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HERMITE-HADAMARD INEQUALITY THROUGH PREQUASIINVEX FUNCTIONS

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Abstract. In this paper we extend some estimates of the right hand side of a Hermite- Hadamard type inequality for prequasiinvex functions. Then, a generalization to functions of several variables on invex subsets of $\mathbb{R}^n$ is introduced.

Keywords: Hermite-Hadamard inequality, invex sets, prequasiinvex functions

1. INTRODUCTION AND PRELIMINARY

Let $I = [c, d]$ be an interval on the real line $\mathbb{R}$, let $f : I \to \mathbb{R}$ be a convex function and let $a, b \in [c, d], a < b$. We consider the well-known Hadamard's inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2} \tag{1.1}$$

Both inequalities hold in the reversed direction if f is concave. We note that Hadamard's inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. Hadamard's inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been found (see, for example, [2, 3, 4, 5]).

The classical Hermite-Hadamard inequality provides estimates of the mean value of a continuous convex function $f : [a, b] \to \mathbb{R}$. Ion in [7] presented some estimates of the right hand side of a Hermite- Hadamard type inequality in which some quasi-convex functions are involved. The main results of [7] are given by the following theorems.

Theorem 1. *Assume $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ is a differentiable function on (a, b). If $|f'|$ is quasi-convex on $[a, b]$ then the following inequality holds*

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true

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{(b-a)\sup\{|f'(a)|,|f'(b)|\}}{4}. \tag{1.2}$$

Theorem 2. *Assume $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ is a differentiable function on (a, b). Assume $p \in \mathbb{R}$ with $p > 1$. If $|f'|^{\frac{p}{p-1}}$ is quasi-convex on $[a, b]$ then the following inequality holds true*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{b-a}{2(p+1)^{1/p}}\cdot\left[\sup\left\{|f'(a)|^{\frac{p}{p-1}},|f'(b)|^{\frac{p}{p-1}}\right\}\right]^{\frac{p-1}{p}}. \tag{1.3}$$

In recent years several extensions and generalizations have been considered for classical convexity. A significant generalization of convex functions is that of invex functions introduced by Hanson in [6]. Pini [8] introduced the concept of prequasiinvex functions as a generalization of invex functions. More recently, prequasiinvex functions, and semistrictly prequasiinvex functions were studied by Yang et al. [10].

Now, we recall some notions in invexity analysis which will be used throughout the paper (see [1, 9, 10] and references therein).

Definition 1. *A set $S \subseteq \mathbb{R}^n$ is said to be invex with respect to the map $\eta : S \times S \to \mathbb{R}^n$, if for every $x, y \in S$ and $t \in [0, 1]$,*

$$y + t\eta(x, y) \in S. \tag{1.4}$$

It is obvious that every convex set is invex with respect to the map $\eta(x, y) = x-y$, but there exist invex sets which are not convex (see [1]).

Definition 2. *Let $S \subseteq \mathbb{R}^n$ b e an invex set with respect to $\eta : S \times S \to \mathbb{R}^n$. Then, the function $f : S \to \mathbb{R}$ is said to be prequasiinvex with respect to η, if for every $x, y \in S$ and $t \in [0, 1]$,*

$$f(y + t\eta(x, y)) \le \max\{f(x), f(y)\}. \tag{1.5}$$

Every quasi convex function is a prequasiinvex with respect to the map $\eta(x, y) = x-y$ but the converse does not holds, see Example 1.1 in [10]. For a characterization and some application of prequasiinvex functions see [10] and references therein.

The organization of the paper is as follows:
In section 2 some generalizations of Hermite-Hadamard type inequality for prequasiinvex functions are given. Section 3 is devoted to a generalization to several variable prequasiinvex functions.

2. Prequasiinvexity and Hermite-Hadamard type inequality

Theorem 3. *Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\theta : A \times A \to \mathbb{R}$. Suppose that $f : A \to \mathbb{R}$ is a differentiable function. If $| f' |$ is prequasiinvex on A then, for every $a, b \in A$ the following inequality holds*

$$\begin{aligned}&\left|\frac{f(b)+f(b+\theta(a,b))}{2}-\frac{1}{\theta(a,b)}\int_b^{a+\theta(a,b))} f(x)dx\right| \\ &\le \frac{|\theta(a,b)|}{4}\sup\{|f'(a)|,|f'(b)|\}\end{aligned} \tag{2.1}$$

Proof. Suppose that $a, b \in A$. Since A is an invex set with respect to θ, for every $t \in [0,1]$ we have $b + t\theta(a,b) \in A$. Integrating by parts implies that

$$\begin{aligned} &\int_0^1 (1-2t) f'(b+t\theta(a,b))dt \\ &= \left[\frac{(1-2t)f(b+t\theta(a,b))}{\theta(a,b)}\right]_0^1 + \frac{2}{\theta(a,b)}\int_0^1 f(b+t\theta(a,b))dt \\ &= -\frac{f(b)+f(b+\theta(a,b))}{\theta(a,b)} + \frac{2}{(\theta(a,b))^2}\int_b^{b+\theta(a,b))} f(x)dx. \end{aligned} \tag{2.2}$$

By prequasiinvexity of $|f'|$ and (2.2) we get

$$\begin{aligned} &\left|\frac{f(b)+f(b+\theta(a,b))}{2} - \frac{1}{\theta(a,b)}\int_b^{b+\theta(a,b))} f(x)dx\right| \\ &= \left|\frac{\theta(a,b)}{2}\int_0^1 (1-2t)f'(b+t\theta(a,b))dt\right| \\ &\le \frac{|\theta(a,b)|}{2}\max\{|f'(a)|, |f'(b)|\}\int_0^1 |1-2t|\, dt \\ &= \frac{|\theta(a,b)|}{4}\max\{|f'(a)|, |f'(b)|\}. \end{aligned} \tag{2.3}$$

□

Theorem 4. *Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\theta : A \times A \to \mathbb{R}$. Suppose that $f : A \to \mathbb{R}$ is a differentiable function. Assume $p \in \mathbb{R}$ with $p > 1$. If $|f'|^{p/p-1}$ is prequasiinvex on A then, for every $a, b \in A$ the following inequality holds*

$$\begin{aligned} &\left|\frac{f(b)+f(b+\theta(a,b))}{2} - \frac{1}{\theta(a,b)}\int_b^{b+\theta(a,b))} f(x)dx\right| \\ &\le \frac{|\theta(a,b)|}{2(p+1)^{1/p}}\left[\sup\{|f'(a)|^{p/p-1}, |f'(b)|^{p/p-1}\}\right]^{p/p-1}. \end{aligned} \tag{2.4}$$

Proof. Suppose that $a, b \in A$. By assumption, Hölder's inequality and the proof of Theorem 3 we have

$$
\begin{aligned}
&\left| \frac{f(b) + f(b+\theta(a,b))}{2} - \frac{1}{\theta(a,b)} \int_b^{b+\theta(a,b))} f(x)dx \right| \\
&\leq \frac{|\theta(a,b)|}{2} \int_0^1 |1-2t| \, |f'(b+t\theta(a,b))| \, dt \\
&\leq \frac{|\theta(a,b)|}{2} \left(\int_0^1 |1-2t|^p \, dt \right)^{1/p} \left(\int_0^1 |f'(b+t\theta(a,b))|^q \, dt \right)^{1/q} \\
&= \frac{|\theta(a,b)|}{2(p+1)^{1/p}} \left(\int_0^1 |f'(b+t\theta(a,b))|^q \, dt \right)^{1/q} \\
&\leq \frac{|\theta(a,b)|}{2(p+1)^{1/p}} \left(\int_0^1 \sup\{|f'(a)|^q, |f'(a)|^q\} dt \right)^{1/q} \\
&= \frac{|\theta(a,b)|}{2(p+1)^{1/p}} \left(\sup\{|f'(a)|^q, |f'(a)|^q\} \right)^{1/q},
\end{aligned}
\tag{2.5}
$$

where $q := p/(p-1)$. □

Note that if $A = [a, b]$ and $\theta(x, y) = x - y$ for every $x, y \in A$ then, we can deduce Theorems 1 and 2, from Theorems 3 and 4, respectively.

In the following example we introduce a differentiable function which satisfies the assumptions of Theorems 3 and 4.

Example 1. Suppose that $K := (-2, 0) \cup (0, 2)$ and the function $\theta : K \times K \to \mathbb{R}$ is defined by

$$
\theta(x, y) := \begin{cases} x - y & x > 0, \ y > 0, \\ x - y & x < 0, \ y < 0, \\ 0 & \text{otherwise.} \end{cases}
$$

Clearly θ satisfies condition C and K is an open invex set with respect to θ. Assume that $p \in \mathbb{R}$ and $p > 1$. Suppose that the function $f : K \to \mathbb{R}$ defined by

$$
f(x) := (p/p-1)e^{(p-1/p)x}.
$$

Hence, for every $x \in K$ if $g(x) := |f'(x)|^{p/p-1}$ then, $g(x) = e^x$. It is obvious that for every pair of distinct points $x, y \in K$

$$
g'(x)\theta(y, x) > 0 \text{ implies } g'(y)\theta(x, y) < 0. \tag{2.6}
$$

Now, by (2.6) and Theorem 3.1 in [11] the function $g(x) = |f'(x)|^{p/p-1}$ is prequasi-invex with respect to θ.

3. An extension to several variables functions

The aim of this section is to extend the Theorem 4 and Proposition 1 in [7] to functions of several variables defined on invex subsets of $\mathbb{R}^n$.

Let $S \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : S \times S \to \mathbb{R}^n$. For every $x, y \in S$ the $\eta-$path P_{xv} joining the points x and $v := x + \eta(y, x)$ is defined as follows

$$
P_{xv} := \{z : z = x + t\eta(y, x) : t \in [0, 1]\}.
$$

The mapping η is said to be satisfies the condition C if for every $x, y \in S$ and $t \in [0,1]$,

$$\eta(y, y + t\eta(x,y)) = -t\eta(x,y),$$
$$\eta(x, y + t\eta(x,y)) = (1-t)\eta(x,y).$$

Note that for every $x, y \in S$ and every $t_1, t_2 \in [0,1]$ from condition C we have

$$\eta(y + t_2\eta(x,y), y + t_1\eta(x,y)) = (t_2 - t_1)\eta(x,y), \tag{3.1}$$

see [9] for details.

Proposition 1. *Let $S \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : S \times S \to \mathbb{R}^n$ and $f : S \to \mathbb{R}$ is a function. Suppose that η satisfies condition C on S. Then, for every $x, y \in S$ the function f is prequasiinvex with respect to η on $\eta-$path P_{xv} if and only if the function $\varphi : [0,1] \to \mathbb{R}$ defined by*

$$\varphi(t) := f(x + t\eta(y,x)),$$

is quasiconvex on $[0,1]$.

Proof. Suppose that φ is quasiconvex on $[0,1]$ and $z_1 := x + t_1\eta(y,x) \in P_{xv}, z_2 := x + t_2\eta(y,x) \in P_{xv}$. Fix $\lambda \in [0,1]$. By (2.4) we have

$$\begin{aligned} f(z_1 + \lambda\eta(z_2, z_1)) &= f(x + ((1-\lambda)t_1 + \lambda t_2)\eta(y,x)) \\ &= \varphi((1-\lambda)t_1 + \lambda t_2) \\ &\leq \max\{\varphi(t_1), \varphi(t_2)\} \\ &= \max\{f(z_1), f(z_2)\}. \end{aligned} \tag{3.2}$$

Hence, f is prequasiinvex with respect to η on $\eta-$path P_{xv}.

Conversely, let $x, y \in S$ and the function f be prequasiinvex with respect to η on $\eta-$path P_{xv}. Suppose that $t_1, t_2 \in [0,1]$. Then, for every $\lambda \in [0,1]$ we have

$$\begin{aligned} \varphi((1-\lambda)t_1 + \lambda t_2) &= f(x + ((1-\lambda)t_1 + \lambda t_2)\eta(y,x)) \\ &= f\left(x + t_1\eta(y,x) + \lambda\eta(x + t_2\eta(y,x), x + t_1\eta(y,x)\right) \\ &\leq \max\{f(x + t_1\eta(y,x)), f(x + t_1\eta(y,x))\} \\ &= \max\{\varphi(t_1), \varphi(t_2)\}. \end{aligned} \tag{3.3}$$

Therefore, φ is quasiconvex on $[0,1]$. □

The following Theorem is a generalization of Proposition 1 in [7].

Theorem 5. *Let $S \subseteq \mathbb{R}^n$ be an open invex set with respect to $\eta : S \times S \to \mathbb{R}^n$. Assume that η satisfies condition C. Suppose that for every $x, y \in S$ the function $f : S \to \mathbb{R}^+$ is prequasiinvex with respect to η on $\eta-$path P_{xv}. Then, for every $a, b \in (0,1)$ with $a < b$ the following inequality holds,*

$$\begin{aligned} &|\frac{1}{2}\int_0^a f(x + s\eta(y,x))ds + \frac{1}{2}\int_0^b f(x + s\eta(y,x))ds \\ &\qquad - \frac{1}{b-a}\int_a^b \left(\int_0^s f(x + t\eta(y,x))dt\right) ds| \\ &\qquad \leq \frac{b-a}{4}\sup\{f(x + a\eta(y,x)), f(x + b\eta(y,x))\}. \end{aligned} \tag{3.4}$$

Proof. Let $x, y \in S$ and $a, b \in (0,1)$ with $a < b$. Since f is prequasiinvex with respect to η on $\eta-$path P_{xv} by Proposition 1 the function $\varphi : [0,1] \to \mathbb{R}^+$ defined by

$$\varphi(t) := f(x + t\eta(y,x)),$$

is quasiconvex on $[0,1]$. Now, we define the function $\phi : [0,1] \to \mathbb{R}^+$ as follows

$$\phi(t) := \int_0^t \varphi(s)ds = \int_0^t f(x + s\eta(y,x))ds.$$

Obviously for every $t \in (0,1)$ we have

$$\phi'(t) = \varphi(t) = f(x + t\eta(y,x)) \geq 0,$$

hence, $|\phi'(t)| = \phi'(t)$. Applying Theorem 1 to the function ϕ implies that

$$\left|\frac{\phi(a) + \phi(b)}{2} - \frac{1}{b-a}\int_a^b \phi(s)ds\right| \leq \frac{b-a}{4} \sup\{\phi'(a), \phi'(b)\},$$

and we deduce that (3.4) holds. □

Remark 1. Let $\phi : [0,1] \to \mathbb{R}^+$ be a function and q a positive real number, then ϕ is quasiconvex if and only if the function $\phi^q : [0,1] \to \mathbb{R}^+$ is quasiconvex. Indeed, for every $x, y \in [0,1]$ it is easy to see that

$$(\max\{\phi(x), \phi(y)\})^q = \max\{\phi^q(x), \phi^q(y)\}. \tag{3.5}$$

Therefore if $t \in [0,1]$ we have

$\phi(tx+(1-t)y) \leq \max\{\phi(x), \phi(y)\}$ if and only if $\phi^q(tx+(1-t)y) \leq \max\{\phi^q(x), \phi^q(y)\}$.

The following theorem is a generalization of Theorem 4 to functions of several variables.

Theorem 6. *Let $S \subseteq \mathbb{R}^n$ be an open invex set with respect to $\eta : S \times S \to \mathbb{R}^n$. Assume that η satisfies condition C. Suppose that for every $x, y \in S$ the function $f : S \to \mathbb{R}^+$ is prequasiinvex with respect to η on $\eta-$path P_{xv}. Then, for every $p > 1$ and $a, b \in (0,1)$ with $a < b$ the following inequality holds,*

$$\begin{aligned} &|\frac{1}{2}\int_0^a f(x + s\eta(y,x))ds + \frac{1}{2}\int_0^b f(x + s\eta(y,x))ds \\ &- \frac{1}{b-a}\int_a^b \left(\int_0^s f(x + t\eta(y,x))dt\right) ds| \\ &\leq \frac{b-a}{2(p+1)^{1/p}} \sup\{[f(x + a\eta(y,x))]^{p/p-1}, [f(x + b\eta(y,x))]^{p/p-1}\}^{p/p-1}. \end{aligned} \tag{3.6}$$

Proof. Let $x, y \in S$ and $a, b \in (0,1)$ with $a < b$. Suppose that ϕ and φ are the functions which are defined in the proof of Theorem 5. Since $|\phi'(t)| : [0,1] \to \mathbb{R}^+$ is quasiconvex on $[0,1]$, by Remark 1 the function $|\phi'|^{p/p-1}$ is also quasiconvex on $[0,1]$.

Now, by applying Theorem 2 to function ϕ we get

$$\left|\frac{\phi(a) + \phi(b)}{2} - \frac{1}{b-a}\int_a^b \phi(s)ds\right| \leq \frac{b-a}{2(p+1)^{1/p}} \sup\{[\phi'(a)]^{p/p-1}, [\phi'(b)]^{p/p-1}\}^{p/p-1},$$

and we deduce that (3.6) holds and proof is complete. □

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