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# ON THE INEQUALITY $R_p < R$ OF THE PEDAL TRIANGLE

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Abstract. In this paper we give a simple proof of the pedal triangle inequality $R_p < R$ and prove a strengthened result. We also establish a refinement of this inequality. Some related conjectures checked by the computer are put forward.

## 1. Introduction and main results

Let $P$ be an arbitrary interior point of the $\triangle ABC$ and $D, E, F$ the feet of perpendiculars from $P$ to the sides $BC, CA, AB$ respectively. $S, R$ and $r$ are the area, circumradius and inradius of $\triangle ABC$ respectively, $S_p, R_p$ and $r_p$ denote the corresponding elements of pedal triangle $DEF$. Put $BC = a, CA = b, AB = c, PA = R_1, PB = R_2, PC = R_3, PD = r_1, PE = r_2, PF = r_3$.

It is well known that the following inequality holds between $S_p$ and $S$:

$$S_p \leq \frac{1}{4}S, \tag{1.1}$$

with equality if and only if $P$ coincide with the circumcenter $O$ of $\triangle ABC$. This inequality is a consequence of the following Gerngonne formula (see [1]):

$$S_p = \frac{1}{4}(1 - \frac{PO^2}{R^2})S, \tag{1.2}$$

which holds actually for arbitrary interior point $P$ of the circumcircle of $\triangle ABC$.

Of course, we can consider other inequalities between pedal triangle $DEF$ and triangle $ABC$. For instance, comparing the circumradius $R_p$ of pedal triangle $DEF$ and the circumradius $R$ of $\triangle ABC$, we can find the following conclusion:

**Theorem 1.** *For an arbitrary interior point $P$ of $\triangle ABC$, we have*

$$R_p < R. \tag{1.3}$$

In fact, the author has already pointed out the inequality (1.3) without proof in 1992 in [2]. But it seems that nobody has studied this inequality since then. In this paper we shall give a simple proof of inequality (1.3) and also prove the following stronger result:

**Theorem 2.** *For an arbitrary interior point $P$ of $\triangle ABC$, we have*

$$R_p + \frac{3\sqrt{3}r_1r_2r_3}{4S_p} \leq R, \tag{1.4}$$

*with equality if and only if $\triangle ABC$ is equilateral and $P$ is its center.*

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In my recent paper [3], I proved linear inequality:

$$R_1 + R_2 + R_3 - r_1 - r_2 - r_3 \geq 6r_p. \tag{1.5}$$

This inequality inspires the author to find the following refinement of (1.3):

**Theorem 3.** *For an arbitrary interior point $P$ of $\triangle ABC$, we have*

$$R_p < \frac{1}{2}(R_1 + R_2 + R_3 - r_1 - r_2 - r_3) < R. \tag{1.6}$$

## 2. Proofs of the theorems

### 2.1. Proof of Theorem 1

*Proof.* We shall show first that inequality

$$ar_2r_3 + br_3r_1 + cr_1r_2 \geq (r_2 + r_3)(r_3 + r_1)(r_1 + r_2) \tag{2.1}$$

holds for arbitrary interior point $P$ of $\triangle ABC$. To do so, we prove again the following weighted inequality:

$$(xa + yb + zc)(ayz + bzx + cxy) > 2(y + z)(z + x)(x + y)S. \tag{2.2}$$

where $x, y, z$ are positive real number $x, y, z$.

Since

$$\begin{aligned}
&(xa + yb + zc)(ayz + bzx + cxy) - 2(y + z)(z + x)(x + y)S \\
&= xyz(a^2 + b^2 + c^2) + a(bz + cy)x^2 + b(cx + az)y^2 + c(ay + bx)z^2 \\
&\quad -2\left[2xyz + x(y^2 + z^2) + y(z^2 + x^2) + z(x^2 + y^2)\right] S \\
&= xyz(a^2 + b^2 + c^2 - 4S) + (bc - 2S)x(y^2 + z^2) \\
&\quad +(ca - 2S)y(z^2 + x^2) + (ab - 2S)z(x^2 + y^2) > 0,
\end{aligned}$$

the claimed inequality (2.2) is proved.

In (2.2), we take $x = r_1, y = r_2, z = r_3$, then using the identity:

$$ar_1 + br_2 + cr_3 = 2S, \tag{2.3}$$

One get inequality (2.2). Form (2.2) and the triangle inequality $r_2 + r_3 \geq EF$ etc., we have

$$ar_2r_3 + br_3r_1 + cr_1r_2 > EF \cdot FD \cdot DE. \tag{2.4}$$

It is easy to get that

$$ar_2r_3 + br_3r_1 + cr_1r_2 = 4RS_p, \tag{2.5}$$

Thus we have

$$EF \cdot FD \cdot DE < 4RS_p. \tag{2.6}$$

Note that $EF \cdot FD \cdot DE = 4S_pR_p$. Thus inequality (1.3) follows from (2.6) immediately. The proof of Theorem 1 is completed. □

**Remark 2.1.** *The best inequality of the type $R_p < kR$ is $R_p < R$. To show this, we suppose that the following inequality holds for positive real numbers $x, y, z$:*

$$(xa + yb + zc)(ayz + bzx + cxy) > k(y + z)(z + x)(x + y)S. \tag{2.7}$$

*For $x = 0, y = z = 1$, we get $(b + c)a > 2kS$. Since $a(b + c) < 4S$, hence $k < 2$. This means the constant 2 in the right hand side of (2.2) is best possible. Therefore, the inequality $R_p < R$ is also best possible.*

**Remark 2.2.** *From the inequality $s > 2R$ of the acute-angled triangle and Theorem 1 we have*

$$R_p < \frac{1}{2}s. \tag{2.8}$$

*By using the method to prove $R_p < R$, we can easily prove that (2.8) holds for obtuse-angled $\triangle ABC$. From this and Theorem 1 we deduce inequality (2.7) holds for arbitrary $\triangle ABC$.*

### 2.2. Proof of Theorem 2

To prove Theorem 2, we need several lemmas.

**Lemma 1.** *Let $x, y, z$ be three real numbers such that $y+z > 0, z+x > 0, x+y > 0$ and $yz + zx + xy > 0$. Then the following inequality holds for all $\triangle ABC$:*

$$xa^2 + yb^2 + zc^2 \geq 4\sqrt{yz + zx + xy}S, \tag{2.9}$$

*with equality if and only if $x : y : z = (b^2+c^2-a^2) : (c^2+a^2-b^2) : (a^2+b^2-c^2)$.*

*Proof.* When real numbers $x, y, z$ satisfy $y + z > 0, z + x > 0, x + y > 0$ and $yz + zx + xy > 0$, we easily prove that $\sqrt{y+z}, \sqrt{z+x}, \sqrt{x+y}$ form a triangle $A_0B_0C_0$ with area $\frac{1}{2}\sqrt{yz+zx+xy}$. It is well known that the following famous Neuberg-Pedoe inequality holds (see [4]):

$$a^2(b'^2 + c'^2 - a'^2) + b^2(c'^2 + a'^2 - b'^2) + c^2(a'^2 + b'^2 - c'^2) \geq 16SS', \tag{2.10}$$

where $a', b', c'$ are the sides of $\triangle A'B'C'$ and $S'$ is its area (Equality in (2.10) holds if and only if two triangles are similar). If we employ this inequality for $\triangle A_0B_0C_0$ and $\triangle ABC$, we obtain the inequality (2.9) at once and easily know its equality condition. □

**Lemma 2.** *For arbitrary interior point of $\triangle ABC$, we have*

$$\frac{a}{r_1} + \frac{b}{r_2} + \frac{c}{r_3} - \frac{a}{R_1} - \frac{b}{R_2} - \frac{c}{R_3} \geq 3\sqrt{3}, \tag{2.11}$$

*with equality if and only if $\triangle ABC$ is equilateral and $P$ is its center.*

*Proof.* Since the area of the quadrilateral is less than or equal to the half of product of two diagonals, so we have

$$S_{\triangle PCA} + S_{\triangle PAB} \leq \frac{1}{2}aR_1,$$

Namely,

$$br_2 + cr_3 \leq aR_1, \tag{2.12}$$

and two similar relations are valid. Therefore, to prove (2.11) we need to prove that

$$\frac{a}{r_1} + \frac{b}{r_2} + \frac{c}{r_3} - \frac{a^2}{br_2 + cr_3} - \frac{b^2}{cr_3 + ar_1} - \frac{c^2}{ar_1 + br_2} \geq 3\sqrt{3}. \tag{2.13}$$

Now we shall show first that the equivalent weighted inequality:

$$(x+y+z)\left(\frac{a^2}{x} + \frac{b^2}{y} + \frac{c^2}{z} - \frac{a^2}{y+z} - \frac{b^2}{z+x} - \frac{c^2}{x+y}\right) \geq 6\sqrt{3}S, \tag{2.14}$$

or

$$\frac{y+z-x}{x(y+z)}a^2 + \frac{z+x-y}{y(z+x)}b^2 + \frac{x+y-z}{z(x+y)}c^2 \geq \frac{6\sqrt{3}}{x+y+z}S \tag{2.15}$$

holds for positive real numbers $x, y, z$.

Note that

$$\frac{z+x-y}{y(z+x)}+\frac{x+y-z}{z(x+y)}=\frac{x(y^2+z^2+xy+xz)}{yz(z+x)(x+y)}>0,$$

and

$$\frac{(z+x-y)(x+y-z)}{yz(z+x)(x+y)}+\frac{(x+y-z)(z+x-y)}{zx(x+y)(y+z)}$$
$$+\frac{(y+z-x)(z+x-y)}{xy(y+z)(z+x)}=\frac{2(x+y+z)}{(y+z)(z+x)(x+y)}>0.$$

Thus according to Lemma 1, to prove (2.15) we need only to prove that

$$4\sqrt{\frac{2(x+y+z)}{(y+z)(z+x)(x+y)}}\geq\frac{6\sqrt{3}}{x+y+z},$$

which is equivalent to

$$8(x+y+z)^3\geq 27(y+z)(z+x)(x+y).$$

This follows from AM-GM inequality obviously. Hence, inequalities (2.15), (2.14) are proved.

In (2.14), put $x=ar_1, y=br_2, z=cr_3$, then using identity (2.3) we obtain (19). Hence (2.11) is proved. It is easy to see that the equality conditions of (2.11) is just as mentions as in Lemma 2. This completes the proof of Lemma 2. □

**Lemma 3.** *For arbitrary interior point $P$, the following inequality holds:*

$$aR_2R_3+bR_3R_1+cR_1R_2\geq abc \tag{2.16}$$

*with equality if and only if $\triangle ABC$ is acute and $P$ is its orthocenter.*

Inequality (2.16) is due to T.Hayashi and holds for arbitrary point $P$ actually, see [4],[5].

**Lemma 4.** *If the following inequality holds for an arbitrary point $P$ of the plane of $\triangle ABC$:*

$$f(a,b,c,R_1,R_2,R_3,r_1,r_2,r_3)\geq 0, \tag{2.17}$$

*Then the inequality holds after making transformation $K$:*

$$(a,b,c,R_1,R_2,R_3,r_1,r_2,r_3)$$
$$\rightarrow\left(\frac{aR_1}{2r_2r_3R},\frac{bR_2}{2r_3r_1R},\frac{cR_3}{2r_1r_2R},\frac{1}{r_1},\frac{1}{r_2},\frac{1}{r_3},\frac{1}{R_1}.\frac{1}{R_1},\frac{1}{R_1}\right).$$

The $K$ transformation is called reciprocation transformation, see [6],[7],[2].

**Lemma 5.** *For arbitrary interior point $P$ of $\triangle ABC$, we have*

$$ar_1R_1^2+br_2R_2^2+cr_3R_3^2=8R^2S_p. \tag{2.18}$$

The identity (2.18) is very important and is equivalent to

$$ar_1R_1^2+br_2R_2^2+cr_3R_3^2=2R(ar_2r_3+br_3r_1+cr_1r_2), \tag{2.19}$$

which is given in [8] by M.S.Klamkin. The author [2] pointed out general identity:

$$\vec{S}_{\triangle PBC}PA^2+\vec{S}_{\triangle PCA}PB^2+\vec{S}_{\triangle PAB}PC^2=4R^2\vec{S}_{\triangle DEF} \tag{2.20}$$

holds for any point $P$ in the plane of $\triangle ABC$.

Next, we prove Theorem 2.

*Proof.* By Lemma 3, we have

$$\frac{a}{R_1}+\frac{b}{R_2}+\frac{c}{R_3}\geq\frac{abc}{R_1R_2R_3}. \tag{2.21}$$

From this and the inequality (2.11) of Lemma 2, we deduce that

$$\frac{a}{r_1}+\frac{b}{r_2}+\frac{c}{r_3}-\frac{abc}{R_1R_2R_3}\geq 3\sqrt{3}. \tag{2.22}$$

Applying Lemma 4 to (2.22), we obtain

$$\frac{aR_1^2}{2r_2r_3R}+\frac{bR_2^2}{2r_3r_1R}+\frac{cR_3^2}{2r_1r_2R}-\frac{abcR_1R_2R_3}{8r_1r_2r_3R^3}\geq 3\sqrt{3}, \tag{2.23}$$

Namely

$$\frac{ar_1R_1^2+br_2R_2^2+cr_3R_3^2}{2r_2r_3R}-\frac{R_1R_2R_3\sin A\sin B\sin C}{r_1r_2r_3}\geq 3\sqrt{3}.$$

Then using identity (2.18) and noticing that

$$R_1R_2R_3\sin A\sin B\sin C=4S_pR_p, \tag{2.24}$$

we get

$$\frac{4RS_p}{r_1r_2r_3}-\frac{4S_pR_p}{r_1r_2r_3}\geq 3\sqrt{3}.$$

which evidently implies inequality (1.4). It is easy to see that the equality in (1.4) holds if and only if $\triangle ABC$ is equilateral and $P$ is its center. This completes the proof of Theorem 2. □

### 2.3. Proof of Theorem 3

First, we prove the following lemma:

**Lemma 6.** *For any positive real numbers $x, y, z, u, v, w$, we have*

$$p_1u^2+p_2v^2+p_3w^2>q_1vw+q_2wu+q_3uv, \tag{2.25}$$

*where*

$$\begin{aligned}
p_1&=x^2(y+z)(y^2+xy+zx+z^2),\\
p_2&=y^2(z+x)(z^2+yz+yx+x^2),\\
p_3&=z^2(x+y)(x^2+zx+zy+y^2),\\
q_1&=yz[2x^3+2(y+z)x^2-(y^2+z^2)x-2yz(y+z)],\\
q_2&=zx[2y^3+2(z+x)y^2-(z^2+x^2)y-2zx(z+x)],\\
q_3&=xy[2z^3+2(x+y)z^2-(x^2+y^2)z-2xy(x+y)].
\end{aligned}$$

In the sequel we will apply the method of the Difference Substitution (see [9]-[11]) to prove inequality (2.25).

*Proof.* Putting

$$Q=p_1u^2+p_2v^2+p_3w^2-(q_1vw+q_2wu+q_3uv). \tag{2.26}$$

Because of the symmetry in (2.25) without loss of generality suppose that $u\geq v\geq w>0$. Also, we put

$$\begin{cases} v=w+p, & (p\geq 0)\\ u=w+p+q & (q\geq 0).\end{cases} \tag{2.27}$$

Under these hypotheses, we shall divide our argument into the following six cases:

**Case 1.** The positive real numbers $x, y, z$ satisfy $x \geq y \geq z$.

In this case, we set

$$\begin{cases} y = z + m, & (m \geq 0) \\ x = z + m + n & (n \geq 0). \end{cases} \tag{2.28}$$

Plugging (2.27) and (2.28) into the expression of $Q$, with the help of Maple software we obtain easily the following identity:

$$\begin{aligned} Q \;=\;& (16wq + 8q^2 + 32wp + 24w^2 + 16p^2 + 16pq)z^5 + (36np^2 + 40nw^2 \\ & +66mwq + 52nwq + 72mp^2 + 132mwp + 30mq^2 + 72mpq + 80mw^2 \\ & +20nq^2 + 66nwp + 46npq)z^4 + (56mnq^2 + 226m^2wp + 154mnwq \\ & +128mnp^2 + 50n^2wq + 114mnw^2 + 150mnpq + 128m^2p^2 + 16n^2q^2 \\ & +114m^2w^2 + 59n^2wp + 128m^2pq + 41n^2pq + 45m^2q^2 + 226mnwp \\ & +30n^2p^2 + 113m^2wq + 34n^2w^2)z^3 + (72n^2mp^2 + 100m^3wq \\ & +112m^3pq + 11n^3pq + 138m^2nw^2 + 4n^3q^2 + 34m^3q^2 + 8n^3p^2 \\ & +112m^3p^2 + 300m^2nwp + 59m^2nq^2 + 10n^3w^2 + 184m^2npq \\ & +92m^3w^2 + 134n^2mwp + 17n^3wp + 14n^3wq + 168m^2np^2 \\ & +179m^2nwq + 88n^2mpq + 30n^2mq^2 + 66n^2mw^2 + 200m^3wp \\ & +96n^2mwq)z^2 + m(n + m)(42m^2w^2 + 13m^2q^2 + 90m^2wp \\ & +45m^2wq + 48m^2p^2 + 48m^2pq + 15mnq^2 + 42mnw^2 + 52mnwq \\ & +52mnpq + 90mnwp + 48mnp^2 + 10n^2w^2 + 11n^2p^2 + 13n^2pq \\ & +14n^2wq + 4n^2q^2 + 21n^2wp)z + m^2(2m + n)(n + m)^2(2p + 2w + q)^2. \end{aligned}$$

Since $u > 0, v > 0, w > 0, p \geq 0, q \geq 0, m \geq 0, n \geq 0$, hence we know $Q > 0$.

**Case 2.** The positive real numbers $x, y, z$ satisfy $x \geq z \geq y$.

We set

$$\begin{cases} z = y + m, & (m \geq 0) \\ x = y + m + n & (n \geq 0). \end{cases} \tag{2.29}$$

Plugging (2.27) and (2.29) into the expression of $Q$, we can get

$$\begin{aligned} Q \;=\;& (8q^2 + 16p^2 + 16wq + 32wp + 16pq + 24w^2)y^5 + (40nw^2 + 30mq^2 \\ & +20nq^2 + 44mp^2 + 52nwq + 94mwp + 36np^2 + 54mpq + 66nwp \\ & +66mwq + 80mw^2 + 46npq)y^4 + (114m^2w^2 + 46m^2p^2 + 45m^2q^2 \\ & +50n^2wq + 41n^2pq + 34n^2w^2 + 16n^2q^2 + 113m^2wq + 154mnwq \\ & +59n^2wp + 114mnw^2 + 156mnwp + 76mnp^2 + 116mnpq + 30n^2p^2 \\ & +115m^2wp + 56mnq^2 + 75m^2pq)y^3 + (42n^2mp^2 + 34m^3q^2 \\ & +92m^3w^2 + 56m^3pq + 14n^3wq + 179m^2nwq + 68n^2mpq \\ & +94n^2mwp + 8n^3p^2 + 96n^2mwq + 11n^3pq + 100m^3wq + 17n^3wp \\ & +66n^2mw^2 + 155m^2nwp + 138m^2nw^2 + 113m^2npq + 4n^3q^2 \\ & +10n^3w^2 + 30n^2mq^2 + 84m^3wp + 60m^2np^2 + 26m^3p^2 \\ & +59m^2nq^2)y^2 + m(m + n)(45m^2wq + 10m^2p^2 + 42m^2w^2 \\ & +39m^2wp + 23m^2pq + 13m^2q^2 + 15mnp^2 + 46mnwp + 15mnq^2 \\ & +30mnpq + 52mnwq + 42mnw^2 + 10n^2w^2 + 5n^2p^2 + 4n^2q^2 \\ & +9n^2pq + 14n^2wq + 13n^2wp)y + m^2(2m + n)(m + n)^2(2w + p + q)^2. \end{aligned}$$

Clearly, we also have $Q > 0$ in the second case.

**Case 3.** The positive real numbers $x, y, z$ satisfy $y \geq z \geq x$.
Letting

$$\begin{cases} z = x + m, & (m \geq 0) \\ y = x + m + n & (n \geq 0). \end{cases} \tag{2.30}$$

Plugging relations (2.27) and (2.30) into the expression of $Q$, we can get

$$Q = Q_1 + Q_2 \tag{2.31}$$

where

$$\begin{aligned} Q_1 &= (16p^2 + 16pq + 32pw + 8q^2 + 16qw + 24w^2)x^5 + (44mp^2 + 34mpq \\ &+94mpw + 20mq^2 + 28mqw + 80mw^2 + 36np^2 + 26npq + 66npw \\ &+10nq^2 + 14nqw + 40nw^2)x^4 + (46m^2p^2 + 17m^2pq + 115m^2pw \\ &+16m^2q^2 + 2m^2qw + 114m^2w^2 + 76mnp^2 + 36mnpq + 156mnpw \\ &+16mnq^2 + 2mnqw + 114mnw^2 + 30n^2p^2 + 19n^2pq + 59n^2pw \\ &+5n^2q^2 + 9n^2qw + 34n^2w^2)x^3, \\ Q_2 &= a_1x^2 + b_1x + c_1, \end{aligned}$$

and

$$\begin{aligned} a_1 &= 12(2p + 7w)m^3p + (60p + 7q + 155w)m^2np + 2(21p + 8q \\ &+47w)mn^2p + (8p^2 + 5pq + 17pw + q^2 + 3qw + 10w^2)n^3 \\ &+2[q^2 - 8qw + 46w^2 + 2(p - q)^2]m^3 + 6(q^2 - 4qw + 23w^2)m^2n \\ &+2(2q^2 - qw + 33w^2)mn^2 \\ b_1 &= m(m + n)(10m^2p^2 - 3m^2pq + 39m^2pw - 6m^2qw + 42m^2w^2 \\ &+15mnp^2 + 46mnpw - 6mnqw + 42mnw^2 + 5n^2p^2 + n^2pq \\ &+13n^2pw - n^2qw + 10n^2w^2), \\ c_1 &= m^2(2m + n)(m + n)^2(p + 2w)^2. \end{aligned}$$

Since $q^2 - 8qw + 46w^2 > 0, q^2 - 4qw + 23w^2 > 0$ and $2q^2 - qw + 33w^2 > 0$, thus we have $a_1 \geq 0$. Therefore, in order to prove $Q_2 > 0$ we need to prove that $D_1 = 4a_1c_1 - b_1^2 > 0$. After making the calculations we obtain

$$D_1 = m^2(m + n)^2(M_1 + M_2), \tag{2.32}$$

where

$$\begin{aligned} M_1 &= (2m + n)(54m + 7n)(m + n)^2p^4 + 2(2m + n)(7m^3q + 181m^3w \\ &+29m^2nq + 365m^2nw + 27mn^2q + 229mn^2w + 5n^3q + 33n^3w)p^3 \\ &+(23m^4q^2 + 98m^4qw + 1895m^4w^2 + 64m^3nq^2 + 480m^3nqw \\ &+4424m^3nw^2 + 62m^2n^2q^2 + 772m^2n^2qw + 3862m^2n^2w^2 \\ &+24mn^3q^2 + 430mn^3qw + 1404mn^3w^2 + 3n^4q^2 + 76n^4qw \\ &+171n^4w^2)p^2 + 2(46m^4q^2 + 40m^4qw + 1178m^4w^2 + 110m^3nq^2 \\ &+204m^3nqw + 2526m^3nw^2 + 115m^2n^2q^2 + 469m^2n^2qw \\ &+2036m^2n^2w^2 + 54mn^3q^2 + 322mn^3qw + 706mn^3w^2 + 9n^4q^2 \\ &+67n^4qw + 86n^4w^2)pw, \end{aligned}$$

$$\begin{aligned} M_2 &= (92q^2-8qw+1180w^2)m^4+8(23q^2-2qw+295w^2)m^3n \\ &+4(44q^2+65qw+429w^2)m^2n^2+4(21q^2+67qw+134w^2)mn^3 \\ &+(3q+10w)(5q+6w)n^4. \end{aligned}$$

Inequality $M_1>0$ is clearly true. Since $92q^2-8qw+1180w^2>0, 23q^2-2qw+295w^2>0$, we know $M_2>0$ and thus $D_1>0$. Hence inequality $Q_2>0$ and $Q>0$ are proved.

**Case 4.** The positive real numbers $x,y,z$ satisfy $y\geq x\geq z$.

In this case we put

$$\begin{cases} x=z+m, & (m\geq 0) \\ y=z+m+n & (n\geq 0). \end{cases} \tag{2.33}$$

Plugging (2.27) and (2.33) into the expression of $Q$, we now have the identity:

$$\begin{aligned} Q &= (16pq+16wq+32wp+24w^2+16p^2+8q^2)z^5+(72mpq+14nwq \\ &+40nw^2+26npq+72mp^2+80mw^2+66nwp+66mwq+132mwp \\ &+10nq^2+36np^2+30mq^2)z^4+(128m^2p^2+72mnwq+30n^2p^2 \\ &+9n^2wq+114m^2w^2+226m^2wp+19n^2pq+106mnpq+45m^2q^2 \\ &+114mnw^2+226mnwp+34n^2w^2+5n^2q^2+128m^2pq+113m^2wq \\ &+59n^2wp+128mnp^2+34mnq^2)z^3+(138m^2nw^2+17n^3wp \\ &+92m^3w^2+3n^3wq+38mn^2wq+n^3q^2+134mn^2wp+168m^2np^2 \\ &+56mn^2pq+34m^3q^2+112m^3p^2+152m^2npq+72mn^2p^2 \\ &+66mn^2w^2+8n^3p^2+10n^3w^2+14mn^2q^2+112m^3pq+43m^2nq^2 \\ &+121m^2nwq+5n^3pq+100m^3wq+200m^3wp+300m^2nwp)z^2 \\ &+m(n+m)(21n^2wp+9n^2pq+7n^2wq+10n^2w^2+11n^2p^2 \\ &+2n^2q^2+38mnwq+48mnp^2+42mnw^2+44mnpq+11mnq^2 \\ &+90mnwp+45m^2wq+42m^2w^2+90m^2wp+48m^2p^2+48m^2pq \\ &+13m^2q^2)z+m^2(n+2m)(n+m)^2(q+2p+2w)^2. \end{aligned}$$

Therefore, inequality $Q>0$ is valid.

**Case 5.** The positive real numbers $x,y,z$ satisfy $z\geq x\geq y$.

Put

$$\begin{cases} x=y+m, & (m\geq 0) \\ z=y+m+n & (n\geq 0). \end{cases} \tag{2.34}$$

Plugging (2.27) and(2.34) into the expression of $Q$ and making the calculations, we obtain

$$\begin{aligned} Q &= (16p^2+16pq+32pw+8q^2+16qw+24w^2)y^5+(44mp^2+54mpq \\ &+94mpw+30mq^2+66mqw+80mw^2+8np^2+8npq+28npw \\ &+10nq^2+14nqw+40nw^2)y^4+(46m^2p^2+75m^2pq+115m^2pw \\ &+45m^2q^2+113m^2qw+114m^2w^2+16mnp^2+34mnpq+74mnpw \\ &+34mnq^2+72mnqw+114mnw^2+18n^2pw+5n^2q^2+9n^2qw \\ &+34n^2w^2)y^3+(26m^3p^2+56m^3pq+84m^3pw+34m^3q^2 \\ &+100m^3qw+92m^3w^2+18m^2np^2+55m^2npq+97m^2npw \end{aligned}$$

$$+43m^2nq^2 + 121m^2nqw + 138m^2nw^2 + 10mn^2pq + 36mn^2pw$$
$$+14mn^2q^2 + 38mn^2qw + 66mn^2w^2 + 6n^3pw + n^3q^2 + 3n^3qw$$
$$+10n^3w^2)y^2 + (m+n)(10m^2p^2 + 23m^2pq + 39m^2pw + 13m^2q^2$$
$$+45m^2qw + 42m^2w^2 + 5mnp^2 + 16mnpq + 32mnpw + 11mnq^2$$
$$+38mnqw + 42mnw^2 + 2n^2pq + 6n^2pw + 2n^2q^2 + 7n^2qw$$
$$+10n^2w^2)ym + (2m+n)(m+n)^2(p+q+2w)^2m^2.$$

Inequality $Q > 0$ also holds obviously.

**Case 6.** The positive real numbers $x, y, z$ satisfy $z \geq y \geq x$.

We put

$$\begin{cases} y = x + m, & (m \geq 0) \\ z = x + m + n & (n \geq 0). \end{cases} \tag{2.35}$$

Plugging (2.27) and (2.35) into the expression of $Q$, then we have

$$Q = a_2q^2 + b_2q + c_2, \tag{2.36}$$

where

$$\begin{aligned}
a_2 &= x^2(2x + 2m + n)(4x^2 + 6xm + 2m^2 + 3xn + 2mn + n^2), \\
b_2 &= (16p + 16w)x^5 + (34mp + 28mw + 8np + 14nw)x^4 \\
&\quad +(17m^2p + 2m^2w - 2mnp + 2mnw + 9n^2w)x^3 \\
&\quad +(-4m^3p - 16m^3w - 19m^2np - 24m^2nw - 10mn^2p \\
&\quad -2mn^2w + 3n^3w)x^2 - m(m+n)(3m^2p + 6m^2w + 6mnp \\
&\quad +6mnw + 2n^2p + n^2w)x, \\
c_2 &= (16p^2 + 32pw + 24w^2)x^5 + (44mp^2 + 94mpw + 80mw^2 \\
&\quad +8np^2 + 28npw + 40nw^2)x^4 + (46m^2p^2 + 115m^2pw \\
&\quad +114m^2w^2 + 16mnp^2 + 74mnpw + 114mnw^2 + 18n^2pw \\
&\quad +34n^2w^2)x^3 + (26m^3p^2 + 84m^3pw + 92m^3w^2 \\
&\quad +18m^2np^2 + 97m^2npw + 138m^2nw^2 + 36mn^2pw + 66mn^2w^2 \\
&\quad +6n^3pw + 10n^3w^2)x^2 + (m+n)(10m^2p^2 + 39m^2pw \\
&\quad +42m^2w^2 + 5mnp^2 + 32mnpw + 42mnw^2 + 6n^2pw + 10n^2w^2)xm \\
&\quad +(2m+n)(m+n)^2(p+2w)^2m^2.
\end{aligned}$$

Noticing that $a_2 > 0, c_2 > 0$, to prove $Q > 0$ we need to prove $D_2 = 4a_2c_2 - b_2^2 > 0$. Using Maple Software, we easily obtain the following identity:

$$D_2 = (x+m)(x+m+n)x^2D_3, \tag{2.37}$$

where

$$\begin{aligned}
D_3 &= (512x^6 + 2560x^5m + 1280x^5n + 5104x^4m^2 + 5104x^4mn \\
&\quad +1404x^4n^2 + 5200x^3m^3 + 7800x^3m^2n + 4248x^3mn^2 \\
&\quad +824x^3n^3 + 2860x^2m^4 + 5720x^2m^3n + 4620x^2m^2n^2 \\
&\quad +1760x^2mn^3 + 251x^2n^4 + 808xm^5 + 2020xm^4n + 2144xm^3n^2 \\
&\quad +1196xm^2n^3 + 326xmn^4 + 31xn^5 + 92m^6 + 276m^5n + 360m^4n^2 \\
&\quad +260m^3n^3 + 99m^2n^4 + 15mn^5)w^2 + 2(2x+m)(128x^5 + 576x^4m \\
&\quad +240x^4n + 988x^3m^2 + 886x^3mn + 216x^3n^2 + 806x^2m^3
\end{aligned}$$

$$+1150x^2m^2n+597x^2mn^2+124x^2n^3+312xm^4+624xm^3n$$
$$+510xm^2n^2+221xmn^3+42xn^4+46m^5+120m^4n$$
$$+135m^3n^2+91m^2n^3+36mn^4+6n^5)wp$$
$$+(x+m)(2x+m)^2(64x^3+144x^2m+96x^2n+103xm^2$$
$$+144xmn+48xn^2+23m^3+51m^2n+36mn^2+8n^3)p^2.$$

Inequalities $D_2>0, D_3>0$ are valid. Therefore, $Q>0$ holds at the last case.

Combining the argumentations of the above six case, we deduce that inequality $Q>0$ holds for positive real numbers $x,y,z,u,v,w$. This completes the proof of Lemma 6. □

**Lemma 7.** [2] *For any $\triangle ABC$ and positive real numbers $x,y,z$, we have*

$$\frac{s-a}{x}+\frac{s-b}{y}+\frac{s-c}{z}\geq\frac{s(xa+yb+zc)}{yza+zxb+xyc}, \tag{2.38}$$

*where $s=(a+b+c)/2$. Equality holds if and only if $x=y=z$.*

Next, we prove Theorem 3.

*Proof.* We first prove the right hand inequality of (1.6):

$$\frac{1}{2}(R_1+R_2+R_3-r_1-r_2-r_3)<R. \tag{2.39}$$

According to obvious inequality:

$$R_1\leq\frac{r_2+r_3}{\sin A} \tag{2.40}$$

and two analogies, it is enough to show that

$$\frac{r_2+r_3}{\sin A}+\frac{r_3+r_1}{\sin B}+\frac{r_1+r_2}{\sin C}-r_1-r_2-r_3<2R,$$

Namely,

$$r_1\left(\frac{1}{\sin B}+\frac{1}{\sin C}-1\right)+r_2\left(\frac{1}{\sin C}+\frac{1}{\sin A}-1\right)$$
$$+r_3\left(\frac{1}{\sin A}+\frac{1}{\sin B}-1\right)<2R.$$

From this and previous identity (2.3), we need to prove that

$$\frac{1}{2R}\left(\frac{1}{\sin B}+\frac{1}{\sin C}-1\right)\leq\frac{a}{2S} \tag{2.41}$$

and two analogous inequalities. Because of symmetry, we only need to prove (2.41). Since

$$\frac{a}{2S}=\frac{1}{h_a}=\frac{1}{2R\sin B\sin C},$$

where $h_a$ is the altitude of side $BC$. Hence (2.41) is equivalent to trigonometry inequality:

$$\frac{1}{\sin B}+\frac{1}{\sin C}-1\leq\frac{1}{\sin B\sin C},$$

or

$$\sin B+\sin C-\sin B\sin C+1\geq 0. \tag{2.42}$$

It is easy to prove the following identity:

$$\sin B+\sin C-\sin B\sin C+1=\left(\cos\frac{A}{2}-\cos\frac{B-C}{2}\right)^2. \tag{2.43}$$

Therefore we see that inequality (2.42) holds true, hence inequality (2.39) is proved. As the equalities of (2.41) and its two analogies can not occur at the same time, so inequality (2.39) is strict. This competes the proof of (2.39).

Secondly, we prove the left hand inequality of (1.6):

$$R_p < \frac{1}{2}(R_1 + R_2 + R_3 - r_1 - r_2 - r_3). \tag{2.44}$$

To do so, we shall first prove the following weighted inequality with positive real numbers $x, y, z$:

$$\begin{aligned}&\frac{a}{x} + \frac{b}{y} + \frac{c}{z} - \left(\frac{a}{y+z} + \frac{b}{z+x} + \frac{c}{x+y}\right)\\ &> \frac{1}{s}\left[\frac{a(s-a)}{x} + \frac{b(s-b)}{y} + \frac{c(s-c)}{z}\right].\end{aligned} \tag{2.45}$$

where $s = (a+b+c)/2$. By using Maple software we get the identity:

$$\begin{aligned}&\frac{a}{x} + \frac{b}{y} + \frac{c}{z} - \left(\frac{a}{y+z} + \frac{b}{z+x} + \frac{c}{x+y}\right)\\ &-\frac{1}{s}\left[\frac{a(s-a)}{x} + \frac{b(s-b)}{y} + \frac{c(s-c)}{z}\right]\\ &= \frac{m_1a^2 + m_2b^2 + m_3c^2 - (n_1bc + n_2ca + n_3ab)}{xyz(y+z)(z+x)(x+y)(a+b+c)}.\end{aligned} \tag{2.46}$$

where

$$\begin{aligned}m_1 &= yz(z+x)(x+y)(2y+2z-x),\\ m_2 &= zx(x+y)(y+z)(2z+2x-y),\\ m_3 &= xy(y+z)(z+x)(2x+2y-z),\\ n_1 &= xyz(y+z)(y+z+2x),\\ n_2 &= xyz(z+x)(z+x+2y),\\ n_3 &= xyz(x+y)(x+y+2z).\end{aligned}$$

Therefore, to prove (2.45) we need to prove that

$$Q_0 \equiv m_1a^2 + m_2b^2 + m_3c^2 - (n_1bc + n_2ca + n_3ab) > 0. \tag{2.47}$$

Putting $s-a = u, s-b = v, s-c = w$, then $a = v+w, b = w+u, c = u+v$, and

$$\begin{aligned}Q_0 \;=\; & m_1(v+w)^2 + m_2(w+u)^2 + m_3(u+v)^2\\ & -[n_1(w+u)(u+v) + n_2(u+v)(v+w) + n_3(v+w)(w+u)].\end{aligned} \tag{2.48}$$

Plugging $m_1, m_2, m_3, n_1, n_2, n_3$ into (2.48), we can obtain

$$\frac{1}{2}Q_0 = Q = p_1u^2 + p_2v^2 + p_3w^2 - (q_1vw + q_2wu + q_3uv), \tag{2.49}$$

where $p_1, p_2, p_3, q_1, q_2, q_3$ are the same as in Lemma 6. From (2.49) and Lemma 6 we see $Q_0 > 0$, thus we finish the proof of inequality (2.45).

Making substitutions $x \to xa, y \to yb, z \to zc$ in (2.45), then we get

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} - \left(\frac{a}{yb+zc} + \frac{b}{zc+xa} + \frac{c}{xa+yb}\right) > \frac{1}{s}\left(\frac{s-a}{x} + \frac{s-b}{y} + \frac{s-c}{z}\right). \tag{2.50}$$

This and the inequality (2.38) of Lemma 7 imply that

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-\frac{a}{yb+zz}-\frac{b}{zc+xa}-\frac{c}{xa+yb}>\frac{xa+yb+zc}{yza+zxb+xyc}. \tag{2.51}$$

For $x=r_1, y=r_2, z=r_3$ in (2.51), then using identities (2.3),(2.5) and previous inequality (2.12) and so on, one has

$$\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}-\frac{1}{R_1}-\frac{1}{R_2}-\frac{1}{R_3}>\frac{S}{2RS_p}. \tag{2.52}$$

Finally, we make use of the $K$ transformation of Lemma 4 to inequality (2.52) and use the following relations under the $K$ transformation (see [3]):

$$S \to \frac{S}{2r_1r_2r_3R},\ R \to \frac{R_1R_2R_3}{4r_1r_2r_3R},\ S_p \to \frac{S}{2R_1R_2R_3R_p}.$$

After simplifying, we get

$$R_1+R_2+R_3-r_1-r_2-r_3>2R_p.$$

Hence the desired inequality (2.44) is proved. The proof of Theorem 3 comes to the end. □

## 3. **Some related conjectures**

Considering the stronger inequality of $R_p<R$, we first conjecture that the following linear inequality holds:

$$R_p+2r_p\le R. \tag{3.1}$$

where $r_p$ is the inradius of pedal triangle $DEF$. But, after verifying by the computer, we find this inequality is not true. However, the author thinks the following strict inequality (3.2) is valid.

**Conjecture 1.** *For any interior point $P$ of $\triangle ABC$, we have*

$$R_p+\sqrt{2}r_p<R. \tag{3.2}$$

On the other hand, we propose two similar conjectures:

**Conjecture 2.** *For any interior point $P$ of $\triangle ABC$, we have*

$$R_p+\frac{8r_p^2}{R}\le R. \tag{3.3}$$

**Conjecture 3.** *For any interior point $P$ of $\triangle ABC$, we have*

$$R_p+\frac{4r_p^2}{R_p}\le R. \tag{3.4}$$

We also find it is likely that the following inequality holds:

$$R_p^2+12r_p^2\ge R^2, \tag{3.5}$$

which is stronger than $R_p<R$. More generally, we propose the following exponent generalization:

**Conjecture 4.** *Let $k\ge 2$ be a real number, then we have*

$$R_p^k+2^k(2^k-1)r_p^k\le R^k. \tag{3.6}$$

By previous identity (2.24) and $S = 2R^2 \sin A \sin B \sin C$, we derive that inequality $R_p < R$ is equivalent to

$$\frac{R_1R_2R_3}{8R^3} < \frac{S_p}{S}. \tag{3.7}$$

This prompts the author to present the stronger inequality:

**Conjecture 5.** *For any interior point $P$ of $\triangle ABC$, we have*

$$\frac{8r_1r_2r_3 + R_1R_2R_3}{8R^3} \le \frac{S_p}{S}. \tag{3.8}$$

If (3.8) is valid, then one obtain by area inequality (1.1) that

$$8r_1r_2r_3 + R_1R_2R_3 \le 2R^3, \tag{3.9}$$

which is not proved yet.

For strict inequality (2.52), we pose the following strengthening conjecture:

**Conjecture 6.** *For any interior point $P$ of $\triangle ABC$, we have*

$$\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}-\frac{1}{R_1}-\frac{1}{R_2}-\frac{1}{R_3} \ge \left(1+\frac{S}{2S_p}\right)\frac{1}{R}. \tag{3.10}$$

If the inequality holds true, then it follows from (1.1) that

$$\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}-\frac{1}{R_1}-\frac{1}{R_2}-\frac{1}{R_3} \ge \frac{3}{R}. \tag{3.11}$$

The author can not prove this weaker inequality and believes it can be strengthened to the following:

**Conjecture 7.** *For any interior point $P$ of $\triangle ABC$, we have*

$$\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}-\frac{1}{R_1}-\frac{1}{R_2}-\frac{1}{R_3} \ge \frac{3}{2r}. \tag{3.12}$$

Euler's inequality $R \ge 2r$ shows that (3.12) is stronger than (3.11).

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