STABILIZABILITY OF THE STOLARSKY MEAN AND ITS APPROXIMATION IN TERMS OF THE POWER BINOMIAL MEAN

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ABSTRACT. We prove that the Stolarsky mean $E_{p,q}$ of order (p,q) is (B_{q-p}, B_p)-stabilizable, where B_p denotes the power binomial mean. This allows us to approximate the nonstable Stolarsky mean by an iterative algorithm involving the stable power binomial mean.

1. INTRODUCTION

Throughout this paper, we understand by binary mean a map m between two positive real numbers satisfying the following statements.
(i) $m(a,a) = a$, for all $a > 0$;
(ii) $m(a,b) = m(b,a)$, for all $a, b > 0$;
(iii) $m(ta, tb) = tm(a,b)$, for all $a, b, t > 0$;
(iv) $m(a,b)$ is an increasing function in a (and in b);
(v) $m(a,b)$ is a continuous function of a and b.

A binary mean is also called mean with two variables. Henceforth, we shortly call mean instead of binary mean. The definition of mean with three or more variables can be stated in a similar manner. The set of all (binary) means can be equipped with a partial ordering, called point-wise order, defined by: $m_1 \leq m_2$ if and only if $m_1(a,b) \leq m_2(a,b)$ for every $a, b > 0$.

The standard examples of means satisfying the above requirements are recalled in the following.

$$A := A(a,b) = \frac{a+b}{2}; \ G := G(a,b) = \sqrt{ab}; \ H := H(a,b) = \frac{2ab}{a+b};$$

$$L := L(a,b) = \frac{b-a}{ln\ b - ln\ a}; \ L(a,a) = a, \ I := I(a,b) = \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/(b-a)}, \ I(a,a) = a,$$

respectively called the arithmetic, geometric, harmonic, logarithmic and identric means. These means satisfy the following inequalities

$$min \leq H \leq G \leq L \leq I \leq A \leq max,$$

where min and max are the trivial means $(a,b) \longmapsto min(a,b)$ and $(a,b) \longmapsto max(a,b)$.

For a given mean m, we set

$$m^*(a,b) = \left(m\left(a^{-1}, b^{-1}\right)\right)^{-1},$$

and it is easy to see that m^* is also a mean, called the dual mean of m. The symmetry and homogeneity axioms (ii),(iii) yield $m^*(a,b) = \frac{ab}{m(a,b)}$ which we briefly write $m^* = G^2/m$. Every mean m satisfies $m^{**} = m$ and, if m_1 and m_2 are two means such that $m_1 \leq m_2$

then $m_1^* \geq m_2^*$. A mean m is called self-dual if $m^* = m$. It is clear that the arithmetic and harmonic means are mutually dual and the geometric mean is the unique self-dual mean.

The dual of the logarithmic mean is given by

$$L^* := L^*(a,b) = ab\frac{\ln b - \ln a}{b-a}, \quad L^*(a,a) = a,$$

while that of the identric mean is

$$I^* := I^*(a,b) = e\left(\frac{a^b}{b^a}\right)^{1/b-a}, \quad I^*(a,a) = a.$$

The following inequalities are immediate from the above.

$$min \leq H \leq I^* \leq L^* \leq G \leq L \leq I \leq A \leq max.$$

A mean m is called strict mean if $m(a,b)$ is strictly increasing in a (and in b). Also, every strict mean m satisfies that, $m(a,b) = a \Longrightarrow a = b$. It is easy to see that the trivial means are not strict, while A, G, H, L, L^*, I, I^* are strict means.

In the literature, there are some families of means, called power means, which include the above familiar means. Precisely, let p and q be two real numbers, the Stolarsky mean $E_{p,q}$ of order (p,q) is defined by, [8, 9].

$$E_{p,q} := E_{p,q}(a,b) = \left(\frac{p}{q}\frac{b^q - a^q}{b^p - a^p}\right)^{1/(q-p)}, \quad E_{p,q}(a,a) = a.$$

It is understood that this family of means includes some of the most of particular cases in the following sense:

• The power binomial mean:

$$\begin{cases} E_{p,2p}(a,b) := B_p(a,b) = B_p = \left(\frac{a^p + b^p}{2}\right)^{1/p}, \\ \\ B_{-\infty} = min, \ B_{-1} = H, \ B_1 = A, \ B_0 := lim_{p\to 0}B_p = G, \ B_\infty = max. \end{cases}$$

• The power logarithmic mean:

$$\begin{cases} E_{1,p+1} := L_p(a,b) = L_p = \left(\frac{a^{p+1} - b^{p+1}}{(p+1)(a-b)}\right)^{1/p}, \ L_p(a,a) = a, \\ \\ L_{-\infty} = min, \ L_{-2} = G, \ L_{-1} = L, \ L_0 = I, \ L_1 = A, \ L_\infty = max. \end{cases}$$

• The power difference mean:

$$\begin{cases} E_{p,p+1}(a,b) := D_p(a,b) = D_p = \frac{p}{p+1}\frac{a^{p+1} - b^{p+1}}{a^p - b^p}, \ D_p(a,a) = a, \\ \\ D_{-\infty} = min, \ D_{-2} = H, \ D_{-1} = L^*, \ D_{-1/2} = G, \ D_0 = L, \ D_1 = A, \ D_\infty = max. \end{cases}$$

• The power exponential mean:

$$\begin{cases} E_{p,p}(a,b) := I_p(a,b) = I_p = exp\left(-\frac{1}{p} + \frac{a^p \ln a - b^p \ln b}{a^p - b^p}\right), \ I_p(a,a) = a, \\ \\ I_{-\infty} = min, \ I_{-1} = I^*, \ I_0 = G, \ I_1 = I, \ I_{+\infty} = max. \end{cases}$$

- The second power logarithmic mean:

$$\begin{cases} E_{0,p}(a,b) := l_p(a,b) = l_p = \left(\dfrac{1}{p} \dfrac{b^p - a^p}{\ln b - \ln a} \right)^{1/p} , \ l_p(a,a) = a, \\[3mm] l_{-\infty} = min, \ l_{-1} = L^*, \ l_0 = G, \ l_1 = L, \ l_{+\infty} = max. \end{cases}$$

It is easy to see that $E_{p,q}$ is symmetric in p and q. Further, it is well known that $E_{p,q}$ is monotone increasing with respect to p and q. In particular the power means B_p, L_p, D_p, I_p are monotonic increasing with respect to p. Otherwise, $E_{p,q}$ is a strict mean for all real numbers p and q and so B_p, L_p, D_p, I_p are strict means for each real number p.

2. Background Material about Stabilizable Means

Recently, the author introduced [6] two new concepts, namely the stability and stabilizability notions for means as itemized in what follows.

Definition 2.1. Let m_1, m_2, m_3 be three given means. For $a, b > 0$, define

$$\mathcal{R}(m_1, m_2, m_3)(a,b) = m_1\Big(m_2\big(a, m_3(a,b)\big), m_2\big(m_3(a,b), b\big)\Big),$$

called the resultant mean-map of m_1, m_2 and m_3.

A detailed study of the properties of the resultant mean-map can be found in [6]. In particular, we recall the following result which will be needed later.

Proposition 2.1. *The map* $(a,b) \longmapsto \mathcal{R}(m_1, m_2, m_3)(a,b)$ *defines a mean, with the following properties:*
(i) For every means m_1, m_2, m_3 *we have*

$$\Big(\mathcal{R}(m_1, m_2, m_3)\Big)^* = \mathcal{R}\big(m_1^*, m_2^*, m_3^*\big).$$

(ii) The mean-map $\mathcal{R}$ *is point-wisely increasing with respect to each of its mean variables, that is,*

$$\Big(m_1 \leq m_1', \ m_2 \leq m_2', \ m_3 \leq m_3', \ \Big) \Longrightarrow \mathcal{R}(m_1, m_2, m_3) \leq \mathcal{R}(m_1', m_2', m_3').$$

The resultant mean-map stems its importance in the fact that it is an useful tool for introducing the following definition, [6].

Definition 2.2. A mean m is said to be:
- Stable if $\mathcal{R}(m, m, m) = m$.
- Stabilizable if there exist two nontrivial stable means m_1 and m_2 satisfying the relation $\mathcal{R}(m_1, m, m_2) = m$. We then say that m is (m_1, m_2)-stabilizable.

In [6] the author proved that if m is stable then so is m^* and, if m is (m_1, m_2)-stabilizable then m^* is (m_1^*, m_2^*)-stabilizable. He also studied the stability and stabilizability of the standard means H, G, L, I, A and that of the power means B_p, L_p, I_p, D_p, l_p. For the Stolarsky mean $E_{p,q}$ he putted the following open problem:
Problem. Determine the set of all couples (p,q) such that the Stolarsky mean $E_{p,q}$ of order (p,q) is stable or stabilizable.

The first fundamental goal of the present paper is to give a positive answer to this problem. We then establish that $E_{p,q}$ is (B_{q-p}, B_p)-stabilizable. In particular, we immediately obtain the stability of B_p and the stabilizability of L_p, D_p, I_p, l_p already differently discussed in [6].

In another part, in [5] the author introduced another concept for means as itemized in the following.

Definition 2.3. Let m_1 and m_2 be two means. The tensor product of m_1 and m_2 is the map, denoted $m_1 \otimes m_2$, defined by

$$\forall a, b, c, d > 0 \qquad m_1 \otimes m_2(a, b, c, d) = m_1\Big(m_2(a, b), m_2(c, d)\Big).$$

A binary mean m will be called cross mean if $m^{\otimes 2} := m \otimes m$ is a mean with four variables, that is,

$$\forall a, b, c, d > 0 \qquad m^{\otimes 2}(a, b, c, d) = m^{\otimes 2}(a, c, b, d).$$

As proved in [5], for all real number p, the power binomial mean B_p is a cross mean while L_p and D_p are not always cross means. We can also verify that I_p and l_p are not always cross means. In particular, A, G, H are cross means while L, I, L^*, I^* are not. Every cross mean is a stable mean and the reverse implication putted in [6] as an open problem still open.

The concept of cross mean has been used [5] for constructing some iterative algorithms involving the stable power binomial mean and converging, respectively, to the stabilizable power logarithmic and difference means. The following open problem has also been putted, [5]:

Problem. Is it possible to approximate $E_{p,q}$ by an iterative algorithm involving only the stable power binomial mean?

The second aim of this work is to give a positive answer to this question. We then state two adjacent iterative algorithms involving the stable power binomial mean and both converging to the Stolarsky mean. The algorithms of [5], converging to L_p and D_p, are here immediately deduced. We also obtain iterative algorithms converging to I_p and l_p from which we derive an interesting explicit formulae of l_p in terms of infinite products.

3. Stabilizability of the Stolarsky Mean

As already pointed, we wish to establish the stabilizability of the Stolarsky mean as recited in the following result.

Theorem 3.1. *For all real numbers p and q, the Stolarsky mean $E_{p,q}$ is (B_{q-p}, B_p)-stabilizable.*

Proof. According to the definition of the mean-map $\mathcal{R}$ and that of the stabilizability, we may show the following

$$(3.1) \quad \mathcal{R}\Big(B_{q-p}, E_{p,q}, B_p\Big)(a, b) = B_{q-p}\Big(E_{p,q}\big(a, B_p(a, b)\big), E_{p,q}\big(B_p(a, b), b\big)\Big) = E_{p,q}(a, b),$$

for all $a, b > 0$. By virtue of the explicit form of $E_{p,q}$ we have, for all $q \neq 0$, $p \neq 0$, $p \neq q$ and $a, b > 0$, $a \neq b$,

$$E_{p,q}(a, B_p(a, b)) = \left(\frac{p}{q} \frac{B_p^q(a, b) - a^q}{\frac{a^p + b^p}{2} - a^p}\right)^{1/(q-p)} = \left(\frac{2p}{q} \frac{B_p^q(a, b) - a^q}{b^p - a^p}\right)^{1/(q-p)}.$$

Similarly, we obtain

$$E_{p,q}(B_p(a, b), b) = \left(\frac{2p}{q} \frac{b^q - B_p^q(a, b)}{b^p - a^p}\right)^{1/(q-p)}.$$

Substituting these two latter expressions in the second side of (3.1), with the explicit form of B_{q-p}, we obtain the announced result for p, q such that $p, q \neq 0$, $p \neq q$. In the contrary

case, the desired result follows from the above with an argument of continuity. The proof of the theorem is complete. □

From the above theorem we immediately find the following result already stated by the author in [6].

Corollary 3.2. *For all real number p, the following statements are met:*
(i) The power binomial mean B_p is stable.
(ii) The power logarithmic mean L_p is (B_p, A)-stabilizable while the power difference mean D_p is (A, B_p)-stabilizable.
(iii) The power exponential mean I_p is (G, B_p)-stabilizable while the second power logarithmic mean l_p is (B_p, G)-stabilizable.

Proof. (i) Take $q = 2p$ in the above theorem, with $E_{p,2p} = B_p$.
(ii) Comes, respectively, from the fact that $L_p = E_{1,p+1}$ and $D_p = E_{p,p+1}$, with $B_1 = A$.
(iii) Since $I_p = E_{p,p}$ and $l_p = E_{0,p}$ with $B_0 = G$, we deduce, respectively, the announced result. □

4. ITERATIVE ALGORITHM CONVERGING TO $E_{p,q}$

As already pointed before, the fundamental goal of this section is to approximate the Stolarsky mean by iterative algorithm. We first state the following.

Proposition 4.1. *Let p and q be two real numbers such that $q \leq 2p$, then the following inequalities holds true*

$$B_{q-p} \leq \mathcal{R}\big(B_{q-p}, B_{q-p}, B_p\big) \leq E_{p,q} \leq \mathcal{R}\big(B_{q-p}, B_p, B_p\big) \leq B_p.$$

If $q \geq 2p$ then the above inequalities are reversed, with equalities for $q = 2p$.

Proof. If $q \leq 2p$ then, by the increase monotonicity of $E_{p,q}$ in p and q, we have $B_{q-p} \leq E_{p,q} \leq B_p$. This, with Proposition 2.1,(ii) and the fact that $E_{p,q}$ is (B_{q-p}, B_p)-stabilizable, yields the desired result. The rest of the corollary follows by the same arguments, so completes the proof. □

Inspired by the above proposition, we now are in position to construct iterative process for approaching the nonstable Stolarsky mean in terms of the stable power binomial mean. Precisely, for all positive real numbers a, b and all fixed real numbers p, q, define the following iterative algorithms.

$$(4.1) \qquad \begin{cases} \Lambda_{p,q}^{n+1}(a,b) = \mathcal{R}\Big(B_{q-p}, \Lambda_{p,q}^n, B_p\Big)(a,b) \\[2mm] \Lambda_{p,q}^0(a,b) = B_{q-p}(a,b) \end{cases}$$

$$(4.2) \qquad \begin{cases} V_{p,q}^{n+1}(a,b) = \mathcal{R}\Big(B_{q-p}, V_{p,q}^n, B_p\Big)(a,b) \\[2mm] V_{p,q}^0(a,b) = B_p(a,b). \end{cases}$$

By a mathematical induction it is easy to see that $\Lambda_{p,q}^n$ and $V_{p,q}^n$ are means for all $n \geq 0$. In what follows, we will study the convergence of the above algorithms. We start with the next result giving a link between the two sequences $\big(\Lambda_{p,q}^n(a,b)\big)_n$ and $\big(V_{p,q}^n(a,b)\big)_n$.

Proposition 4.2. *With the above, the sequences $\left(\Lambda_{p,q}^n(a,b)\right)_n$ and $\left(V_{p,q}^n(a,b)\right)_n$ satisfy the following relationship*

$$\Lambda_{p,q}^{n+1}(a,b) = B_{q-p}\Big(\Lambda_{p,q}^n(a,b), V_{p,q}^n(a,b)\Big)$$

for all $a,b > 0$ and every $n \geq 0$.

Proof. For $n = 0$, relations (4.1) give

$$\Lambda_{p,q}^1(a,b) = B_{q-p}\Big(B_{q-p}\big(a, B_p(a,b)\big), B_{q-p}\big(B_p(a,b), b\big)\Big).$$

This, with the fact that B_p is a cross mean for all real number p, yields

$$\Lambda_{p,q}^1(a,b) = B_{q-p}\Big(B_{q-p}(a,b), B_{q-p}\big(B_p(a,b), B_p(a,b)\big)\Big) = B_{q-p}\Big(B_{q-p}(a,b), B_p(a,b)\Big).$$

This, with (4.1) and (4.2), gives

$$\Lambda_{p,q}^1(a,b) = B_{q-p}\Big(\Lambda_{p,q}^0(a,b), V_{p,q}^0(a,b)\Big).$$

By a mathematical induction, the desired result follows with the same arguments as previous, so completes the proof. □

Proposition 4.3. *Assume that $q \leq 2p$. Then, for all $a,b > 0$ and every $n \geq 0$, we have*

$$(4.3) \quad B_{q-p}(a,b) \leq ... \leq \Lambda_{p,q}^{n-1}(a,b) \leq \Lambda_{p,q}^n(a,b) \leq V_{p,q}^n(a,b) \leq V_{p,q}^{n-1}(a,b) \leq ... \leq B_p(a,b).$$

If $q \geq 2p$ then the above inequalities are reversed with equalities if $q = 2p$.

Proof. Assume that $q \leq 2p$. Since $p \longmapsto B_p(a,b)$ is increasing then $B_{q-p}(a,b) \leq B_p(a,b)$ and thus $\Lambda_{p,q}^0(a,b) \leq V_{p,q}^0(a,b)$ for all $a,b > 0$. By a mathematical induction, with (4.1) and (4.2), we easily prove that,

$$\Lambda_{p,q}^n(a,b) \leq V_{p,q}^n(a,b),$$

for all $a,b > 0$ and every $n \geq 0$. According to the above proposition and the monotonicity axiom of B_{q-p}, one has

$$\Lambda_{p,q}^{n+1}(a,b) \geq B_{q-p}\Big(\Lambda_{p,q}^n(a,b), \Lambda_{p,q}^n(a,b)\Big) = \Lambda_{p,q}^n(a,b),$$

for each $n \geq 0$. It follows that, the sequence $\left(\Lambda_{p,q}^n(a,b)\right)_n$ is monotone increasing. Now, let us show the decrease monotonicity of $\left(V_{p,n}(a,b)\right)_n$. By (4.2) we obtain

$$V_{p,q}^1(a,b) = B_{q-p}\Big(B_p\big(a, B_p(a,b)\big), B_p\big(B_p(a,b), b\big)\Big),$$

which, with $B_{q-p}(a,b) \leq B_p(a,b)$, becomes

$$V_{p,q}^1(a,b) \leq B_p\Big(B_p\big(a, B_p(a,b)\big), B_p\big(B_p(a,b), b\big)\Big).$$

This, with the fact that B_p is a cross mean for all p, yields

$$V_{p,q}^1(a,b) \leq B_p\Big(B_p(a,b), B_p\big(B_p(a,b), B_p(a,b)\big)\Big) = B_p(a,b) = V_{p,q}^0(a,b),$$

for all $a,b > 0$. A simple mathematical induction, gives the decrease monotonicity of $\left(V_{p,q}^n(a,b)\right)_n$ and the proof of inequalities (4.3) is complete. If $q \geq 2p$, all inequalities in the above are reversed and remain equalities for $q = 2p$, so completes the proof. □

Theorem 4.4. *The sequences $\left(\Lambda_{p,q}^n(a,b)\right)_n$ and $\left(V_{p,q}^n(a,b)\right)_n$ both converge to the same limit $E_{p,q}(a,b)$, Stolarsky mean of a and b, with the following estimations,*

$$(4.4) \qquad \forall n \geq 0 \qquad B_{q-p}(a,b) \leq ... \leq \Lambda_{p,q}^n(a,b) \leq E_{p,q}(a,b) \leq V_{p,q}^n(a,b) \leq ... \leq B_p(a,b)$$

if $q \leq 2p$, with reversed inequalities if $q \geq 2p$ and equalities if $q = 2p$.

Proof. Assume that $q \leq 2p$. By virtue of the above proposition, the sequences $\left(\Lambda_{p,q}^n(a,b)\right)_n$ and $\left(V_{p,q}^n(a,b)\right)_n$ are monotone and bounded and so they converge. Calling $m_{p,q}(a,b)$ and $M_{p,q}(a,b)$ their limits, respectively, we deduce from Proposition 4.2, with an argument of continuity, that

$$m_{p,q}(a,b) = B_{q-p}\Big(m_{p,q}(a,b), M_{p,q}(a,b)\Big).$$

This, with the fact B_{q-p} is a strict mean for all real numbers p,q, yields $m_{p,q}(a,b) = M_{p,q}(a,b)$, that is, $\left(\Lambda_{p,q}^n(a,b)\right)_n$ and $\left(V_{p,q}^n(a,b)\right)_n$ converge with the same limit. Let us prove that this common limit is exactly $E_{p,q}(a,b)$. It is sufficient to show that $E_{p,q}(a,b)$ is an intermediary mean between $\Lambda_{p,q}^n(a,b)$ and $V_{p,q}^n(a,b)$, for all $n \geq 0$, i.e.

$$(4.5) \qquad \Lambda_{p,q}^n(a,b) \leq E_{p,q}(a,b) \leq V_{p,q}^n(a,b)$$

for all $a,b > 0$ and every $n \geq 0$. We wish to establish (4.5) by a mathematical induction. Since the map $(p,q) \longmapsto E_{p,q}(a,b)$ is increasing in p and q then we easily deduce that (with $q \leq 2p$),

$$B_{q-p}(a,b) \leq E_{p,q}(a,b) \leq B_p(a,b),$$

and so

$$\Lambda_{p,q}^0(a,b) \leq E_{p,q}(a,b) \leq V_{p,q}^0(a,b),$$

for all $a,b > 0$. Assume that (4.5) is true for n. By the recursive schemes (4.1) and (4.2), with Proposition 2.1,(ii), we obtain

$$\Lambda_{p,q}^{n+1}(a,b) = \mathcal{R}\big(B_{q-p}, \Lambda_{p,q}^n, B_p\big)(a,b) \leq \mathcal{R}\big(B_{q-p}, E_{p,q}, B_p\big)(a,b)$$
$$\leq \mathcal{R}\big(B_{q-p}, V_{p,q}^n, B_p\big)(a,b) = V_{p,q}^{n+1}(a,b).$$

This, with the fact that $E_{p,q}$ is (B_{q-p}, B_p)-stabilizable, that is $E_{p,q} = \mathcal{R}\big(B_{q-p}, E_{p,q}, B_p\big)$, completes the proof of (4.5). Now, letting $n \to +\infty$ in (4.5) we obtain

$$m_{p,q}(a,b) \leq E_{p,q}(a,b) \leq M_{p,q}(a,b).$$

This, with the fact that $m_{p,q}(a,b) = M_{p,q}(a,b)$, yields the desired results for $q \leq 2p$. If $q \geq 2p$ the above inequalities are reversed and the proof of the theorem is complete. $\quad\square$

By virtue of the relationships $E_{1,p+1} = L_p$ and $E_{p,p+1} = D_p$, the above algorithms approaching $E_{p,q}(a,b)$ immediately give those converging, respectively, to $L_p(a,b)$ and $D_p(a,b)$, already stated by the author in [5]. We left to the reader the routine task for formulating the two adjacent algorithms converging to $I_p(a,b)$ power identric mean of a and b. However, we may state the following result which gives an interesting explicit formulae of $l_p(a,b)$ in terms of infinite products.

Corollary 4.5. *The sequences $\left(\Lambda_{0,p}^n(a,b)\right)_n$ and $\left(V_{0,p}^n(a,b)\right)_n$ both converge to the same limit $l_p(a,b)$ the second power logarithmic mean of a and b. Further the following formulae*

$$(4.6) \qquad l_p(a,b) = \prod_{n=1}^{\infty} B_p\Big(a^{1/2^n}, b^{1/2^n}\Big),$$

holds for all $a,b > 0$ and every real number p.

Proof. The first part of the corollary follows from the above theorem with the fact that $E_{0,p}(a,b) = l_p(a,b)$. Let us prove the second part. Since $\Lambda_{0,p}^n$ is a mean for all $n \geq 0$, the homogeneity axiom with (4.1) yield

$$\Lambda_{0,p}^{n+1}(a,b) = B_p(\sqrt{a}, \sqrt{b})\Lambda_{0,p}^n(\sqrt{a}, \sqrt{b})$$

for all $n \geq 0$, with similar recursive relation for $(V_{0,p}^n)_n$. By mathematical induction, with $\Lambda_{0,p}^0(a,b) = B_p(a,b)$ and $V_{0,p}^0(a,b) = \sqrt{ab}$, we easily deduce that

$$\Lambda_{0,p}^n(a,b) = B_p\left(a^{1/2^n}, b^{1/2^n}\right) \prod_{i=1}^n B_p\left(a^{1/2^i}, b^{1/2^i}\right)$$

and

$$V_{0,p}^n(a,b) = (ab)^{1/2^{n+1}} \prod_{i=1}^n B_p\left(a^{1/2^i}, b^{1/2^i}\right)$$

for every $n \geq 0$. This, when combined with the first part, gives the desired result so completes the proof. $\qquad\square$

Finally, we pay attention to formulae (4.6) which makes appear interesting information as itemizing in what follows:

First, (4.6) gives a simple recursive process for the approximation computation of $l_p(a,b)$ when the real numbers a, b and p are given. Moreover, at least for $p \in \mathbb{Q}$, (4.6) contains elementary operations (sum, product, root) relatively to the initial expression of $l_p(a,b)$ which contains logarithms.

Secondly, (4.6) stems another type of importance giving us an idea for extending $l_p(a,b)$ from two variables to three or more ones. We can immediately suggest that a reasonable analogue of $l_p(a,b)$ for k arguments is given by

$$(4.7) \qquad l_p(a_1, a_2, ..., a_k) = \prod_{n=1}^\infty B_p\left(a_1^{1/2^n}, a_2^{1/2^n}, ..., a_k^{1/2^n}\right),$$

where

$$B_p(a_1, a_2, ..., a_k) = \left(\frac{\sum_{i=1}^k a_i^p}{k}\right)^{1/p}$$

is the power binomial mean with k variables. In particular, taking $p = 1$ in (4.7) we can propose as logarithmic mean of k arguments $a_1, a_2, ..., a_k$ the next expansion

$$(4.8) \qquad L(a_1, a_2, ..., a_k) = \prod_{n=1}^\infty A\left(a_1^{1/2^n}, a_2^{1/2^n}, ..., a_k^{1/2^n}\right),$$

where $A(a_1, a_2, ..., a_k)$ is the arithmetic mean of $a_1, a_2, ..., a_k$. The question to compare (4.8) to some other definitions of the logarithmic mean with several arguments, as that given in [1, 2, 3, 4], is not obvious and appears to be interesting. For more details about this latter point, we indicate the recent paper [7].

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