t)^n \, df^{(n)}(t) \right|$, for $c, \lambda \in [a,b]$.

By the property (i) from Lemma 1 applied for $f^{(n)}$ we have for $c < \lambda$ that

$$B(\lambda, c) \leq \max_{t \in [c,\lambda]} |\lambda - t|^n \bigvee_c^\lambda \left(f^{(n)}\right) = (\lambda - c)^n \bigvee_c^\lambda \left(f^{(n)}\right)$$

and for $c > \lambda$ that

$$B(\lambda, c) \leq \max_{t \in [\lambda,c]} |\lambda - t|^n \bigvee_\lambda^c \left(f^{(n)}\right) = (c - \lambda)^n \bigvee_\lambda^c \left(f^{(n)}\right)$$

which gives that

$$B(\lambda, c) \leq |\lambda - c|^n \left| \bigvee_c^\lambda \left(f^{(n)}\right) \right| \leq |\lambda - c|^n \bigvee_a^b \left(f^{(n)}\right)$$

for $c, \lambda \in [a, b]$.

This implies that

$$\int_a^b B(\lambda, c)\, d\lambda \leq \int_a^b |\lambda - c|^n \left| \bigvee_c^\lambda \left(f^{(n)}\right) \right| d\lambda \tag{3.15}$$
$$\leq \bigvee_a^b \left(f^{(n)}\right) \int_a^b |\lambda - c|^n\, d\lambda$$
$$= \frac{1}{n+1}\left[(b-c)^{n+1} + (c-a)^{n+1}\right] \bigvee_a^b \left(f^{(n)}\right)$$

for $c \in [a, b]$.

Making use of (3.14) and (3.15) we deduce the desired inequality (3.12).

By the property (ii) from Lemma 1 applied for $f^{(n)}$ we have that

$$B(\lambda, c) \leq L_n \left| \int_c^\lambda |\lambda - t|^n\, dt \right| = \frac{L_n}{n+1} |\lambda - c|^{n+1}$$

$c, \lambda \in [a, b]$, which produces the bound

$$\int_a^b B(\lambda, c)\, d\lambda \leq \frac{L_n}{n+1} \int_a^b |\lambda - c|^{n+1}\, d\lambda \tag{3.16}$$
$$= \frac{L_n}{(n+1)(n+2)}\left[(b-c)^{n+2} + (c-a)^{n+2}\right]$$

for $c \in [a, b]$.

Utilizing (3.14) and (3.16) we deduce the desired inequality (3.13). □

The following particular case provides the best error bounds:

Corollary 2. *Under the assumptions of Theorem 3 we have the representation (3.9) where $M_n(f, u, a, b)$ is defined in (2.14) and the error ${}_M R_n(f, u, a, b)$ satisfies the bound*

$$|{}_M R_n(f, u, a, b)| \leq \frac{1}{n!} K \int_a^b \left| \lambda - \frac{a+b}{2} \right|^n \left| \bigvee_{\frac{a+b}{2}}^\lambda \left(f^{(n)}\right) \right| d\lambda \tag{3.17}$$
$$\leq \frac{1}{2^n (n+1)!} K (b-a)^{n+1} \bigvee_a^b \left(f^{(n)}\right).$$

Moreover, if the n-th derivative $f^{(n)}$ is Lipschitzian with the constant $L_n > 0$ on $[a, b]$, then we have

$$|{}_M R_n(f, u, a, b)| \leq \frac{1}{2^{n+1}(n+2)!} K L_n (b-a)^{n+2}. \tag{3.18}$$

4. Applications

1. We consider the following *finite Laplace-Stieltjes transform* defined by

$$\left(\mathcal{L}_{[a,b]}g\right)(s):=\int_a^b e^{-st}dg(t) \tag{4.1}$$

where a,b are real numbers with $a<b$, s is a complex number and $g:[a,b]\to\mathbb{C}$ is a function of bounded variation.

Since the function $f_s:[a,b]\to\mathbb{C},\ f_s(t):=e^{-st}$ is continuous for any $s\in\mathbb{C}$, the transform (4.1) is well defined for any $s\in\mathbb{C}$.

We observe that the function f_s has derivatives of all orders and

$$f_s^{(k)}(t)=(-1)^k s^k e^{-st} \text{ for any } s\in\mathbb{C},\ t\in[a,b] \text{ and } k\geq 0. \tag{4.2}$$

We also observe that

$$\begin{aligned}\left\|f_s^{(n+1)}\right\|_{[a,b],\infty} &:= \sup_{t\in[a,b]}\left|f_s^{(n+1)}(t)\right| = |s|^{n+1}\sup_{t\in[a,b]}\left|e^{-st}\right| \\ &= |s|^{n+1}\sup_{t\in[a,b]}e^{-t\operatorname{Re}s} = |s|^{n+1}\times\begin{cases} e^{-a\operatorname{Re}s} & \text{if } \operatorname{Re}s\geq 0,\\ e^{-b\operatorname{Re}s} & \text{if } \operatorname{Re}s<0.\end{cases}\end{aligned}$$

To simplify the notations, we denote by

$$\beta_{[a,b]}(s):=\begin{cases} e^{-a\operatorname{Re}s} & \text{if } \operatorname{Re}s\geq 0,\\ e^{-b\operatorname{Re}s} & \text{if } \operatorname{Re}s<0.\end{cases} \tag{4.3}$$

On utilizing Theorem 1 we have the representation

$$\left(\mathcal{L}_{[a,b]}g\right)(s)=\mathcal{G}_n(g,a,c,b)(s)+\mathcal{Z}_n(g,a,c,b)(s) \tag{4.4}$$

where

$$\begin{aligned}&\mathcal{G}_n(g,a,c,b)(s)\\ &:=\sum_{k=0}^{n}\frac{(-1)^k}{k!}s^k e^{-sc}\left[(b-c)^k g(b)+(-1)^{k+1}(c-a)^k g(a)\right]\\ &+\sum_{k=0}^{n-1}\frac{(-1)^k}{k!}s^{k+1}e^{-sc}\int_a^b(\lambda-c)^k g(\lambda)\,d\lambda\end{aligned} \tag{4.5}$$

and the remainder $\mathcal{Z}_n(g,a,c,b)(s)$ can be represented as

$$\mathcal{Z}_n(g,a,c,b)(s):=\frac{(-1)^{n+1}}{n!}s^{n+1}\int_a^b\left(\int_c^\lambda(\lambda-t)^n e^{-st}dt\right)dg(\lambda). \tag{4.6}$$

Here $s\in\mathbb{C}$ and $c\in[a,b]$.

Since g is of bounded variation on $[a,b]$ and the derivative $f_s^{(n)}$ is Lipschitzian with the constant

$$L_n:=\left\|f_s^{(n+1)}\right\|_{[a,b],\infty}=|s|^{n+1}\beta_{[a,b]}(s)$$

then by Theorem 2 we have the bound

$$\begin{aligned} (4.7) \qquad & |\mathcal{Z}_n(g,a,c,b)(s)| \\ & \leq \frac{1}{(n+1)!}|s|^{n+1}\beta_{[a,b]}(s)\left[\frac{1}{2}(b-a)+\left|c-\frac{a+b}{2}\right|\right]^{n+1}\bigvee_a^b(g), \end{aligned}$$

for any $s\in\mathbb{C}$ and $c\in[a,b]$.

As above, the best approximation we can get from (4.4) is for $c=\frac{a+b}{2}$, namely, we have the representation

$$(4.8) \qquad \left(\mathcal{L}_{[a,b]}g\right)(s) = {}_M\mathcal{G}_n(g,a,b)(s) + {}_M\mathcal{Z}_n(g,a,b)(s)$$

where

$$\begin{aligned} (4.9) \qquad & {}_M\mathcal{G}_n(g,a,b)(s) \\ & := \sum_{k=0}^{n}\frac{(-1)^k}{2^k k!}s^k e^{-s\frac{a+b}{2}}(b-a)^k\left[g(b)+(-1)^{k+1}g(a)\right] \\ & + \sum_{k=0}^{n-1}\frac{(-1)^k}{k!}s^{k+1}e^{-s\frac{a+b}{2}}\int_a^b\left(\lambda-\frac{a+b}{2}\right)^k g(\lambda)\,d\lambda \end{aligned}$$

and the remainder ${}_M\mathcal{Z}_n(g,a,b)(s)$ can be represented as

$$(4.10) \qquad {}_M\mathcal{Z}_n(g,a,b)(s) := \frac{(-1)^{n+1}}{n!}s^{n+1}\int_a^b\left(\int_{\frac{a+b}{2}}^{\lambda}(\lambda-t)^n e^{-st}dt\right)dg(\lambda).$$

The error ${}_M\mathcal{Z}_n(g,a,b)(s)$ satisfies the bound

$$\begin{aligned} (4.11) \qquad & |{}_M\mathcal{Z}_n(g,a,b)(s)| \\ & \leq \frac{1}{2^{n+1}(n+1)!}|s|^{n+1}\beta_{[a,b]}(s)(b-a)^{n+1}\bigvee_a^b(g), \end{aligned}$$

for any $s\in\mathbb{C}$.

Now, if we restrict the function g to belong to the class of Lipschitzian functions with the constant $K>0$ on the interval $[a,b]$, then the error in the representation (4.4) will satisfy the bound

$$|\mathcal{Z}_n(g,a,c,b)(s)| \leq \frac{1}{(n+2)!}K|s|^{n+1}\beta_{[a,b]}(s)\left[(b-c)^{n+2}+(c-a)^{n+2}\right]$$

for any $s\in\mathbb{C}$ and $c\in[a,b]$.

Finally, the error ${}_M\mathcal{Z}_n(g,a,b)(s)$ from the representation (4.8) satisfies the inequality

$$|{}_M\mathcal{Z}_n(g,a,b)(s)| \leq \frac{1}{2^{n+1}(n+2)!}K|s|^{n+1}\beta_{[a,b]}(s)(b-a)^{n+2}$$

for any $s\in\mathbb{C}$.

2. We consider now the *finite Fourier-Stieltjes sine and cosine transforms* defined by

$$(4.12) \qquad \left(\mathcal{F}_{s,[a,b]}g\right)(u) := \int_a^b \sin(ut)\,dg(t), \quad \left(\mathcal{F}_{c,[a,b]}g\right)(u) := \int_a^b \cos(ut)\,dg(t),$$

where a,b are real numbers with $a<b$, u is a real number and $g:[a,b]\to\mathbb{C}$ is a function of bounded variation.

Since the functions $f_{s;u}, f_{c;u} : [a,b] \to \mathbb{R}$, $f_{s;u}(t) := \sin(ut)$, $f_{c;u}(t) := \cos(ut)$ are continuous for any $u \in \mathbb{R}$, the transforms (4.12) are well defined for any $u \in \mathbb{R}$.

Utilizing the well-know formulae for the n-th derivatives of sine and cosine functions, namely,

$$\text{if } y = \sin(Ax+B) \text{ then } \frac{d^n y}{dx^n} = A^n \sin\left(Ax + B - \frac{n\pi}{2}\right)$$

and

$$\text{if } y = \cos(Ax+B) \text{ then } \frac{d^n y}{dx^n} = A^n \cos\left(Ax + B - \frac{n\pi}{2}\right),$$

then we have

$$f_{s;u}^{(k)}(t) = u^k \sin\left(ut - \frac{k\pi}{2}\right) \text{ and } f_{c;u}^{(k)}(t) = u^k \cos\left(ut - \frac{k\pi}{2}\right)$$

for any $u \in \mathbb{R}$ and $k \geq 0$.

We observe that, in general we have the bounds

$$\left\| f_{s;u}^{(n+1)} \right\|_{[a,b],\infty} = \sup_{t\in[a,b]} \left| u^{n+1} \sin\left(ut - \frac{(n+1)\pi}{2}\right) \right| \leq |u|^{n+1}$$

and

$$\left\| f_{c;u}^{(n+1)} \right\|_{[a,b],\infty} = \sup_{t\in[a,b]} \left| u^{n+1} \cos\left(ut - \frac{(n+1)\pi}{2}\right) \right| \leq |u|^{n+1}$$

for any $u \in \mathbb{R}$, the closed interval $[a,b]$ and $n \geq 0$.

On utilizing Theorem 1 we have the representation

$$\left(\mathcal{F}_{s,[a,b]} g\right)(u) = \mathcal{K}_{s,n}(g,a,c,b)(u) + \mathcal{W}_{s,n}(g,a,c,b)(u) \tag{4.13}$$

where

$$\begin{aligned} &\mathcal{K}_{s,n}(g,a,c,b)(u) \\ &:= \sum_{k=0}^{n} \frac{1}{k!} u^k \sin\left(uc - \frac{k\pi}{2}\right) \left[(b-c)^k g(b) + (-1)^{k+1} (c-a)^k g(a)\right] \\ &- \sum_{k=0}^{n-1} \frac{1}{k!} u^{k+1} \sin\left(uc - \frac{(k+1)\pi}{2}\right) \int_a^b (\lambda - c)^k g(\lambda)\, d\lambda \end{aligned} \tag{4.14}$$

and the remainder $\mathcal{W}_{s,n}(g,a,c,b)(u)$ can be represented as

$$\begin{aligned} &\mathcal{W}_{s,n}(g,a,c,b)(u) \\ &= \frac{1}{n!} u^{n+1} \int_a^b \left(\int_c^\lambda (\lambda - t)^n \sin\left(ut - \frac{(n+1)\pi}{2}\right) dt \right) dg(\lambda). \end{aligned} \tag{4.15}$$

Since g is of bounded variation on $[a,b]$ and the derivative $f_s^{(n)}$ is Lipschitzian with the constant

$$L_n := \left\| f_s^{(n+1)} \right\|_{[a,b],\infty} \leq |u|^{n+1}$$

then by Theorem 2 we have the bound

$$\begin{aligned} &|\mathcal{W}_{s,n}(g,a,c,b)(u)| \\ &\leq \frac{1}{(n+1)!} |u|^{n+1} \left[\frac{1}{2}(b-a) + \left|c - \frac{a+b}{2}\right|\right]^{n+1} \bigvee_a^b (g), \end{aligned} \tag{4.16}$$

for any $u \in \mathbb{R}$ and $c \in [a,b]$.

As above, the best approximation we can get from (4.4) is for $c = \frac{a+b}{2}$, namely, we have the representation

$$\left(\mathcal{F}_{s,[a,b]}g\right)(u) = {}_M\mathcal{K}_{s,n}(g,a,b)(u) + {}_M\mathcal{W}_{s,n}(g,a,b)(u) \tag{4.17}$$

where

$$\begin{aligned} &{}_M\mathcal{K}_{s,n}(g,a,b)(u) \\ &:= \sum_{k=0}^{n} \frac{1}{2^K k!} u^k \sin\left(\frac{a+b}{2}u - \frac{k\pi}{2}\right)(b-a)^k \left[g(b) + (-1)^{k+1} g(a)\right] \\ &- \sum_{k=0}^{n-1} \frac{1}{k!} u^{k+1} \sin\left(\frac{a+b}{2}u - \frac{(k+1)\pi}{2}\right) \int_a^b \left(\lambda - \frac{a+b}{2}\right)^k g(\lambda)\, d\lambda \end{aligned} \tag{4.18}$$

and the remainder ${}_M\mathcal{W}_{s,n}(g,a,b)(u)$ can be represented as

$$\begin{aligned} &{}_M\mathcal{W}_{s,n}(g,a,b)(u) \\ &= \frac{1}{n!} u^{n+1} \int_a^b \left(\int_{\frac{a+b}{2}}^{\lambda} (\lambda - t)^n \sin\left(ut - \frac{(n+1)\pi}{2}\right) dt\right) dg(\lambda). \end{aligned} \tag{4.19}$$

for any $u \in \mathbb{R}$.

Here the error satisfies the bound

$$\left|{}_M\mathcal{W}_{s,n}(g,a,b)(u)\right| \le \frac{1}{2^{n+1}(n+1)!} |u|^{n+1} (b-a)^{n+1} \bigvee_a^b (g) \tag{4.20}$$

for any $u \in \mathbb{R}$.

Now, if we restrict the function g to belong to the class of Lipschitzian functions with the constant $K > 0$ on the interval $[a,b]$, then the error in the representation (4.17) will satisfy the bound:

$$\left|\mathcal{W}_{s,n}(g,a,c,b)(u)\right| \le \frac{1}{(n+2)!} K |u|^{n+1} \left[(b-c)^{n+2} + (c-a)^{n+2}\right], \tag{4.21}$$

for any $u \in \mathbb{R}$ and $c \in [a,b]$.

Finally, the error from the representation (4.17) satisfies the inequality

$$\left|{}_M\mathcal{W}_{s,n}(g,a,b)(u)\right| \le \frac{1}{2^{n+1}(n+2)!} K |u|^{n+1} (b-a)^{n+2} \tag{4.22}$$

for any $u \in \mathbb{R}$.

Similar results may be stated for the finite Fourier-Stieltjes cosine transform, however the details are left to the interested reader.

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