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PARAMETERIZED LOGARITHMIC MEAN

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ABSTRACT. In this paper, a parameterized logarithmic mean is introduced. Some new means are derived and open problems are putted.

1. INTRODUCTION

Throughout this paper, we understand by mean a binary map m between positive real numbers satisfying the following statements:
(i) $m(a,a)=a$, for all $a>0$ (normalization axiom);
(ii) $m(ta,tb)=tm(a,b)$, for all $a,b,t>0$ (homogeneity axiom);
(iii) $m(a,b)$ is an increasing function in a (and in b) (monotonicity axiom).
A mean m is called symmetric if the following axiom is further satisfied:
(iv) $m(a,b)=m(b,a)$, for all $a,b>0$ (symmetry axiom).

For two means m_1 and m_2 we write $m_1 \leq m_2$ (resp. $m_1 < m_2$) if and only if $m_1(a,b) \leq m_2(a,b)$ for all $a,b>0$ (resp. $m_1(a,b) < m_2(a,b)$ for all $a,b>0$ with $a \neq b$). Two trivial (symmetric) means are $(a,b) \longmapsto min(a,b)$ and $(a,b) \longmapsto max(a,b)$, and every (symmetric or not) mean m satisfies

$$min(a,b) \leq m(a,b) \leq max(a,b),$$

for all $a,b>0$. We denote min and max the two trivial means which we call lower and upper means respectively. The standard examples of (symmetric) means satisfying the above requirements are recalled in the following.

- Arithmetic mean, $A(a,b)=\dfrac{a+b}{2}$;
- Geometric mean, $G(a,b)=\sqrt{ab}$;
- Harmonic mean, $H(a,b)=\dfrac{2ab}{a+b}$;
- Logarithmic mean, $L(a,b)=\dfrac{a-b}{ln\ a-ln\ b},\ a\neq b,\ \ L(a,a)=a$;
- Identric (or Exponential) mean, $I(a,b)=\dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{1/b-a},\ a\neq b,\ \ I(a,a)=a$.

It is well known that, the above standard means satisfy the following inequalities

$$min < H < G < L < I < A < max. \tag{1.1}$$

Now, let $\alpha \in [0,1]$ and $a,b>0$ be given real numbers. The following binary maps

$$H_\alpha(a,b)=\Big((1-\alpha)a^{-1}+\alpha b^{-1}\Big)^{-1};\ \ G_\alpha(a,b)=a^{1-\alpha}b^\alpha;\ \ A_\alpha(a,b)=(1-\alpha)a+\alpha b$$

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are known in the literature as the parameterized harmonic, geometric and arithmetic means, respectively. These means are generally not symmetric and they satisfy the following axiom (called here the conjugate symmetry axiom or the joint-symmetry axiom)

$$m_\alpha(a,b) = m_{1-\alpha}(b,a)$$

for all $\alpha \in [0,1]$ and $a,b > 0$, where m_α stands for one of $H_\alpha, G_\alpha, A_\alpha$. Further, the following inequalities

$$min(a,b) < H_\alpha(a,b) < G_\alpha(a,b) < A_\alpha(a,b) < max(a,b) \tag{1.2}$$

hold true for every $\alpha \in]0,1[$ and all $a,b > 0$ with $a \neq b$. The above parameterized means extend the familiar harmonic, geometric and arithmetic means in the sense that if $\alpha = 1/2$ then they are reduced to $H(a,b), G(a,b)$ and $A(a,b)$, respectively.

After this, a question arises from the above:

Question 1. Does exist a parameterized mean $L_\alpha(a,b)$ satisfying the following assertions:
(1) $L_\alpha(a,b)$ extends $L(a,b)$ in the sense

$$L_{1/2}(a,b) = L(a,b)$$

holds for all $a,b > 0$;
(2) $L_\alpha(a,b)$ satisfies the conjugate symmetry axiom, that is,

$$L_\alpha(a,b) = L_{1-\alpha}(b,a)$$

is valid for each $\alpha \in [0,1]$ and all $a,b > 0$;
(3) $L_\alpha(a,b)$ interpolates $A_\alpha(a,b)$ and $G_\alpha(a,b)$, that is to say,

$$H_\alpha(a,b) < G_\alpha(a,b) < L_\alpha(a,b) < A_\alpha(a,b)$$

holds true for every $\alpha \in]0,1[$ and all $a,b > 0$ with $a \neq b$.

The fundamental goal of the present paper is to give an affirmative answer for the above question. This allows us to derive some other means whose the links between those known in the literature remain as open problems.

2. Parameterized Logarithmic Mean

In the aim to explain our key idea for the reader, we begin this section by recalling various expansions of the logarithmic mean $L(a,b)$ differently introduced in the literature.

- $L(a,b) = \dfrac{b-a}{ln\, b - ln\, a}, \quad L(a,a) = a;$
- $\Big(L(a,b)\Big)^{-1} = \dfrac{1}{b-a}\displaystyle\int_a^b \frac{1}{t}dt = \int_0^1 \Big((1-t)a + tb\Big)^{-1}dt = \int_0^1 \Big(A_t(a,b)\Big)^{-1}dt;$
- $\Big(L(a,b)\Big)^{-1} = \displaystyle\int_0^{+\infty} \frac{dt}{(t+a)(t+b)};$
- $L(a,b) = \displaystyle\int_0^1 a^{1-t}b^t dt = \int_0^1 G_t(a,b)dt.$

As well known (and it is easy to verify that) the above expressions of $L(a,b)$ have the same value. The above question saying that how to obtain the desired parameterized logarithmic mean $L_\alpha(a,b)$ from the above expressions of $L(a,b)$ appears to us not obvious. For this, we need another equivalent expression of $L(a,b)$ having appropriate form for our purpose. We recall the following result which has been derived by the author from some results about new concepts for means, [1, 2, 3].

Theorem 2.1. *The following expansion*

$$L(a,b)=\prod_{n=1}^{\infty}A\Big(a^{1/2^n},b^{1/2^n}\Big):=\prod_{n=1}^{\infty}\frac{a^{1/2^n}+b^{1/2^n}}{2} \tag{2.1}$$

holds true for all $a,b>0$.

The above theorem makes appear new information telling us that the logarithmic mean $L(a,b)$ can be expressed in terms of infinite products involving the (simple) arithmetic mean whose the associate parameterized arithmetic mean $A_\alpha(a,b)=(1-\alpha)a+\alpha b$ has also an easy expression. In another way, the expression of $L(a,b)$ given by the above theorem gives us the key idea for formulating the desired parameterized logarithmic mean. Precisely, we may state the following.

Definition 2.1. Let $\alpha\in[0,1]$ and $a,b>0$. We set

$$L_\alpha(a,b)=\prod_{n=1}^{\infty}A_\alpha\Big(a^{1/2^n},b^{1/2^n}\Big):=\prod_{n=1}^{\infty}\Big((1-\alpha)a^{1/2^n}+\alpha b^{1/2^n}\Big), \tag{2.2}$$

which will be called the parameterized logarithmic mean.

Of course, it is easy to see that the infinite products defining $L_\alpha(a,b)$ converges. Further, $L_\alpha(a,b)$ satisfies the two assertions (1) and (2), since the relations

$$L_{1/2}(a,b)=L(a,b)\quad\text{and}\quad L_\alpha(a,b)=L_{1-\alpha}(b,a)$$

are obviously satisfied for each $\alpha\in[0,1]$ and all $a,b>0$. Our parameterized logarithmic mean satisfies also the assertion (3) as confirmed by the following statement.

Theorem 2.2. *Let $\alpha\in]0,1[$ and all $a,b>0$ with $a\neq b$. Then the following inequalities hold*

$$\Big(H_\alpha(a,b)<\Big)G_\alpha(a,b)<L_\alpha(a,b)<A_\alpha(a,b). \tag{2.3}$$

Proof. For fixed integer $n\geq 1$, the map $x\longmapsto x^{1/2^n}$ is strictly concave on $]0,+\infty[$. Then we have

$$L_\alpha(a,b)=\prod_{n=1}^{\infty}\Big((1-\alpha)a^{1/2^n}+\alpha b^{1/2^n}\Big)<\prod_{n=1}^{\infty}\Big((1-\alpha)a+\alpha b\Big)^{1/2^n},$$

and consequently

$$L_\alpha(a,b)<\Big((1-\alpha)a+\alpha b\Big)^{\sum_{i=1}^{\infty}(1/2^n)}.$$

A simple computation yields $\sum_{i=1}^{\infty}(1/2^n)=1$ and so the right side of (2.3) is obtained. Now, according to (2.2) with (1.2) we have

$$L_\alpha(a,b)>\prod_{n=1}^{\infty}a^{\frac{1-\alpha}{2^n}}b^{\frac{\alpha}{2^n}},$$

or again

$$L_\alpha(a,b)>\big(a^{1-\alpha}b^\alpha\big)^{\sum_{i=1}^{\infty}(1/2^n)}$$

which by the same arguments as previous gives $L_\alpha(a,b)>G_\alpha(a,b)$, so completes the proof. □

3. Some Means and Open Problems

In this section we will display the construction of means derived from the above parameterized means. Preserving the same notations as previous, we first introduce the following.

Proposition 3.1. *For all fixed real numbers $a, b > 0$, the function $t \longmapsto L_t(a,b)$ is continuous on $[0,1]$.*

Proof. As previously, for fixed $a, b > 0$ we have for each $t \in [0,1]$

$$L_t(a,b) \leq \prod_{n=1}^{\infty}\Big((1-t)a+tb\Big)^{1/2^n} \leq \prod_{n=1}^{\infty}\Big(max(a,b)\Big)^{1/2^n}.$$

It follows that the infinite products defining $L_t(a,b)$ converges normally then uniformly and so the announced result follows. □

Now, we are in position to state the next definition.

Definition 3.1. For $a, b > 0$, we set

$$E(a,b) = \int_0^1 L_t(a,b)dt.$$

Proposition 3.2. *The map $(a,b) \longmapsto E(a,b)$ defines a symmetric mean and the following double inequality*

$$L(a,b) < E(a,b) < A(a,b)$$

holds true for all $a, b > 0$ with $a \neq b$.

Proof. The fact that $E(a,b)$ is a mean is immediate and detail is omitted for the reader. For proving the symmetry axiom it is sufficient to use the change variable $t = 1-s$ with the help of the relation $L_t(a,b) = L_{1-t}(b,a)$. Now, starting from the double inequality

$$G_t(a,b) < L_t(a,b) < A_t(a,b),$$

valid for all $a, b > 0$, $a \neq b$ and $t \in]0,1[$, and according to Proposition 3.1, we can integrate the three sides of the above double inequality over $t = 0$ to $t = 1$. The desired result follows after elementary computations. □

Definition 3.2. For $a, b > 0$, we set

$$F(a,b) = exp\int_0^1 ln\Big(L_t(a,b)\Big)dt. \tag{3.1}$$

Since the real-map $x \longmapsto exp\ x$ is strictly convex, the integral Jensen inequality immediately gives $F(a,b) < E(a,b)$ for all $a, b > 0, a \neq b$. Further, the next result may be stated.

Proposition 3.3. *The map $(a,b) \longmapsto F(a,b)$ defines a symmetric mean and the following double inequality*

$$G(a,b) < F(a,b) < I(a,b)$$

is valid for all $a, b > 0$ with $a \neq b$.

Proof. Let $a, b > 0$ with $a \neq b$ and $t \in]0,1[$. Starting from the double inequality

$$G_t(a,b) < L_t(a,b) < A_t(a,b),$$

using the strict increase monotonicity of the real-map $x \longmapsto ln\ x,\ x > 0$ and integrating side to side we obtain

$$\int_0^1 ln\ G_t(a,b)dt < \int_0^1 ln\ L_t(a,b)dt < \int_0^1 ln\ A_t(a,b)dt.$$

Elementary computations lead to

$$\int_0^1 ln\ G_t(a,b)dt = \frac{ln\ a + ln\ b}{2} = ln\ \sqrt{ab},$$

$$\int_0^1 ln\ A_t(a,b)dt = ln\ I(a,b). \tag{3.2}$$

Substituting these in the above, with the strict increase monotonicity of $x \longmapsto exp\ x$, we obtain the desired result. □

As in the above, Proposition 3.3 with (1.1) allows us to ask if the means $F(a,b)$ and $L(a,b)$ coincide or not. Before giving an affirmative answer for this question, we may state the following result which gives an expansion of $F(a,b)$ in terms of infinite products involving the identric mean.

Theorem 3.4. *For all $a,b > 0$, the following expansion holds true*

$$F(a,b) = \prod_{n=1}^{\infty} I\Big(a^{1/2^n}, b^{1/2^n}\Big). \tag{3.3}$$

Proof. According to (3.1) and (2.2) we have

$$\begin{aligned} F(a,b) &= exp\int_0^1 \sum_{n=1}^{\infty} ln\ A_t\big(a^{1/2^n}, b^{1/2^n}\big)dt \\ &= exp\sum_{n=1}^{\infty}\int_0^1 ln\ A_t\big(a^{1/2^n}, b^{1/2^n}\big)dt = \prod_{n=1}^{\infty} exp\int_0^1 ln\ A_t\big(a^{1/2^n}, b^{1/2^n}\big)dt. \end{aligned} \tag{3.4}$$

This, with (3.2), gives the desired result. □

The next corollary gives a response for the above question as recited in what follows.

Corollary 3.5. *The mean F interpolates the means G and L, that is, the following inequalities*

$$G(a,b) < F(a,b) < L(a,b) \tag{3.5}$$

hold true for all $a,b > 0$ with $a \neq b$.

Proof. By (3.3) with (1.1) we obtain

$$F(a,b) < \prod_{n=1}^{\infty} A\Big(a^{1/2^n}, b^{1/2^n}\Big),$$

which, when combined with (2.1), yields the right inequality of (3.5). Again (3.3), with (1.1), gives

$$F(a,b) > \prod_{n=1}^{\infty} G\big(a^{1/2^n}, b^{1/2^n}\big) = \prod_{n=1}^{\infty} (ab)^{1/2^{n+1}} = \sqrt{ab}.$$

The proof is complete. □

Summarizing the above results, the following chains of inequalities are met:

$$G < F < L < E < A,$$

$$G < F < L < I < A.$$

We can then arise the next open question.

Question 2. Prove or disprove that the means E and I are different. We conjecture that E interpolates L and I, i.e. $L < E < I$.

Finally, we end this section by putting the next open problem about a parameterized identric mean.

Question 3. Does exit a parameterized mean $I_\alpha(a,b),\ \alpha \in [0,1]$, such that:
(j) $I_{1/2}(a,b) = I(a,b)$ for all $a,b > 0$;
(jj) $I_\alpha(a,b) = I_{1-\alpha}(b,a)$ for all $a,b > 0$ and $\alpha \in [0,1]$;
(jjj) $L_\alpha(a,b) < I_\alpha(a,b) < A_\alpha(a,b)$ for all $a,b > 0$ with $a \neq b$ and $\alpha \in]0,1[$.

References

[1] M. Raïssouli, Approaching the Power Logarithmic and Difference Means by Algorithms involving the Power Binomial Mean, International Journal of Mathematics and Mathematical Sciences, Volume 2011 (2011), Article ID 687825, 12 pages.

[2] M. Raïssouli, Stability and Stabilizability for Means, Applied Mathematics E-Notes, 11 (2011), 159-174.

[3] M. Raïssouli, Stabilizability of the Stolarsky Mean and its Approximation in terms of the Power Binomial Mean. Submitted, 2011.

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