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NEW OSTROWSKI TYPE INEQUALITIES FOR CO-ORDINATED CONVEX FUNCTIONS

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Abstract. In this paper some new Owstroski type inequalites for co-ordinated convex functions are obtained.

1. Introduction

In 1938, A. Ostrowski proved the following interesting inequality [8]:

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) whose derivative $f':(a,b)\to\mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < \infty$. The we have the inequality*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\|f'\|_\infty, \qquad (1.1)$$

for all $x\in[a,b]$. The constant $\frac{1}{4}$ is the best possible.

The inequality (1.1) can be rewritten in equivalent form as:

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \left[\frac{(x-a)^2+(b-x)^2}{2(b-a)}\right]\|f'\|_\infty .$$

Since 1938 when A. Ostrowski proved his famous inequality, see [8], many mathematicians have been working about and around it, in many different directions and with a lot of applications in Numerical Analysis and Probability, etc.

Several generalizations of the Ostrowski integral inequality for mappings of bounded variation, Lipschitzian, monotonic, absolutely continuous, convex mappings and n-times differentiable mappings with error estimates for some special means and for some numerical quadrature rules are considered by many authors. For recent results and generalizations concerning Ostrowski's inequality see [1]-[13] and the references therein.

Let us consider now a bidimensional interval $\Delta =: [a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b$ and $c<d$, a mapping $f:\Delta\to\mathbb{R}$ is said to be convex on Δ if the inequality

$$f(\lambda x+(1-\alpha)z, \lambda y+(1-\alpha)w) \le \lambda f(x,y)+(1-\lambda)f(z,w),$$

holds for all $(x,y),(z,w)\in\Delta$ and $\lambda\in[0,1]$. The mapping f is said to be concave on the co-ordinates on Δ if the above inequality holds in reversed direction, for all $(x,y),(z,w)\in\Delta$ and $\lambda\in[0,1]$.

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A modification for convex (concave) functions on Δ, which are also known as co-ordinated convex (concave) functions, was introduced by S. S. Dragomir [7] as follows:

A function $f:\Delta\to\mathbb{R}$ is said to be convex (concave) on the co-ordinates on Δ if the partial mappings $f_y:[a,b]\to\mathbb{R}, f_y(u)=f(u,y)$ and $f_x:[c,d]\to\mathbb{R}, f_x(v)=f(x,v)$ are convex (concave) where defined for all $x\in[a,b], y\in[c,d]$.

A formal definition for co-ordinated convex (concave) functions may be stated in:

Definition 1. *A mapping $f:\Delta\to\mathbb{R}$ is said to be convex on the co-ordinates on Δ if the inequality*

$$\begin{aligned}&f(tx+(1-t)y,su+(1-s)w)\\&\leq tsf(x,u)+t(1-s)f(x,w)+s(1-t)f(y,u)+(1-t)(1-s)f(y,w),\end{aligned}\tag{1.2}$$

holds for all $t,s\in[0,1]$ and $(x,y),(u,w)\in\Delta$. The mapping f is concave on the co-ordinates on Δ if the inequality (1.2) holds in reversed direction for all $t,s\in[0,1]$ and $(x,y),(u,w)\in\Delta$.

Clearly, every convex (concave) mapping $f:\Delta\to\mathbb{R}$ is convex (concave) on the co-ordinates. Furthermore, there exists co-ordinated convex (concave) function which is not convex (concave), (see for instance [7]).

Here we also quote the following result form [7] to be used in the sequel of the paper:

Theorem 2. *Suppose that $f:\Delta\to\mathbb{R}$ is co-ordinated convex on Δ. Then one has the inequalities:*

$$\begin{aligned}&f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\&\leq\frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx+\frac{1}{d-c}\int_a^d f\left(\frac{a+b}{2},y\right)dy\right]\\&\leq\frac{1}{(b-a)(d-c)}\int_a^b\int_a^d f(x,y)\,dydx\\&\leq\frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x,c)\,dx+\frac{1}{b-a}\int_a^b f(x,d)\,dx\right.\\&\left.+\frac{1}{d-c}\int_c^d f(a,y)\,dy+\frac{1}{d-c}\int_c^d f(b,y)\,dy\right]\\&\leq\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}.\end{aligned}\tag{1.3}$$

The above inequalities are sharp. The inequalities in (1.3) hold in reverse direction if the mapping f is concave.

The main aim of this paper is to establish some new Ostrowski type inequalities for co-ordinated convex functions in Section 2.

2. Main Results

To establish our main results we need the following identity:

Lemma 1. *Let $f:\Delta\to\mathbb{R}$ be a twice partial differentiable mapping on Δ°. If $\frac{\partial^2 f}{\partial s\partial t}\in L(\Delta)$, then the following identity holds:*

$$\begin{aligned}
&f(x,y)+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(u,v)\,dvdu-A\\
&=\frac{(x-a)^2(y-c)^2}{(b-a)(d-c)}\int_0^1\int_0^1 st\frac{\partial^2}{\partial s\partial t}f(tx+(1-t)a,sy+(1-s)c)\,dsdt\\
&-\frac{(x-a)^2(d-y)^2}{(b-a)(d-c)}\int_0^1\int_0^1 st\frac{\partial^2}{\partial s\partial t}f(tx+(1-t)a,sy+(1-s)d)\,dsdt\\
&-\frac{(b-x)^2(y-c)^2}{(b-a)(d-c)}\int_0^1\int_0^1 st\frac{\partial^2}{\partial s\partial t}f(tx+(1-t)b,sy+(1-s)c)\,dsdt\\
&+\frac{(b-x)^2(d-y)^2}{(b-a)(d-c)}\int_0^1\int_0^1 st\frac{\partial^2}{\partial s\partial t}f(tx+(1-t)b,sy+(1-s)d)\,dsdt, \qquad (2.1)
\end{aligned}$$

for all $(x,y)\in\Delta$, where

$$A=\frac{1}{d-c}\int_c^d f(x,v)\,dv+\frac{1}{b-a}\int_a^b f(u,y)\,du.$$

Proof. Using integration by parts, we have

$$\begin{aligned}
&\frac{(x-a)^2(y-c)^2}{(b-a)(d-c)}\int_0^1\int_0^1 st\frac{\partial^2}{\partial s\partial t}f(tx+(1-t)a,sy+(1-s)c)\,dsdt\\
&=\frac{(x-a)^2(y-c)^2}{(b-a)(d-c)}\int_0^1 t\left[s\frac{\partial^2}{\partial s\partial t}f(tx+(1-t)a,sy+(1-s)c)\,ds\right]dt\\
&=\frac{(x-a)(y-c)}{(b-a)(d-c)}f(x,y)-\frac{x-a}{(b-a)(d-c)}\int_c^y f(x,v)\,dv\\
&-\frac{y-c}{(b-a)(d-c)}\int_a^x f(u,y)\,du+\frac{1}{(b-a)(d-c)}\int_a^x\int_c^y f(u,v)\,dvdu \qquad (2.2)
\end{aligned}$$

Similarly, by integration by parts, we also have

$$\begin{aligned}
&\frac{(x-a)^2(d-y)^2}{(b-a)(d-c)}\int_0^1\int_0^1 st\frac{\partial^2}{\partial s\partial t}f(tx+(1-t)a,sy+(1-s)d)\,dsdt\\
&=-\frac{(x-a)(d-y)}{(b-a)(d-c)}f(x,y)-\frac{x-a}{(b-a)(d-c)}\int_d^y f(x,v)\,dv\\
&+\frac{d-y}{(b-a)(d-c)}\int_a^x f(u,y)\,du+\frac{1}{(b-a)(d-c)}\int_a^x\int_d^y f(u,v)\,dvdu, \qquad (2.3)
\end{aligned}$$

$$\begin{aligned}
&\frac{(b-x)^2(y-c)^2}{(b-a)(d-c)}\int_0^1\int_0^1 st\frac{\partial^2}{\partial s\partial t}f(tx+(1-t)b,sy+(1-s)c)\,dsdt\\
&=-\frac{(b-x)(y-c)}{(b-a)(d-c)}f(x,y)+\frac{b-x}{(b-a)(d-c)}\int_c^y f(x,v)\,dv\\
&-\frac{y-c}{(b-a)(d-c)}\int_b^x f(u,y)\,du+\frac{1}{(b-a)(d-c)}\int_b^x\int_c^y f(u,v)\,dvdu \qquad (2.4)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{(b-x)^2(d-y)^2}{(b-a)(d-c)}\int_0^1\int_0^1 st\frac{\partial^2}{\partial s\partial t}f(tx+(1-t)b,sy+(1-s)d)\,dsdt \\
&=\frac{(b-x)(d-y)}{(b-a)(d-c)}f(x,y)+\frac{b-x}{(b-a)(d-c)}\int_d^y f(x,v)\,dv \\
&+\frac{d-y}{(b-a)(d-c)}\int_b^x f(u,y)\,du+\frac{1}{(b-a)(d-c)}\int_b^x\int_d^y f(u,v)\,dvdu. \qquad (2.5)
\end{aligned}$$

From (2.2)-(2.5), we get (2.1). This completes the proof. □

Theorem 3. *Let $f:\Delta\to\mathbb{R}$ be a twice partial differentiable mapping on Δ° such that $\frac{\partial^2 f}{\partial s\partial t}\in L(\Delta)$. If $\left|\frac{\partial^2 f}{\partial s\partial t}\right|$ is convex on the co-ordinates on Δ and $\left|\frac{\partial^2}{\partial s\partial t}f(x,y)\right|\le M$, $(x,y)\in\Delta$, then the following inequality holds:*

$$\begin{aligned}
&\left|f(x,y)+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(u,v)\,dvdu-A\right| \\
&\le M\left[\frac{(x-a)^2+(b-x)^2}{2(b-a)}\right]\left[\frac{(y-c)^2+(d-y)^2}{2(d-c)}\right], \qquad (2.6)
\end{aligned}$$

for all $(x,y)\in\Delta$, where A is defined in Lemma 1.

Proof. By Lemma 1, we have that the following inequality holds:

$$\begin{aligned}
&\left|f(x,y)+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(u,v)\,dvdu-A\right| \\
&\le\frac{(x-a)^2(y-c)^2}{(b-a)(d-c)}\int_0^1\int_0^1 st\left|\frac{\partial^2}{\partial s\partial t}f(tx+(1-t)a,sy+(1-s)c)\right|dsdt \\
&+\frac{(x-a)^2(d-y)^2}{(b-a)(d-c)}\int_0^1\int_0^1 st\left|\frac{\partial^2}{\partial s\partial t}f(tx+(1-t)a,sy+(1-s)d)\right|dsdt \\
&+\frac{(b-x)^2(y-c)^2}{(b-a)(d-c)}\int_0^1\int_0^1 st\left|\frac{\partial^2}{\partial s\partial t}f(tx+(1-t)b,sy+(1-s)c)\right|dsdt \\
&+\frac{(b-x)^2(d-y)^2}{(b-a)(d-c)}\int_0^1\int_0^1 st\left|\frac{\partial^2}{\partial s\partial t}f(tx+(1-t)b,sy+(1-s)d)\right|dsdt, \qquad (2.7)
\end{aligned}$$

for all $(x,y)\in\Delta$.

Using the co-ordinated convexity of $\left|\frac{\partial^2 f}{\partial s\partial t}\right|$, we get the following inequality holds:

$$\begin{aligned}
&\int_0^1\int_0^1 st\left|\frac{\partial^2}{\partial s\partial t}f(tx+(1-t)a,sy+(1-s)c)\right|dsdt \\
&\le\left|\frac{\partial^2}{\partial s\partial t}f(x,y)\right|\int_0^1\int_0^1 t^2s^2\,dsdt+\left|\frac{\partial^2}{\partial s\partial t}f(x,c)\right|\int_0^1\int_0^1 t^2s(1-s)\,dsdt \\
&+\left|\frac{\partial^2}{\partial s\partial t}f(a,y)\right|\int_0^1\int_0^1 s^2t(1-t)\,dsdt+\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|\int_0^1\int_0^1 st(1-t)(1-s)\,dsdt. \qquad (2.8)
\end{aligned}$$

Since

$$\int_0^1\int_0^1 t^2s^2dsdt=\frac{1}{9},\ \int_0^1\int_0^1 t^2s\left(1-s\right)dsdt=\int_0^1\int_0^1 s^2t\left(1-t\right)dsdt=\frac{1}{18},$$

$$\int_0^1\int_0^1 st\left(1-t\right)\left(1-s\right)dsdt=\frac{1}{36}\text{ and }\left|\frac{\partial^2}{\partial s\partial t}f\left(x,y\right)\right|\leq M,(x,y)\in\Delta.$$

Hence from (2.8), we obtain

$$\int_0^1\int_0^1 st\left|\frac{\partial^2}{\partial s\partial t}f\left(tx+\left(1-t\right)a,sy+\left(1-s\right)c\right)\right|dsdt\leq\frac{1}{4}M. \tag{2.9}$$

Analogously, we also have

$$\int_0^1\int_0^1 st\left|\frac{\partial^2}{\partial s\partial t}f\left(tx+\left(1-t\right)a,sy+\left(1-s\right)d\right)\right|dsdt\leq\frac{1}{4}M, \tag{2.10}$$

$$\int_0^1\int_0^1 st\left|\frac{\partial^2}{\partial s\partial t}f\left(tx+\left(1-t\right)b,sy+\left(1-s\right)c\right)\right|dsdt\leq\frac{1}{4}M \tag{2.11}$$

and

$$\int_0^1\int_0^1 st\left|\frac{\partial^2}{\partial s\partial t}f\left(tx+\left(1-t\right)b,sy+\left(1-s\right)d\right)\right|dsdt\leq\frac{1}{4}M. \tag{2.12}$$

Now by making use of the inequalities (2.9)-(2.12) and the fact that

$$\begin{aligned}&\left(x-a\right)^2\left(y-c\right)^2+\left(x-a\right)^2\left(d-y\right)^2+\left(b-x\right)^2\left(y-c\right)^2+\left(b-x\right)^2\left(d-y\right)^2\\&=\left[\left(x-a\right)^2+\left(b-x\right)^2\right]\left[\left(y-c\right)^2+\left(d-y\right)^2\right],\end{aligned}$$

we get the inequality (2.6). This completes the proof. □

The following result is about the powers of the absolute value of the partial derivatives:

Theorem 4. *Let $f:\Delta\to\mathbb{R}$ be a twice partial differentiable mapping on Δ° such that $\frac{\partial^2 f}{\partial s\partial t}\in L\left(\Delta\right)$. If $\left|\frac{\partial^2 f}{\partial s\partial t}\right|^q$ is convex on the co-ordinates on Δ, p, $q>1$, $\frac{1}{p}+\frac{1}{q}=1$ and $\left|\frac{\partial^2}{\partial s\partial t}f\left(x,y\right)\right|\leq M$, $(x,y)\in\Delta$, then the following inequality holds:*

$$\begin{aligned}&\left|f\left(x,y\right)+\frac{1}{\left(b-a\right)\left(d-c\right)}\int_a^b\int_c^d f\left(u,v\right)dvdu-A\right|\\&\leq\frac{M}{\left(1+p\right)^{\frac{2}{p}}}\left[\frac{\left(x-a\right)^2+\left(b-x\right)^2}{b-a}\right]\left[\frac{\left(y-c\right)^2+\left(d-y\right)^2}{d-c}\right],\end{aligned} \tag{2.13}$$

for all $(x,y)\in\Delta$, where A is defined in Lemma 1.

Proof. By Lemma 1 and using the Hölder inequality for double integrals, we have that inequality holds:

$$\left|f(x,y)+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(u,v)\,dvdu-A\right|\le\left(\int_0^1\int_0^1 s^pt^p\right)^{\frac{1}{p}}$$
$$\times\left[\frac{(x-a)^2(y-c)^2}{(b-a)(d-c)}\left(\int_0^1\int_0^1\left|\frac{\partial^2}{\partial s\partial t}f(tx+(1-t)a,sy+(1-s)c)\right|^q dsdt\right)^{\frac{1}{q}}\right.$$
$$+\frac{(x-a)^2(d-y)^2}{(b-a)(d-c)}\left(\int_0^1\int_0^1\left|\frac{\partial^2}{\partial s\partial t}f(tx+(1-t)a,sy+(1-s)d)\right|^q dsdt\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)^2(y-c)^2}{(b-a)(d-c)}\left(\int_0^1\int_0^1\left|\frac{\partial^2}{\partial s\partial t}f(tx+(1-t)b,sy+(1-s)c)\right|^q dsdt\right)^{\frac{1}{q}}$$
$$\left.+\frac{(b-x)^2(d-y)^2}{(b-a)(d-c)}\left(\int_0^1\int_0^1\left|\frac{\partial^2}{\partial s\partial t}f(tx+(1-t)b,sy+(1-s)d)\right|^q dsdt\right)^{\frac{1}{q}}\right],\tag{2.14}$$

for all $(x,y)\in\Delta$.

Since $\left|\frac{\partial^2 f}{\partial s\partial t}\right|^q$ is convex on the co-ordinates on Δ and $\left|\frac{\partial^2}{\partial s\partial t}f(x,y)\right|\le M$, $(x,y)\in\Delta$, we have

$$\int_0^1\int_0^1\left|\frac{\partial^2}{\partial s\partial t}f(tx+(1-t)a,sy+(1-s)c)\right|^q dsdt$$
$$\le\left|\frac{\partial^2}{\partial s\partial t}f(a,y)\right|^q\int_0^1\int_0^1 s(1-t)\,dsdt+\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q\int_0^1\int_0^1(1-t)(1-s)\,dsdt$$
$$+\left|\frac{\partial^2}{\partial s\partial t}f(x,y)\right|^q\int_0^1\int_0^1 tsdsdt+\left|\frac{\partial^2}{\partial s\partial t}f(x,c)\right|^q\int_0^1\int_0^1 t(1-s)\,dsdt=M^q.$$

Similarly, we also have the following inequalities:

$$\int_0^1\int_0^1\left|\frac{\partial^2}{\partial s\partial t}f(tx+(1-t)a,sy+(1-s)d)\right|^q dsdt\le M^q,$$

$$\int_0^1\int_0^1\left|\frac{\partial^2}{\partial s\partial t}f(tx+(1-t)b,sy+(1-s)c)\right|^q dsdt\le M^q$$

and

$$\int_0^1\int_0^1\left|\frac{\partial^2}{\partial s\partial t}f(tx+(1-t)b,sy+(1-s)d)\right|^q dsdt\le M^q.$$

Using the fact

$$\int_0^1\int_0^1 s^pt^p=\frac{1}{(1+p)^{\frac{1}{p}}}$$

and the above inequalities in (2.14), we get (2.13). This completes the proof of the theorem. □

A different approach leads us to the following result:

Theorem 5. *Let $f:\Delta\rightarrow\mathbb{R}$ be a twice partial differentiable mapping on Δ° such that $\frac{\partial^2 f}{\partial s\partial t}\in L(\Delta)$. If $\left|\frac{\partial^2 f}{\partial s\partial t}\right|^q$ is convex on the co-ordinates on Δ, $q\geq 1$ and $\left|\frac{\partial^2}{\partial s\partial t}f(x,y)\right|\leq M$, $(x,y)\in\Delta$, then the following inequality holds:*

$$\left|f(x,y)+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(u,v)\,dvdu-A\right| \leq \frac{M}{4}\left[\frac{(x-a)^2-(b-x)^2}{b-a}\right]\left[\frac{(y-c)^2-(d-y)^2}{d-c}\right], \tag{2.15}$$

for all $(x,y)\in\Delta$, where A is defined in Lemma 1.

Proof. Suppose $q\geq 1$. From Lemma 1 and using the power mean inequality for double integrals, we have

$$\begin{aligned}\left|f(x,y)+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(u,v)\,dvdu-A\right| &\leq \left(\int_0^1\int_0^1 st\right)^{1-\frac{1}{q}}\\ \times\Bigg[\frac{(x-a)^2(y-c)^2}{(b-a)(d-c)}&\left(\int_0^1\int_0^1 ts\left|\frac{\partial^2}{\partial s\partial t}f(tx+(1-t)a,sy+(1-s)c)\right|^q dsdt\right)^{\frac{1}{q}}\\ +\frac{(x-a)^2(y-d)^2}{(b-a)(d-c)}&\left(\int_0^1\int_0^1 ts\left|\frac{\partial^2}{\partial s\partial t}f(tx+(1-t)a,sy+(1-s)d)\right|^q dsdt\right)^{\frac{1}{q}}\\ +\frac{(b-x)^2(y-c)^2}{(b-a)(d-c)}&\left(\int_0^1\int_0^1 ts\left|\frac{\partial^2}{\partial s\partial t}f(tx+(1-t)b,sy+(1-s)c)\right|^q dsdt\right)^{\frac{1}{q}}\\ +\frac{(b-x)^2(d-y)^2}{(b-a)(d-c)}&\left(\int_0^1\int_0^1 ts\left|\frac{\partial^2}{\partial s\partial t}f(tx+(1-t)b,sy+(1-s)d)\right|^q dsdt\right)^{\frac{1}{q}}\Bigg],\end{aligned} \tag{2.16}$$

for all $(x,y)\in\Delta$.

By similar argument as in Theorem 4 that $\left|\frac{\partial^2 f}{\partial s\partial t}\right|^q$ is convex on the co-ordinates on Δ and $\left|\frac{\partial^2}{\partial s\partial t}f(x,y)\right|\leq M$, $(x,y)\in\Delta$, we have

$$\int_0^1\int_0^1 ts\left|\frac{\partial^2}{\partial s\partial t}f(tx+(1-t)a,sy+(1-s)c)\right|^q dsdt\leq\frac{1}{4}M^q,$$

$$\int_0^1\int_0^1 ts\left|\frac{\partial^2}{\partial s\partial t}f(tx+(1-t)a,sy+(1-s)d)\right|^q dsdt\leq\frac{1}{4}M^q,$$

$$\int_0^1\int_0^1 ts\left|\frac{\partial^2}{\partial s\partial t}f(tx+(1-t)b,sy+(1-s)c)\right|^q dsdt\leq\frac{1}{4}M^q$$

and

$$\int_0^1\int_0^1 ts\left|\frac{\partial^2}{\partial s\partial t}f(tx+(1-t)b,sy+(1-s)d)\right|^q dsdt\leq\frac{1}{4}M^q.$$

Now using the above inequalities and

$$\int_0^1\int_0^1 st=\frac{1}{4}$$

in (2.16), we get the desired inequality (2.15). This completes the proof. □

Remark 1. *Since* $(1+p)^{\frac{1}{p}} < 2$, $p > 1$ *and accordingly, we have*

$$\frac{1}{2} < \frac{1}{(1+p)^{\frac{1}{p}}},\ p > 1$$

which gives

$$\frac{1}{4} < \frac{1}{(1+p)^{\frac{2}{p}}},\ p > 1.$$

This reveals that the the inequality (2.15) gives tighter estimate than that of the inequality (2.13).

Remark 2. *From the inequalities proved above in Theorem 3-Theorem 5, one can get several midpoint type inequalities by setting* $x = \frac{a+b}{2}$ *and* $y = \frac{c+d}{2}$. *However the details are left to the interested reader.*

Now we drive some results with co-ordinated concavity property instead of co-ordinated convexity.

Theorem 6. *Let* $f : \Delta \to \mathbb{R}$ *be a twice partial differentiable mapping on* Δ° *such that* $\frac{\partial^2 f}{\partial s \partial t} \in L(\Delta)$. *If* $\left|\frac{\partial^2 f}{\partial s \partial t}\right|^q$ *is concave on the co-ordinates on* Δ *and* $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, *then the inequality*

$$\left| f(x,y) + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(u,v)\, dv du - A \right| \leq \frac{1}{(1+p)^{\frac{2}{p}} (b-a)(d-c)}$$
$$\times \left[(x-a)^2 \left\{ (y-c)^2 \left| \frac{\partial^2}{\partial s \partial t} f\left(\frac{x+a}{2}, \frac{d+y}{2} \right) \right| + (d-y)^2 \left| \frac{\partial^2}{\partial s \partial t} f\left(\frac{x+a}{2}, \frac{d+y}{2} \right) \right| \right\} \right.$$
$$\left. + (b-x)^2 \left\{ (y-c)^2 \left| \frac{\partial^2}{\partial s \partial t} f\left(\frac{b+a}{2}, \frac{y+c}{2} \right) \right| + (d-y)^2 \left| \frac{\partial^2}{\partial s \partial t} f\left(\frac{b+x}{2}, \frac{d+y}{2} \right) \right| \right\} \right], \tag{2.17}$$

hods for all $(x,y) \in \Delta$, *where* A *is defined in Lemma 1.*

Proof. From Lemma 1 and using the Hölder inequality for double integrals, we have that inequality holds:

$$\left| f(x,y) + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(u,v)\, dv du - A \right| \leq \left(\int_0^1 \int_0^1 s^p t^p \right)^{\frac{1}{p}}$$
$$\times \left[\frac{(x-a)^2 (y-c)^2}{(b-a)(d-c)} \left(\int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)a, sy + (1-s)c) \right|^q ds dt \right)^{\frac{1}{q}} \right.$$
$$+ \frac{(x-a)^2 (d-y)^2}{(b-a)(d-c)} \left(\int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)a, sy + (1-s)d) \right|^q ds dt \right)^{\frac{1}{q}}$$
$$+ \frac{(b-x)^2 (y-c)^2}{(b-a)(d-c)} \left(\int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)b, sy + (1-s)c) \right|^q ds dt \right)^{\frac{1}{q}}$$
$$\left. + \frac{(b-x)^2 (d-y)^2}{(b-a)(d-c)} \left(\int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)b, sy + (1-s)d) \right|^q ds dt \right)^{\frac{1}{q}} \right], \tag{2.18}$$

for all $(x, y) \in \Delta$.

Since $\left|\frac{\partial^2 f}{\partial s \partial t}\right|^q$ is concave on the co-ordinates on Δ, so an application of (1.3) with inequalities in reversed direction, gives us the following inequalities:

$$\begin{aligned}
&\int_0^1 \int_0^1 \left|\frac{\partial^2}{\partial s \partial t} f\left(tx + (1-t)a, sy + (1-s)c\right)\right|^q dsdt \\
&\leq \frac{1}{2}\left[\int_0^1 \left|\frac{\partial^2}{\partial s \partial t} f\left(tx + (1-t)a, \frac{y+c}{2}\right)\right|^q dt + \int_0^1 \left|\frac{\partial^2}{\partial s \partial t} f\left(\frac{x+a}{2}, sy + (1-s)c\right)\right|^q ds\right] \\
&\leq \left|\frac{\partial^2}{\partial s \partial t} f\left(\frac{x+a}{2}, \frac{y+c}{2}\right)\right|^q, \qquad (2.19)
\end{aligned}$$

$$\begin{aligned}
&\int_0^1 \int_0^1 \left|\frac{\partial^2}{\partial s \partial t} f\left(tx + (1-t)a, sy + (1-s)d\right)\right|^q dsdt \\
&\leq \frac{1}{2}\left[\int_0^1 \left|\frac{\partial^2}{\partial s \partial t} f\left(tx + (1-t)a, \frac{d+y}{2}\right)\right|^q dt + \int_0^1 \left|\frac{\partial^2}{\partial s \partial t} f\left(\frac{x+a}{2}, sy + (1-s)c\right)\right|^q ds\right] \\
&\leq \left|\frac{\partial^2}{\partial s \partial t} f\left(\frac{x+a}{2}, \frac{d+y}{2}\right)\right|^q, \qquad (2.20)
\end{aligned}$$

$$\begin{aligned}
&\int_0^1 \int_0^1 \left|\frac{\partial^2}{\partial s \partial t} f\left(tx + (1-t)b, sy + (1-s)c\right)\right|^q dsdt \\
&\leq \frac{1}{2}\left[\int_0^1 \left|\frac{\partial^2}{\partial s \partial t} f\left(tx + (1-t)a, \frac{y+c}{2}\right)\right|^q dt + \int_0^1 \left|\frac{\partial^2}{\partial s \partial t} f\left(\frac{b+x}{2}, sy + (1-s)c\right)\right|^q ds\right] \\
&\leq \left|\frac{\partial^2}{\partial s \partial t} f\left(\frac{b+a}{2}, \frac{y+c}{2}\right)\right|^q \qquad (2.21)
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 \int_0^1 \left|\frac{\partial^2}{\partial s \partial t} f\left(tx + (1-t)b, sy + (1-s)d\right)\right|^q dsdt \\
&\leq \frac{1}{2}\left[\int_0^1 \left|\frac{\partial^2}{\partial s \partial t} f\left(tx + (1-t)b, \frac{d+y}{2}\right)\right|^q dt + \int_0^1 \left|\frac{\partial^2}{\partial s \partial t} f\left(\frac{b+x}{2}, sy + (1-s)d\right)\right|^q ds\right] \\
&\leq \left|\frac{\partial^2}{\partial s \partial t} f\left(\frac{b+x}{2}, \frac{d+y}{2}\right)\right|^q. \qquad (2.22)
\end{aligned}$$

By making use of (2.19)-(2.22) in (2.18), we obtain (2.17). Thus the proof of the theorem is complete. □

Theorem 7. *Let $f : \Delta \rightarrow \mathbb{R}$ be a twice partial differentiable mapping on Δ° such that $\frac{\partial^2 f}{\partial s \partial t} \in L(\Delta)$. If $\left|\frac{\partial^2 f}{\partial s \partial t}\right|^q$ is concave on the co-ordinates on Δ and $q \geq 1$, then* *the inequality*

$$\left|f(x,y)+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(u,v)\,dvdu-A\right|$$
$$\leq\frac{1}{4(b-a)(d-c)}\left[(x-a)^2\left\{(y-c)^2\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{3x+a}{3},\frac{3y+c}{3}\right)\right|\right.\right.$$
$$\left.+(d-y)^2\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{3x+a}{3},\frac{3y+d}{3}\right)\right|\right\}+(b-x)^2\left\{(y-c)^2\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{3x+b}{3},\frac{3y+c}{3}\right)\right|\right.$$
$$\left.\left.+(d-y)^2\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{3x+b}{3},\frac{3y+d}{3}\right)\right|\right\}\right],\tag{2.23}$$

hods for all $(x,y)\in\Delta$, where A is defined in Lemma 1.

Proof. By the concavity of $\left|\frac{\partial^2 f}{\partial s\partial t}\right|^q$ on the co-ordinates on Δ and power mean inequality, we note that the following inequality holds:

$$\left|\frac{\partial^2}{\partial s\partial t}f(\lambda x+(1-\lambda)y,v)\right|^q\geq\lambda\left|\frac{\partial^2}{\partial s\partial t}f(x,v)\right|^q+(1-\lambda)\left|\frac{\partial^2}{\partial s\partial t}f(y,v)\right|^q$$
$$\geq\left(\lambda\left|\frac{\partial^2}{\partial s\partial t}f(x,v)\right|+(1-\lambda)\left|\frac{\partial^2}{\partial s\partial t}f(y,v)\right|\right)^q,$$

for all x, $y\in[a,b]$, $\lambda\in[0,1]$ and for fixed $v\in[c,d]$.
Hence,

$$\left|\frac{\partial^2}{\partial s\partial t}f(\lambda x+(1-\lambda)y,v)\right|\geq\lambda\left|\frac{\partial^2}{\partial s\partial t}f(x,v)\right|+(1-\lambda)\left|\frac{\partial^2}{\partial s\partial t}f(y,v)\right|,$$

for all x, $y\in[a,b]$, $\lambda\in[0,1]$ and for fixed $v\in[c,d]$.
Similarly, we can show that

$$\left|\frac{\partial^2}{\partial s\partial t}f(u,\lambda z+(1-\lambda)w)\right|\geq\lambda\left|\frac{\partial^2}{\partial s\partial t}f(u,z)\right|+(1-\lambda)\left|\frac{\partial^2}{\partial s\partial t}f(u,w)\right|,$$

for all z, $w\in[c,d]$, $\lambda\in[0,1]$ and for fixed $u\in[a,d]$, thus $\left|\frac{\partial^2 f}{\partial s\partial t}\right|$ is concave on the co-ordinates on Δ.

From Lemma 1, we have

$$\left|f(x,y)+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(u,v)\,dvdu-A\right|$$
$$\leq\frac{(x-a)^2(y-c)^2}{(b-a)(d-c)}\int_0^1\int_0^1 st\left|\frac{\partial^2}{\partial s\partial t}f(tx+(1-t)a,sy+(1-s)c)\right|dsdt$$
$$+\frac{(x-a)^2(d-y)^2}{(b-a)(d-c)}\int_0^1\int_0^1 st\left|\frac{\partial^2}{\partial s\partial t}f(tx+(1-t)a,sy+(1-s)d)\right|dsdt$$
$$+\frac{(b-x)^2(y-c)^2}{(b-a)(d-c)}\int_0^1\int_0^1 st\left|\frac{\partial^2}{\partial s\partial t}f(tx+(1-t)b,sy+(1-s)c)\right|^q dsdt$$
$$+\frac{(b-x)^2(d-y)^2}{(b-a)(d-c)}\int_0^1\int_0^1 st\left|\frac{\partial^2}{\partial s\partial t}f(tx+(1-t)b,sy+(1-s)d)\right|^q dsdt,\tag{2.24}$$

for all $(x,y)\in\Delta$.

Since $\left|\dfrac{\partial^2 f}{\partial s\partial t}\right|$ is concave on the co-ordinates on Δ, we have, by the Jensens's inequality for integrals, that

$$\begin{aligned}
&\int_0^1\int_0^1 st\left|\frac{\partial^2}{\partial s\partial t}f\left(tx+(1-t)\,a,sy+(1-s)\,c\right)\right|dsdt\\
&=\int_0^1 t\left[\int_0^1 s\left|\frac{\partial^2}{\partial s\partial t}f\left(tx+(1-t)\,a,sy+(1-s)\,c\right)\right|ds\right]dt\\
&\le\int_0^1 t\left[\left(\int_0^1 sds\right)\left|\frac{\partial^2}{\partial s\partial t}f\left(tx+(1-t)\,a,\frac{\int_0^1 s\left(sy+(1-s)\,c\right)ds}{\int_0^1 sds}\right)\right|\right]dt\\
&=\frac{1}{2}\int_0^1 t\left|\frac{\partial^2}{\partial s\partial t}f\left(tx+(1-t)\,a,\frac{3y+c}{3}\right)\right|dt\\
&\le\frac{1}{2}\left(\int_0^1 tdt\right)\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{\int_0^1 t\left(tx+(1-t)\,a\right)dt}{2},\frac{3y+c}{3}\right)\right|\\
&=\frac{1}{4}\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{3x+a}{3},\frac{3y+c}{3}\right)\right|. \qquad (2.25)
\end{aligned}$$

Analogously, we also have

$$\int_0^1\int_0^1 st\left|\frac{\partial^2}{\partial s\partial t}f\left(tx+(1-t)\,a,sy+(1-s)\,d\right)\right|dsdt\le\frac{1}{4}\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{3x+a}{3},\frac{3y+d}{3}\right)\right|, \qquad (2.26)$$

$$\int_0^1\int_0^1 st\left|\frac{\partial^2}{\partial s\partial t}f\left(tx+(1-t)\,b,sy+(1-s)\,c\right)\right|dsdt\le\frac{1}{4}\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{3x+b}{3},\frac{3y+c}{3}\right)\right| \qquad (2.27)$$

and

$$\int_0^1\int_0^1 st\left|\frac{\partial^2}{\partial s\partial t}f\left(tx+(1-t)\,b,sy+(1-s)\,d\right)\right|dsdt\le\frac{1}{4}\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{3x+b}{3},\frac{3y+d}{3}\right)\right|. \qquad (2.28)$$

Using (2.25)-(2.28) in (2.24), we obtain the required inequality (2.23) and hence the theorem is complete. □

References

[1] M. Alomari, M. Darus, S.S. Dragomir and P. Cerone, Ostrowski type inequalities for functions whose derivatives are s-convex in the second sense, Applied Mathematics Letters Volume 23, Issue 9, September 2010, Pages 1071-1076.

[2] M. Alomari and M. Darus, Some Ostrowski's type inequalities for convex functions with applications, RGMIA, 13(1) (2010), Article 3. [ONLINE: http://ajmaa.org/RGMIA/v13n1.php].

[3] N. S. Barnett and S. S. Dragomir, An Ostrowski type inequality for double integrals and applications for cubature formulae, Soochow J. Math., 27(1), (2001), 109-114.

[4] P. Cerone and S.S. Dragomir, Ostrowski type inequalities for functions whose derivatives satisfy certain convexity assumptions, Demonstratio Math., 37 (2004), no. 2, 299–308.

[5] S.S. Dragomir and A. Sofo, Ostrowski type inequalities for functions whose derivatives are convex, Proceedings of the 4th International Conference on Modelling and Simulation, November 11-13, 2002. Victoria University, Melbourne, Australia. RGMIA Res. Rep. Coll., 5 (2002), Supplement, Article 30. [ONLINE: http://rgmia.vu.edu.au/v5(E).html]

[6] S. S. Dragomir, N. S. Barnett and P. Cerone, An n-dimensional version of Ostrowski's inequality for mappings of Hölder type, RGMIA Res. Pep. Coll., 2(2), (1999), 169-180.

[7] S.S. Dragomir and C.E.M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications,* RGMIA Monographs, Victoria University, 2000. Online: [http://www.staff.vu.edu.au/RGMIA/monographs/hermite_hadamard.html].

[8] A. M. Ostrowski, Über die absolutabweichung einer differentiebaren funktion von ihrem integralmitelwert, Comment. Math. Helv. 10(1938), 226-227.

[9] B. G. Pachpatte, On an inequality of Ostrowski type in three independent variables, J. Math. Anal. Appl., 249(2000), 583-591.

[10] B. G. Pachpatte, On a new Ostrowski type inequality in two independent variables, Tamkang J. Math., 32(1), (2001), 45-49

[11] B. G. Pachpatte, A new Ostrowski type inequality for double integrals, Soochow J. Math., 32(2), (2006), 317-322.

[12] M. Z. Sarikaya, On the Ostrowski type integral inequality, Acta Math. Univ. Comenianae, Vol. LXXIX, 1(2010), pp. 129-134.

[13] N. Ujević, Some double integral inequalities and applications, Appl. Math. E-Notes, 7(2007), 93-101.

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