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ON SOME NEW INEQUALITIES FOR DIFFERENTIABLE CO-ORDINATED CONVEX FUNCTIONS

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Abstract. Several new inequalities for differentiabale co-ordinated convex and concave functions in two variables which are related to the left side of Hermite-Hadamard type inequality for co-ordinated convex functions in two variables are obtained.

1. Introduction

The following definition is well known in literature:

A function $f : I \to \mathbb{R}$, $\emptyset \neq I \subseteq \mathbb{R}$, is said to be convex on I if the inequality

$$f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y),$$

holds for all x, $y \in I$ and $[0, 1]$.

Many important inequalities have been established for the class of convex functions but the most famous is the Hermite-Hadamard's inequality (see for instance [12]). This double inequality is stated as:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}, \tag{1.1}$$

where $f : I \to \mathbb{R}$, $\emptyset \neq I \subseteq \mathbb{R}$ a convex function, a, $b \in I$ with $a < b$. The inequalities in (1.1) are in reversed order if f a concave function.

The inequalities (1.1) have become an important cornerstone in mathematical analysis and optimization and many uses of these inequalities have been discovered in a variety of settings. Moreover, many inequalities of special means can be obtained for a particular choice of the function f. Due to the rich geometrical significance of Hermite-Hadamard's inequality (1.1), there is growing literature providing its new proofs, extensions, refinements and generalizations, see for example [2, 4, 7, 11] and the references therein.

Let us consider now a bidimensional interval $\Delta =: [a, b] \times [c, d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$, a mapping $f : \Delta \to \mathbb{R}$ is said to be convex on Δ if the inequality

$$f(\lambda x + (1-\alpha)z, \lambda y + (1-\alpha)w) \leq \lambda f(x, y) + (1-\lambda) f(z, w),$$

holds for all $(x, y), (z, w) \in \Delta$ and $\lambda \in [0, 1]$.

A modification for convex functions on Δ, which are also known as co-ordinated convex functions, was introduced by S. S. Dragomir [5, 6] as follow:

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A function $f:\Delta\to\mathbb{R}$ is said to be convex on the co-ordinates on Δ if the partial mappings $f_y:[a,b]\to\mathbb{R}, f_y(u)=f(u,y)$ and $f_x:[c,d]\to\mathbb{R}, f_x(v)=f(x,v)$ are convex where defined for all $x\in[a,b], y\in[c,d]$.

A formal definition for co-ordinated convex functions may be stated as follow:

Definition 1. *A function $f:\Delta\to\mathbb{R}$ is said to be convex on the co-ordinates on Δ if the inequality*

$$\begin{aligned}&f(tx+(1-t)y,su+(1-s)w)\\&\leq tsf(x,u)+t(1-s)f(x,w)+s(1-t)f(y,u)+(1-t)(1-s)f(y,w),\end{aligned}$$

holds for all $t,s\in[0,1]$ and $(x,y),(u,w)\in\Delta$.

Clearly, every convex mapping $f:\Delta\to\mathbb{R}$ is convex on the co-ordinates. Furthermore, there exists co-ordinated convex function which is not convex, (see for example [5, 6]). For recent results on co-ordinated convex functions we refer the interested reader to [1, 3, 5, 8, 9, 10, 13].

The following Hermite-Hadamrd type inequality for co-ordinated convex functions on the rectangle from the plane $\mathbb{R}^2$ was also proved in [5]:

Theorem 1. *Suppose that $f:\Delta\to\mathbb{R}$ is co-ordinated convex on Δ. Then one has the inequalities:*

$$\begin{aligned}&f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\&\leq\frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx+\frac{1}{d-c}\int_a^d f\left(\frac{a+b}{2},y\right)dy\right]\\&\leq\frac{1}{(b-a)(d-c)}\int_a^b\int_a^d f(x,y)\,dydx\\&\leq\frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x,c)\,dx+\frac{1}{b-a}\int_a^b f(x,d)\,dx\right.\\&\left.+\frac{1}{d-c}\int_c^d f(a,y)\,dy+\frac{1}{d-c}\int_c^d f(b,y)\,dy\right]\\&\leq\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}.\end{aligned}\tag{1.2}$$

The above inequalities are sharp.

In a recent paper [13], M.Z. Sarikaya et al. proved some new inequalities that give estimate of the difference between the middle and the rightmost terms in (1.2) for differentiable co-ordinated convex functions on rectangle from the plane $\mathbb{R}^2$. Motivated by notion given in [13], in the present paper, we prove some new inequalities which give estimate between the middle and the leftmost terms in (1.2) for differentiable co-ordinated convex functions on rectangle from the plane $\mathbb{R}^2$.

2. Main Results

The following lemma is necessary and plays an important role in establishing our main results:

Lemma 1. *Let $f:\Delta\subseteq\mathbb{R}^2\to\mathbb{R}$ be a partial differentiable mapping on $\Delta:=[a,b]\times[c,d]$ with $a<b$, $c<d$. If $\frac{\partial^2 f}{\partial s\partial t}\in L(\Delta)$, then the following identity holds:*

$$\begin{aligned}
&\frac{1}{(b-a)(d-c)}\int_a^b\int_a^d f(x,y)\,dydx+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\
&-\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx-\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy\\
&=(b-a)(d-c)\int_0^1\int_0^1 K(t,s)\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b,sc+(1-s)d)\,dsdt, \qquad (2.1)
\end{aligned}$$

where

$$K(t,s)=\begin{cases} ts & ,(t,s)\in\left[0,\frac{1}{2}\right]\times\left[0,\frac{1}{2}\right]\\ t(s-1) & ,(t,s)\in\left[0,\frac{1}{2}\right]\times\left(\frac{1}{2},1\right]\\ s(t-1) & ,(t,s)\in\left(\frac{1}{2},1\right]\times\left[0,\frac{1}{2}\right]\\ (t-1)(s-1) & ,(t,s)\in\left(\frac{1}{2},1\right]\times\left(\frac{1}{2},1\right]\end{cases}.$$

Proof. Since

$$\begin{aligned}
&(b-a)(d-c)\int_0^1\int_0^1 K(t,s)\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b,sc+(1-s)d)\,dsdt\\
&=(b-a)(d-c)\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} ts\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b,sc+(1-s)d)\,dsdt\\
&+(b-a)(d-c)\int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^1 t(s-1)\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b,sc+(1-s)d)\,dsdt\\
&+(b-a)(d-c)\int_{\frac{1}{2}}^1\int_0^{\frac{1}{2}} s(t-1)\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b,sc+(1-s)d)\,dsdt\\
&+(b-a)(d-c)\int_{\frac{1}{2}}^1\int_{\frac{1}{2}}^1 (t-1)(s-1)\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b,sc+(1-s)d)\,dsdt\\
&=I_1+I_2+I_3+I_4. \qquad (2.2)
\end{aligned}$$

Now by integration by parts, we have

$$\begin{aligned}
I_1&=(b-a)(d-c)\int_0^{\frac{1}{2}} t\left[\int_0^{\frac{1}{2}} s\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b,sc+(1-s)d)\,ds\right]dt\\
&=\frac{1}{4}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{1}{2}\int_0^{\frac{1}{2}} f\left(ta+(1-t)b,\frac{c+d}{2}\right)dt\\
&-\frac{1}{2}\int_0^{\frac{1}{2}} f\left(\frac{a+b}{2},sc+(1-s)d\right)ds+\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} f(ta+(1-t)b,sc+(1-s)d)\,dsdt. \qquad (2.3)
\end{aligned}$$

If we make use of the substitutions $x = ta + (1-t)\,b$ and $y = sc + (1-s)\,d$, $(t,s) \in [0,1]^2$, in (2.3), we observe that

$$I_1 = \frac{1}{4} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - \frac{1}{2\,(b-a)} \int_{\frac{a+b}{2}}^{b} f\left(x, \frac{c+d}{2}\right) dx$$
$$- \frac{1}{2\,(d-c)} \int_{\frac{c+d}{2}}^{d} f\left(\frac{a+b}{2}, y\right) dy + \frac{1}{(b-a)\,(d-c)} \int_{\frac{a+b}{2}}^{b} \int_{\frac{c+d}{2}}^{d} f\,(x,y)\,dydx.$$

Similarly, by integration by parts, we also have that

$$I_2 = \frac{1}{4} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - \frac{1}{2\,(b-a)} \int_{\frac{a+b}{2}}^{b} f\left(x, \frac{c+d}{2}\right) dx$$
$$- \frac{1}{2\,(d-c)} \int_{c}^{\frac{c+d}{2}} f\left(\frac{a+b}{2}, y\right) dy + \frac{1}{(b-a)\,(d-c)} \int_{\frac{a+b}{2}}^{b} \int_{c}^{\frac{c+d}{2}} f\,(x,y)\,dydx,$$

$$I_3 = \frac{1}{4} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - \frac{1}{2\,(b-a)} \int_{a}^{\frac{a+b}{2}} f\left(x, \frac{c+d}{2}\right) dx$$
$$- \frac{1}{2\,(d-c)} \int_{\frac{c+d}{2}}^{d} f\left(\frac{a+b}{2}, y\right) dy + \frac{1}{(b-a)\,(d-c)} \int_{a}^{\frac{a+b}{2}} \int_{\frac{c+d}{2}}^{d} f\,(x,y)\,dydx$$

and

$$I_4 = \frac{1}{4} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - \frac{1}{2\,(b-a)} \int_{a}^{\frac{a+b}{2}} f\left(x, \frac{c+d}{2}\right) dx$$
$$- \frac{1}{2\,(d-c)} \int_{c}^{\frac{c+d}{2}} f\left(\frac{a+b}{2}, y\right) dy + \frac{1}{(b-a)\,(d-c)} \int_{a}^{\frac{a+b}{2}} \int_{c}^{\frac{c+d}{2}} f\,(x,y)\,dydx.$$

Substitution of the I_1, I_2, I_3 and I_4 in (2.2) gives the desired identity (2.1). □

Theorem 2. *Let $f : \Delta \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a partial differentiable mapping on $\Delta := [a,b] \times [c,d]$ with $a < b$, $c < d$. If $\left|\frac{\partial^2 f}{\partial s \partial t}\right|$ is convex on the co-ordinates on Δ, then the following inequality holds:*

$$\left| \frac{1}{(b-a)\,(d-c)} \int_{a}^{b} \int_{a}^{d} f\,(x,y)\,dydx + f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - A \right|$$

(2.4)

$$\leq \frac{(b-a)\,(d-c)}{16} \left[\frac{\left|\frac{\partial^2}{\partial s \partial t} f\,(a,c)\right| + \left|\frac{\partial^2}{\partial s \partial t} f\,(a,d)\right| + \left|\frac{\partial^2}{\partial s \partial t} f\,(b,c)\right| + \left|\frac{\partial^2}{\partial s \partial t} f\,(b,d)\right|}{4} \right],$$

where

$$A = \frac{1}{b-a} \int_{a}^{b} f\left(x, \frac{c+d}{2}\right) dx + \frac{1}{d-c} \int_{c}^{d} f\left(\frac{a+b}{2}, y\right) dy.$$

Proof. From Lemma 1

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_a^d f(x,y)\,dydx + f\left(\frac{a+b}{2},\frac{c+d}{2}\right) - A\right|$$

(2.5)

$$\leq (b-a)(d-c)\int_0^1\int_0^1 |K(t,s)|\left|\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b, sc+(1-s)d)\right| dsdt$$

Since $\left|\frac{\partial^2 f}{\partial s\partial t}\right|$ is convex on the co-ordinates on Δ, we have

$$\left|\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b, sc+(1-s)d)\right| \leq ts\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right| + t(1-s)\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|$$

(2.6)

$$+s(1-t)\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right| + (1-t)(1-s)\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|.$$

Substitution of (2.6) in (2.5) gives the following inequality:

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_a^d f(x,y)\,dydx + f\left(\frac{a+b}{2},\frac{c+d}{2}\right) - A\right|$$

$$\leq (b-a)(d-c)\int_0^1\int_0^1 |K(t,s)|\left[\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right| ts + \left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right| t(1-s)\right.$$

$$\left.+\left|\frac{\partial^2}{\partial s\partial t}(b,c)\right| s(1-t) + \left|\frac{\partial^2}{\partial s\partial t}(b,d)\right|(1-t)(1-s)\right] dsdt = (b-a)(d-c)$$

$$\times\pi\left\{\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} ts\left[\left|\frac{\partial^2}{\partial s\partial t}(a,c)\right| ts + \left|\frac{\partial^2}{\partial s\partial t}(a,d)\right| t(1-s) + \left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right| s(1-t)\right.\right.$$

$$\left.+\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|(1-t)(1-s)\right] dsdt + \int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^1 t(1-s)\left[\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right| ts\right.$$

$$\left.+\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right| t(1-s) + \left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right| s(1-t) + \left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|(1-t)(1-s)\right] dsdt$$

$$+\int_{\frac{1}{2}}^1\int_0^{\frac{1}{2}} s(1-t)\left[\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right| ts + \left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right| t(1-s) + \left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right| s(1-t)\right.$$

$$\left.+\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|(1-t)(1-s)\right] dsdt + \int_{\frac{1}{2}}^1\int_{\frac{1}{2}}^1 (1-t)(1-s)\left[\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right| ts+\right.$$

(2.7)

$$\left.\left.\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right| t(1-s) + \left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right| s(1-t) + \left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|(1-t)(1-s)\right] dsdt\right\}$$

Evaluating each integral in (2.7) and simplifying, we get (2.4). Hence the proof of the theorem is complete. □

Theorem 3. *Let* $f:\Delta\subseteq\mathbb{R}^2\to\mathbb{R}$ *be a partial differentiable mapping on* $\Delta := [a,b]\times[c,d]$ *with* $a<b$, $c<d$. *If* $\left|\frac{\partial^2 f}{\partial s\partial t}\right|^q$ *is convex on the co-ordinates on* Δ *and*

p, $q>1$, $\frac{1}{p}+\frac{1}{q}=1$, then the following inequality holds:

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_a^d f(x,y)\,dydx+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-A\right|$$

(2.8)

$$\leq\frac{(b-a)(d-c)}{4(p+1)^{\frac{2}{p}}}\left[\frac{\left|\frac{\partial^2}{\partial s\partial t}(a,c)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}(a,d)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}(b,c)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}(b,d)\right|^q}{4}\right]^{\frac{1}{q}},$$

where A is as given in Theorem 2.

Proof. From Lemma 1, we have

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_a^d f(x,y)\,dydx+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-A\right|$$

(2.9)

$$\leq(b-a)(d-c)\int_0^1\int_0^1|K(t,s)|\left|\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b,sc+(1-s)d)\right|dsdt.$$

Now using the well-known Hölder inequality for double integrals, we obtain

$$\int_0^1\int_0^1|K(t,s)|\left|\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b,sc+(1-s)d)\right|dsdt$$

(2.10)

$$\leq\left(\int_0^1\int_0^1|K(t,s)|^p\,dsdt\right)^{\frac{1}{p}}\left(\int_0^1\int_0^1\left|\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b,sc+(1-s)d)\right|^q dsdt\right)^{\frac{1}{q}}.$$

Since $\left|\frac{\partial^2 f}{\partial s\partial t}\right|^q$ is convex on the co-ordinates on Δ, we have

$$\int_0^1\int_0^1\left|\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b,sc+(1-s)d)\right|^q dsdt$$
$$\leq\int_0^1\int_0^1\left[ts\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q+t(1-s)\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q\right.$$
$$\left.+s(1-t)\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q+(1-t)(1-s)\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q\right]dsdt$$

(2.11)

$$=\frac{\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q}{4}.$$

Also, we notice that

$$\begin{aligned}\int_0^1\int_0^1 |K(t,s)|^p\,dsdt &= \int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} t^p s^p\,dsdt + \int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^1 t^p(1-s)^p\,dsdt \\ &\quad + \int_{\frac{1}{2}}^1\int_0^{\frac{1}{2}} s^p(1-t)^p\,dsdt + \int_{\frac{1}{2}}^1\int_{\frac{1}{2}}^1 (1-t)^p(1-s)^p\,dsdt \\ &= \frac{4}{(p+1)^2}\left(\frac{1}{2}\right)^{2(p+1)}. \end{aligned} \tag{2.12}$$

Using (2.11) and (2.12) in (2.10), we obtain

$$\begin{aligned}&\int_0^1\int_0^1 |K(t,s)|\left|\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b, sc+(1-s)d)\right| dsdt \\ &\leq \frac{1}{4(p+1)^{\frac{2}{p}}}\left[\frac{\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q}{4}\right]^{\frac{1}{q}}.\end{aligned}$$

Utilizing the last inequality in (2.9) gives us (2.8). This completes the proof of the theorem. □

Now we state our next result in:

Theorem 4. *Let $f:\Delta\subseteq\mathbb{R}^2\to\mathbb{R}$ be a partial differentiable mapping on $\Delta := [a,b]\times[c,d]$ with $a<b$, $c<d$. If $\left|\frac{\partial^2 f}{\partial s\partial t}\right|^q$ is convex on the co-ordinates on Δ and $q\geq 1$, then the following inequality holds:*

$$\begin{aligned}&\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_a^d f(x,y)\,dydx + f\left(\frac{a+b}{2},\frac{c+d}{2}\right) - A\right| \\ &\leq \frac{(b-a)(d-c)}{16}\left[\frac{\left|\frac{\partial^2}{\partial s\partial t}(a,c)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}(a,d)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}(b,c)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}(b,d)\right|^q}{4}\right]^{\frac{1}{q}},\end{aligned} \tag{2.13}$$

where A is as given in Theorem 2.

Proof. By using Lemma 1, we have that the following inequality:

$$\begin{aligned}&\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_a^d f(x,y)\,dydx + f\left(\frac{a+b}{2},\frac{c+d}{2}\right) - A\right| \\ &\leq (b-a)(d-c)\int_0^1\int_0^1 |K(t,s)|\left|\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b, sc+(1-s)d)\right| dsdt.\end{aligned} \tag{2.14}$$

By the power mean inequality, we have

$$
\begin{aligned}
&\int_0^1\int_0^1 |K(t,s)| \left|\frac{\partial^2}{\partial s\partial t} f(ta+(1-t)b, sc+(1-s)d)\right| dsdt \\
&\le \left(\int_0^1\int_0^1 |K(t,s)|\, dsdt\right)^{1-\frac{1}{q}} \\
&\times \left(\int_0^1\int_0^1 |K(t,s)| \left|\frac{\partial^2}{\partial s\partial t} f(ta+(1-t)b, sc+(1-s)d)\right|^q dsdt\right)^{\frac{1}{q}} \\
&= \left(\frac{1}{16}\right)^{1-\frac{1}{q}} \left(\int_0^1\int_0^1 |K(t,s)| \left|\frac{\partial^2}{\partial s\partial t} f(ta+(1-t)b, sc+(1-s)d)\right|^q dsdt\right)^{\frac{1}{q}}. \qquad (2.15)
\end{aligned}
$$

Using the fact that $\left|\frac{\partial^2 f}{\partial s\partial t}\right|^q$ is convex on the co-ordinates on Δ, we get

$$
\begin{aligned}
&\left|\frac{\partial^2}{\partial s\partial t} f(ta+(1-t)b, sc+(1-s)d)\right|^q \\
&= ts\left|\frac{\partial^2}{\partial s\partial t} f(a,c)\right|^q + t(1-s)\left|\frac{\partial^2}{\partial s\partial t} f(a,d)\right|^q + s(1-t)\left|\frac{\partial^2}{\partial s\partial t} f(b,c)\right|^q \\
&+ (1-t)(1-s)\left|\frac{\partial^2}{\partial s\partial t} f(b,d)\right|^q
\end{aligned}
$$

and hence, we obtain

$$
\begin{aligned}
&\int_0^1\int_0^1 |K(t,s)| \left|\frac{\partial^2}{\partial s\partial t} f(ta+(1-t)b, sc+(1-s)d)\right|^q dsdt \\
&\le \int_0^1\int_0^1 |K(t,s)| \left[ts\left|\frac{\partial^2}{\partial s\partial t} f(a,c)\right|^q + t(1-s)\left|\frac{\partial^2}{\partial s\partial t} f(a,d)\right|^q\right. \\
&\left. +s(1-t)\left|\frac{\partial^2}{\partial s\partial t} f(b,c)\right|^q + (1-t)(1-s)\left|\frac{\partial^2}{\partial s\partial t} f(b,d)\right|^q\right] dsdt \\
&= \frac{1}{64}\left[\left|\frac{\partial^2}{\partial s\partial t} f(a,c)\right|^q + \left|\frac{\partial^2}{\partial s\partial t} f(a,d)\right|^q + \left|\frac{\partial^2}{\partial s\partial t} f(b,c)\right|^q + \left|\frac{\partial^2}{\partial s\partial t} f(b,d)\right|^q\right].
\end{aligned}
$$

Therefore (2.15) becomes

$$
\begin{aligned}
&\int_0^1\int_0^1 |K(t,s)| \left|\frac{\partial^2}{\partial s\partial t} f(ta+(1-t)b, sc+(1-s)d)\right| dsdt \\
&\le \frac{1}{16}\left[\frac{\left|\frac{\partial^2}{\partial s\partial t} f(a,c)\right|^q + \left|\frac{\partial^2}{\partial s\partial t} f(a,d)\right|^q + \left|\frac{\partial^2}{\partial s\partial t} f(b,c)\right|^q + \left|\frac{\partial^2}{\partial s\partial t} f(b,d)\right|^q}{4}\right]^{\frac{1}{q}} \qquad (2.16)
\end{aligned}
$$

Substitution of (2.16) in (2.14), we obtain (2.13). Hence the proof is complete. □

Remark 1. *Since $2^p > p+1$ if $p > 1$ and accordingly*

$$\frac{1}{4} < \frac{1}{2(p+1)^{\frac{1}{p}}}$$

and hence we have that the following inequality:

$$\frac{1}{16}<\frac{1}{4}\cdot\frac{1}{4}<\frac{1}{2\left(p+1\right)^{\frac{1}{p}}}\cdot\frac{1}{2\left(p+1\right)^{\frac{1}{p}}}=\frac{1}{4\left(p+1\right)^{\frac{2}{p}}},$$

and as a consequence we get an improvement of the constant in Theorem 3.

Following theorem is about concave functions on the co-ordinates on Δ:

Theorem 5. *Let $f:\Delta\subseteq\mathbb{R}^2\to\mathbb{R}$ be a partial differentiable mapping on $\Delta:=[a,b]\times[c,d]$ with $a<b$, $c<d$. If $\left|\frac{\partial^2 f}{\partial s\partial t}\right|^q$ is concave on the co-ordinates on Δ and $q\geq 1$, then we have the inequality:*

$$\left|\frac{1}{\left(b-a\right)\left(d-c\right)}\int_a^b\int_a^d f\left(x,y\right)dydx+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-A\right|$$
$$\leq\frac{\left(b-a\right)\left(d-c\right)}{64}\left[\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{a+2b}{3},\frac{c+2d}{3}\right)\right|+\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{a+2b}{3},\frac{2c+d}{3}\right)\right|\right.$$
$$(2.17)\qquad\left.\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{2a+b}{3},\frac{c+2d}{3}\right)\right|+\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{2a+b}{3},\frac{2c+d}{3}\right)\right|\right],$$

where A is as defined in Theorem 2.

Proof. By the concavity of $\left|\frac{\partial^2 f}{\partial s\partial t}\right|^q$ on the co-ordinates on Δ and power mean inequality, we note that the following inequality holds:

$$\left|\frac{\partial^2}{\partial s\partial t}f\left(\lambda x+\left(1-\lambda\right)y,v\right)\right|^q\geq\lambda\left|\frac{\partial^2}{\partial s\partial t}f\left(x,v\right)\right|^q+\left(1-\lambda\right)\left|\frac{\partial^2}{\partial s\partial t}f\left(y,v\right)\right|^q$$
$$\geq\left(\lambda\left|\frac{\partial^2}{\partial s\partial t}f\left(x,v\right)\right|+\left(1-\lambda\right)\left|\frac{\partial^2}{\partial s\partial t}f\left(y,v\right)\right|\right)^q,$$

for all x, $y\in[a,b]$, $\lambda\in[0,1]$ and for fixed $v\in[c,d]$.
Hence,

$$\left|\frac{\partial^2}{\partial s\partial t}f\left(\lambda x+\left(1-\lambda\right)y,v\right)\right|\geq\lambda\left|\frac{\partial^2}{\partial s\partial t}f\left(x,v\right)\right|+\left(1-\lambda\right)\left|\frac{\partial^2}{\partial s\partial t}f\left(y,v\right)\right|,$$

for all x, $y\in[a,b]$, $\lambda\in[0,1]$ and for fixed $v\in[c,d]$.
Similarly, we can show that

$$\left|\frac{\partial^2}{\partial s\partial t}f\left(u,\lambda z+\left(1-\lambda\right)w\right)\right|\geq\lambda\left|\frac{\partial^2}{\partial s\partial t}f\left(u,z\right)\right|+\left(1-\lambda\right)\left|\frac{\partial^2}{\partial s\partial t}f\left(u,w\right)\right|,$$

for all z, $w\in[c,d]$, $\lambda\in[0,1]$ and for fixed $u\in[a,d]$, thus $\left|\frac{\partial^2 f}{\partial s\partial t}\right|$ is concave on the co-ordinates on Δ.

It is clear from Lemma 1 that

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_a^d f(x,y)\,dydx+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-A\right|$$
$$\leq (b-a)(d-c)\int_0^1\int_0^1 |K(t,s)|\left|\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b,sc+(1-s)d)\right|dsdt$$
$$=(b-a)(d-c)\left[\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} st\left|\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b,sc+(1-s)d)\right|dsdt\right.$$
$$+\int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^1 t(1-s)\left|\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b,sc+(1-s)d)\right|dsdt$$
$$+\int_{\frac{1}{2}}^1\int_0^{\frac{1}{2}} s(1-t)\left|\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b,sc+(1-s)d)\right|dsdt$$
$$\left.+\int_{\frac{1}{2}}^1\int_{\frac{1}{2}}^1 (1-t)(1-s)\left|\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b,sc+(1-s)d)\right|dsdt\right]. \tag{2.18}$$

Since $\left|\frac{\partial^2 f}{\partial s\partial t}\right|$ is concave on the co-ordinates, we have, by Jensen's inequality for integrals, that:

$$\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} st\left|\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b,sc+(1-s)d)\right|dsdt$$
$$=\int_0^{\frac{1}{2}} t\left[\int_0^{\frac{1}{2}} s\left|\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b,sc+(1-s)d)\right|ds\right]dt$$
$$\leq\int_0^{\frac{1}{2}} t\left(\int_0^{\frac{1}{2}} sds\right)\left|\frac{\partial^2}{\partial s\partial t}f\left(ta+(1-t)b,\frac{\int_0^{\frac{1}{2}} s(sc+(1-s)d)\,ds}{\int_0^{\frac{1}{2}} sds}\right)\right|dt$$
$$=\frac{1}{8}\int_0^{\frac{1}{2}} t\left|\frac{\partial^2}{\partial s\partial t}f\left(ta+(1-t)b,\frac{c+2d}{3}\right)\right|dt$$
$$\leq\frac{1}{8}\left(\int_0^{\frac{1}{2}} tdt\right)\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{\int_0^{\frac{1}{2}} t(ta+(1-t)b)\,dt}{\int_0^{\frac{1}{2}} tdt},\frac{c+2d}{3}\right)\right|$$
$$=\frac{1}{64}\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{a+2b}{3},\frac{c+2d}{3}\right)\right|. \tag{2.19}$$

In a similar way, we also have that

$$\int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^1 t(1-s)\left|\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b,sc+(1-s)d)\right|dsdt$$
$$\leq\frac{1}{64}\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{a+2b}{3},\frac{2c+d}{3}\right)\right|, \tag{2.20}$$

$$\int_{\frac{1}{2}}^{1}\int_{0}^{\frac{1}{2}} s(1-t)\left|\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b, sc+(1-s)d)\right| dsdt$$
$$\leq \frac{1}{64}\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{2a+b}{3},\frac{c+2d}{3}\right)\right| \tag{2.21}$$

and

$$\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1} (1-t)(1-s)\left|\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b, sc+(1-s)d)\right| dsdt$$
$$\leq \frac{1}{64}\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{2a+b}{3},\frac{2c+d}{3}\right)\right|. \tag{2.22}$$

By making use of (2.19)-(2.22) in (2.18), we get the desired result. This completes the proof. □

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